Phase space representation of density operator: the coherent state and squeezed state

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton

(Date: January 21, 2017)

Here we discuss the phase space representation of the density operator, including Wigner function for the coherent state and squeezed state in quantum optics. The density operator of a given system includes classical as well as quantum mechanical properties.

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1. P function representation for the Density operator

We start with a density operator defined by

$$\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$$

where $P(\alpha)$ is called the Glauber-Sudarshan P function (representation) and $|\alpha\rangle$ is an coherent state. We note that

$$\begin{aligned} Tr[\hat{\rho}] &= Tr\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \\ &= \int d^2\alpha P(\alpha) Tr[|\alpha\rangle\langle\alpha|] \\ &= \int d^2\alpha P(\alpha)\langle\alpha|\alpha\rangle \\ &= \int d^2\alpha P(\alpha) \\ &= 1 \end{aligned}$$

or

$$
\begin{aligned}
Tr[\hat{\rho}] &= Tr\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \\
&= \frac{1}{\pi}\int d^2\alpha P(\alpha)\int d^2\beta\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle \\
&= \frac{1}{\pi}\int d^2\alpha P(\alpha)\int d^2\beta|\langle\beta|\alpha\rangle|^2 \\
&= \frac{1}{\pi}\int d^2\alpha P(\alpha)\int d^2\beta\exp(-|\alpha|^2-|\beta|^2+\alpha\beta^*+\alpha^*\beta) \\
&= \frac{1}{\pi}\int d^2\alpha P(\alpha)\exp(-|\alpha|^2)\int d^2\beta\exp(-|\beta|^2+2\,\mathrm{Re}[\alpha\beta^*])) \\
&= \frac{1}{\pi}\int d^2\alpha P(\alpha)\pi\exp(-|\alpha|^2)\exp(|\alpha^2|) \\
&= \int d^2\alpha P(\alpha) \\
&= 1
\end{aligned}
$$

where

$$\frac{1}{\pi}\int d^2\beta|\beta\rangle\langle\beta| = \hat{1} \qquad \text{(formula)}$$

$$|\langle\beta|\alpha\rangle|^2 = \exp\{-|\alpha|^2-|\beta|^2+2\,\mathrm{Re}[\alpha\beta^*]\}$$

Note that α and β are complex numbers,

$$\beta = x+iy\,, \qquad \alpha = a+ib$$

Since

$$
\begin{aligned}
\mathrm{Re}[\alpha\beta^*] &= \mathrm{Re}[(a+ib)(x-iy)] \\
&= ax+by
\end{aligned}
$$

we get the integral as

$$
\begin{aligned}
\int d^2\beta\exp(-|\beta|^2+2\,\mathrm{Re}[\alpha\beta^*])) &= \int dx\exp(-x^2+2ax)dx\int dy\exp(-y^2+2by) \\
&= \pi\exp(a^2+b^2) \\
&= \pi\exp(|\alpha^2|)
\end{aligned}
$$

We also note that the density operator can be rewritten as

$$\hat{\rho} = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta |\alpha\rangle\langle\alpha|\hat{\rho}|\beta\rangle\langle\beta|$$
$$= \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta \langle\alpha|\hat{\rho}|\beta\rangle|\alpha\rangle\langle\beta|$$

using the formula

$$\frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha| = \hat{1}$$

((**Example**))
The average number of photons can be written as

$$\langle\hat{n}\rangle = \langle\hat{a}^+\hat{a}\rangle$$
$$= Tr[\hat{\rho}\hat{a}^+\hat{a}]$$
$$= Tr[\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|\hat{a}^+\hat{a}]$$
$$= \int d^2\alpha P(\alpha)\langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle$$
$$= \int d^2\alpha P(\alpha)|\alpha|^2$$

So $P(\alpha)$ is normalized as a classical probability distribution.

((**Note**))
Here we note that

$$\langle\beta|\hat{\rho}|\beta\rangle = \int d^2\alpha P(\alpha)\langle\beta|\alpha\rangle\langle\alpha|\beta\rangle$$
$$= \int d^2\alpha P(\alpha)|\langle\beta|\alpha\rangle|^2$$
$$= \int d^2\alpha P(\alpha)\exp[-|\alpha-\beta|^2]$$

since

$$|\langle\beta|\alpha\rangle|^2 = \exp\{-|\alpha|^2 - |\beta|^2 + 2\,\mathrm{Re}[\alpha\beta^*]\} = \exp[-|\alpha-\beta|^2]$$

Since $\exp[-|\alpha-\beta|^2]$ is not a delta function, the diagonal elements

$$\langle \beta | \hat{\rho} | \beta \rangle \neq P(\beta)$$

Although $\langle \beta | \hat{\rho} | \beta \rangle$ must be positive. However, the integral $\int d^2\alpha P(\alpha) \exp[-|\alpha - \beta|^2]$ does not require to be positive.

2. Two-dimensional Dirac delta function of the complex variable

Two-dimensional Dirac delta function of the complex variable, $\alpha = \alpha' + i\alpha''$.

$$\begin{aligned}
\delta^{(2)}(\alpha) &= \delta(\alpha')\delta(\alpha'') \\
&= \frac{1}{(2\pi)^2} \int dx \int dy \exp[i(\alpha' x + \alpha'' y)] \\
&= \frac{1}{\pi^2} \int d^2\gamma \exp(\alpha^* \gamma - \alpha \gamma^*) \\
&= \frac{1}{\pi^2} \int d^2\gamma \exp(\alpha \gamma^* - \alpha^* \gamma)
\end{aligned}$$

or

$$\delta^{(2)}(\alpha) = \frac{1}{\pi^2} \int d^2\gamma \exp(\alpha \gamma^* - \alpha^* \gamma)$$

where

$$\alpha^* \gamma - \alpha \gamma^* = 2i \operatorname{Im}(\alpha^* \gamma) = i(\alpha' x + \alpha'' y)$$

and

$$\gamma = -\frac{y}{2} + i\frac{x}{2} = \frac{1}{2}(-y + ix)$$

We also note that the 2D Dirac delta function is given by

$$\delta^{(2)}(\alpha - \beta) = \frac{1}{\pi^2} \int d^2\gamma \exp[(\alpha - \beta)\gamma^* - (\alpha^* - \beta^*)\gamma]$$

3. Expression of $P(\alpha)$

How can we find the expression of $P(\alpha)$? We consider the density operator which is defined by

$$\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$$

Using the formula for the scalar product for the coherent states,

$$\langle\alpha|\beta\rangle = \exp[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta]$$

we get the matrix element of the density operator as

$$\begin{aligned}\langle -u|\hat{\rho}|u\rangle &= \int d^2\alpha P(\alpha)\langle -u|\alpha\rangle\langle\alpha|u\rangle \\ &= \exp[-|u|^2]\int d^2\alpha P(\alpha)\exp[-|\alpha|^2]\exp[\alpha^* u - \alpha u^*]\end{aligned}$$

or

$$\exp[|u|^2]\langle -u|\hat{\rho}|u\rangle = \int d^2\alpha P(\alpha)\exp[-|\alpha|^2]\exp[\alpha^* u - \alpha u^*]$$

Here we use the 2D Dirac delta function.

$$\begin{aligned}\int d^2u \exp[|u|^2]\langle -u|\hat{\rho}|u\rangle \exp[-\alpha^* u + \alpha u^*] &= \int d^2\beta\, P(\beta)\exp[-|\beta|^2]\int d^2u \exp[-\alpha^* u + \alpha u^*]\exp[\beta^* u - \beta u^*] \\ &= \int d^2\beta\, P(\beta)\exp[-|\beta|^2]\int d^2u \exp[(\alpha-\beta)u^* - (\alpha-\beta)^* u] \\ &= \pi^2\int d^2\beta\, P(\beta)\exp[-|\beta|^2]\delta(\alpha-\beta) \\ &= \pi^2 P(\alpha)\exp[-|\alpha|^2]\end{aligned}$$

or

$$P(\alpha) = \frac{1}{\pi^2}e^{|\alpha|^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle e^{(u^*\alpha - u\alpha^*)}d^2u$$

4. Another expression of $P(\alpha)$)

The expression of $P(\alpha)$ can be also obtained as follows.

$$\left\langle e^{\gamma \hat{a}^+} e^{-\gamma \hat{a}} \right\rangle = Tr[\hat{\rho} e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}}]$$

$$= Tr[\int d^2\beta \, P(\beta) |\beta\rangle\langle\beta| e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}}]$$

$$= \int d^2\beta P(\beta) \langle\beta| e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}} |\beta\rangle$$

$$= \int d^2\beta P(\beta) e^{\gamma\beta^* - \gamma^*\beta}$$

Here we use the expression of the 2D Dirac function as

$$\delta^{(2)}(\alpha - \beta) = \frac{1}{\pi^2} \int d^2\gamma \exp[(\alpha - \beta)\gamma^* - (\alpha^* - \beta^*)\gamma]$$

Thus we get

$$\frac{1}{\pi^2} \int d^2\gamma \left\langle e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}} \right\rangle e^{-\gamma\alpha^* + \gamma^*\alpha} = \frac{1}{\pi^2} \int d^2\gamma (\int d^2\beta \, P(\beta) e^{\gamma\beta^* - \gamma^*\beta})(e^{-\gamma\alpha^* + \gamma^*\alpha})$$

$$= \int d^2\beta \, P(\beta) \frac{1}{\pi^2} \int d^2\gamma e^{-\gamma(\alpha^* - \beta^*) + \gamma^*(\alpha - \beta)}$$

$$= \int d^2\beta \, P(\beta) \delta^{(2)}(\alpha - \beta)$$

$$= P(\alpha)$$

or

$$P(\alpha) = \frac{1}{\pi^2} \int d^2\gamma \left\langle e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}} \right\rangle e^{-\gamma\alpha^* + \gamma^*\alpha}$$

$$= \frac{1}{\pi^2} \int d^2\gamma Tr[\hat{\rho} e^{\gamma(\hat{a}^+ - \alpha^*)} e^{-\gamma^*(\hat{a} - \alpha)}]$$

Here we note that Tr under the integral is characteristic function of the normally ordered $\hat{a}$, $\hat{a}^+$ operators,

$$\chi_N(\gamma) = \left\langle e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}} \right\rangle = Tr[\hat{\rho} e^{\gamma \hat{a}^+} e^{-\gamma^* \hat{a}}]$$

Therefore we have

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \left\langle e^{\gamma\hat{a}^+} e^{-\gamma^*\hat{a}} \right\rangle e^{-\gamma\alpha^* + \gamma^*\alpha}$$
$$= \frac{1}{\pi^2}\int d^2\gamma\, \chi_N(\gamma) e^{-\gamma\alpha^* + \gamma^*\alpha}$$

This relation represents a mapping from the density operator $\hat{\rho}$, which is a function of the two operators $\hat{a}$ and $\hat{a}^+$, to the scalar function $\chi_N(\gamma)$, which is function of the complex variables γ. $P(\alpha)$ is the Fourier transform of the normally ordered characteristic function.

5. *P* function for the coherent state (I)

We consider the density operator for the pure coherent state $|\beta\rangle$;

$$\hat{\rho} = |\beta\rangle\langle\beta|.$$

We note that

$$\langle -u|\hat{\rho}|u\rangle = \langle -u|\beta\rangle\langle\beta|u\rangle$$
$$= \exp[-\frac{1}{2}|u|^2 - \frac{1}{2}|\beta|^2 - u^*\beta]\exp[-\frac{1}{2}|\beta|^2 - \frac{1}{2}|u|^2 + \beta^* u]$$
$$= \exp(-|u|^2 - |\beta|^2 - u^*\beta + \beta^* u)$$

where

$$\langle\beta|u\rangle = \exp[-\frac{1}{2}|\beta|^2 - \frac{1}{2}|u|^2 + \beta^* u]$$

$$\langle -u|\beta\rangle = \exp[-\frac{1}{2}|u|^2 - \frac{1}{2}|\beta|^2 - u^*\beta]$$

Then we get

$$
\begin{aligned}
P(\alpha) &= \frac{1}{\pi^2} e^{|\alpha|^2} \int e^{|u|^2} \langle -u|\hat{\rho}|u\rangle e^{(u^*\alpha - u\alpha^*)} d^2u \\
&= \frac{1}{\pi^2} e^{|\alpha|^2} \int e^{|u|^2} e^{(-|u|^2 - |\beta|^2 - u^*\beta + \beta^* u)} e^{(u^*\alpha - u\alpha^*)} d^2u \\
&= \frac{1}{\pi^2} e^{|\alpha|^2 - |\beta|^2} \int e^{(--u^*\beta + \beta^* u)} e^{(u^*\alpha - u\alpha^*)} d^2u \\
&= \frac{1}{\pi^2} e^{|\alpha|^2 - |\beta|^2} \int e^{(u^*(\alpha-\beta) - u(\alpha^* - \beta^*)} d^2u \\
&= e^{|\alpha|^2 - |\beta|^2} \delta^{(2)}(\alpha - \beta) \\
&= \delta^{(2)}(\alpha - \beta)
\end{aligned}
$$

or

$P(\alpha) = \delta^{(2)}(\alpha - \beta)$ for the density operator (pure state $\hat{\rho} = |\beta\rangle\langle\beta|$).

6. *P* function for the coherent state (II)

We consider the density operator for the pure coherent state $|\beta\rangle$;

$$
\hat{\rho} = |\beta\rangle\langle\beta|.
$$

Thus we get

$$
\begin{aligned}
\chi_N(\gamma) &= Tr[\hat{\rho} e^{\gamma\hat{a}^+} e^{-\gamma^*\hat{a}}] \\
&= Tr[|\beta\rangle\langle\beta| e^{\gamma\hat{a}^+} e^{-\gamma^*\hat{a}}] \\
&= \langle\beta| e^{\gamma\hat{a}^+} e^{-\gamma^*\hat{a}} |\beta\rangle \\
&= e^{\gamma\beta^* - \gamma^*\beta}
\end{aligned}
$$

$$
\begin{aligned}
P(\alpha) &= \frac{1}{\pi^2} \int d^2\gamma \left\langle e^{\gamma\hat{a}^+} e^{-\gamma^*\hat{a}} \right\rangle e^{-\gamma\alpha^* + \gamma^*\alpha} \\
&= \frac{1}{\pi^2} \int d^2\gamma \, \chi_N(\gamma) e^{-\gamma\alpha^* + \gamma^*\alpha}
\end{aligned}
$$

$$
\begin{aligned}
P(\alpha) &= \frac{1}{\pi^2} \int d^2\gamma e^{\gamma\beta^* - \gamma^*\beta} e^{-\gamma\alpha^* + \gamma^*\alpha} \\
&= \frac{1}{\pi^2} \int d^2\gamma \, e^{\gamma^*(\alpha-\beta) - \gamma(\alpha^* - \beta^*)} \\
&= \delta^{(2)}(\alpha - \beta)
\end{aligned}
$$

7. *P* function for the number state $|n\rangle$ (I)

We consider the density operator for the pure coherent state $|n\rangle$;

$$\hat{\rho} = |n\rangle\langle n|.$$

We note that

$$\langle -u|\hat{\rho}|u\rangle = \langle -u|n\rangle\langle n|u\rangle = e^{-|u|^2}\frac{(-u^*u)^n}{n!}$$

Thus we get

$$\begin{aligned}
P(\alpha) &= \frac{1}{\pi^2}e^{|\alpha|^2}\int e^{|u|^2}\langle -u|\hat{\rho}|u\rangle e^{(u^*\alpha - u\alpha^*)}d^2u \\
&= \frac{1}{\pi^2}e^{|\alpha|^2}\int e^{|u|^2}e^{-|u|^2}\frac{(-u^*u)^n}{n!}e^{(u^*\alpha - u\alpha^*)}d^2u \\
&= \frac{e^{|\alpha|^2}}{n!}\frac{1}{\pi^2}\int (-u^*u)^n e^{(u^*\alpha - u\alpha^*)}d^2u
\end{aligned}$$

Formally, we can write

$$\begin{aligned}
P(\alpha) &= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^*}\frac{1}{\pi^2}\int e^{(u^*\alpha - u\alpha^*)}d^2u \\
&= \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta(\alpha)
\end{aligned}$$

8. *P* function for the number state $|n\rangle$ (II)

We consider the density operator for the pure number state $|n\rangle$;

$$\hat{\rho} = |n\rangle\langle n|.$$

Thus we get

$$\begin{aligned}
\chi_N(\gamma) &= Tr[\hat{\rho}e^{\gamma\hat{a}^+}e^{-\gamma^*\hat{a}}] \\
&= Tr[|n\rangle\langle n|e^{\gamma\hat{a}^+}e^{-\gamma^*\hat{a}}] \\
&= \langle n|e^{\gamma\hat{a}^+}e^{-\gamma^*\hat{a}}|n\rangle
\end{aligned}$$

$$
\begin{aligned}
P(\alpha) &= \frac{1}{\pi^2}\int d^2\gamma\ \chi_N(\gamma)e^{-\gamma\alpha^*+\gamma^*\alpha} \\
&= \frac{1}{\pi^2}\int d^2\gamma\ \langle n|e^{\gamma\hat{a}^+}e^{-\gamma^*\hat{a}}|n\rangle e^{-\gamma\alpha^*+\gamma^*\alpha} \\
&= \frac{1}{\pi^3}\int d^2\gamma\ \langle n|e^{\gamma\hat{a}^+}|\beta\rangle\langle\beta|e^{-\gamma^*\hat{a}}|n\rangle e^{-\gamma\alpha^*+\gamma^*\alpha} \\
&= \frac{1}{\pi}\int d^2\beta\langle n|\beta\rangle\langle\beta|n\rangle\frac{1}{\pi^2}\int\ d^2\gamma e^{\gamma^*(\alpha-\beta)-\gamma(\alpha^*-\beta^*)} \\
&= \frac{1}{\pi}\int d^2\beta\langle n|\beta\rangle\langle\beta|n\rangle\delta^{(2)}(\alpha-\beta) \\
&= \frac{1}{\pi}|\langle n|\alpha\rangle|^2
\end{aligned}
$$

or

$$
P(\alpha) = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{\pi n!}e^{-|\alpha|^2}
$$

which is a Poisson function.

where

$$
\langle\beta|e^{-\gamma^*\hat{a}}|n\rangle^* = \langle n|e^{-\gamma\hat{a}^+}|\beta\rangle = e^{-\gamma\beta^*}\langle n|\beta\rangle
$$

$$
\langle\beta|e^{-\gamma^*\hat{a}}|n\rangle = e^{-\gamma^*\beta}\langle\beta|n\rangle
$$

9. Q-representation

There are other orderings possible for the two operators $\hat{a}$ and $\hat{a}^+$. This time we use the antisymmetric ordering and define the antinormally ordered characteristic function

$$
\chi_A(\gamma) = Tr[\hat{\rho}e^{-\gamma^*\hat{a}}e^{\gamma\hat{a}^+}]
$$

Its Fourier transform is called Q representation. This representation is sometimes called the Husimi function.

$$
Q(\alpha) = \frac{1}{\pi^2}\int d^2\gamma\ \chi_A(\gamma)e^{\gamma^*\alpha-\gamma\alpha^*}
$$

and has a simple form

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \geq 0$$

Thus $\pi Q(\alpha)$ is the probability of finding a system in the density operator $\hat{\rho}$ in the coherent state $|\alpha\rangle$. This proof is given below.

$$\begin{aligned}
\chi_A(\gamma) &= Tr[\hat{\rho}e^{-\gamma^*\hat{a}}e^{\gamma\hat{a}^+}] \\
&= \frac{1}{\pi}\int d^2\beta\, Tr[\hat{\rho}e^{-\gamma^*\hat{a}}|\beta\rangle\langle\beta|e^{\gamma\hat{a}^+}
\end{aligned}$$

$$\begin{aligned}
Q(\alpha) &= \frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*\alpha-\gamma\alpha^*}\, \frac{1}{\pi}\int d^2\beta\, Tr[\hat{\rho}e^{-\gamma^*\hat{a}}|\beta\rangle\langle\beta|e^{\gamma\hat{a}^+}] \\
&= \frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*\alpha-\gamma\alpha^*}\, \frac{1}{\pi}\int d^2\beta\, Tr[\hat{\rho}|\beta\rangle\langle\beta|e^{\gamma\hat{a}^+}]e^{-\gamma^*\beta+\gamma\beta^*} \\
&= \frac{1}{\pi}\int d^2\beta\, Tr[\hat{\rho}|\beta\rangle\langle\beta|]\frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*(\alpha-\beta)-\gamma(\alpha^*-\beta^*)} \\
&= \frac{1}{\pi}\int d^2\beta\, [\langle\beta|\hat{\rho}|\beta\rangle\delta^{(2)}(\alpha-\beta) \\
&= \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle
\end{aligned}$$

((**Husimi function**))

Kôdi Husimi (June 29, 1909 – May 8, 2008, Japanese:) was a Japanese theoretical physicist who served as the president of the Science Council of Japan. Husimi trees in graph theory, and the Husimi Q representation in quantum mechanics, are named after him.
https://en.wikipedia.org/wiki/K%C3%B4di_Husimi

Fig. Photo of Prof. Dodi Husimi.
http://www.47news.jp/PN/200805/PN2008050901000355.-.-.CI0003.jpg

10. Q representation for a pure coherent state

For the density operator

$$\hat{\rho} = |\beta\rangle\langle\beta|$$

we have

$$Q(\alpha) = \frac{1}{\pi}|\langle\beta|\alpha\rangle|^2 = \frac{1}{\pi}\exp[-|\alpha|^2 - |\beta|^2 + \alpha\beta^* + \alpha^*\beta]$$

since

$$\langle\beta|\alpha\rangle = \exp[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha\beta^*]$$

11. *Q* representation for the pure number state

For the density operator

$$\hat{\rho} = |n\rangle\langle n|$$

we have

$$Q(\alpha) = \frac{1}{\pi}\left|\langle n|\alpha\rangle\right|^2 = \frac{|\alpha|^{2n}}{\pi n!}\exp(-|\alpha|^2)$$

which is the Poisson function

12. Wigner representation

For symmetric order of two operators $\hat{a}$ and $\hat{a}^+$. The characteristic function is defined as

$$\chi_S(\gamma) = Tr[\hat{\rho}\exp(\gamma\hat{a}^+ - \gamma^*\hat{a})] = Tr[\hat{\rho}\hat{D}(\gamma)] = \langle\hat{D}(\gamma)\rangle$$

Its Fourier transform is called Wigner representation.

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2}\int d^2\gamma\ \chi_S(\gamma)e^{\gamma^*\alpha-\gamma\alpha^*} \\ &= \frac{1}{\pi^2}\int d^2\gamma\ e^{\gamma^*\alpha-\gamma\alpha^*}Tr[\hat{\rho}\hat{D}(\gamma)] \end{aligned}$$

13. Another expression of Wigner function

$$\hat{X} = \frac{\beta}{\sqrt{2}}\hat{x} = \frac{\beta}{\sqrt{2}}\frac{1}{\sqrt{2}\beta}(\hat{a}+\hat{a}^+) = \frac{1}{2}(\hat{a}+\hat{a}^+),$$

$$\hat{Y} = \frac{1}{\sqrt{2}\hbar\beta}\hat{p} = \frac{1}{\sqrt{2}\hbar\beta}\frac{m\omega_0}{\sqrt{2}\beta i}(\hat{a}-\hat{a}^+) = \frac{1}{2i}(\hat{a}-\hat{a}^+)$$

or

$$\hat{a} = \hat{X} + i\hat{Y}, \qquad \hat{a}^+ = \hat{X} - i\hat{Y}.$$

Note that

$$[\hat{X},\hat{Y}] = [\frac{\beta}{\sqrt{2}}\hat{x}, \frac{1}{\sqrt{2}\hbar\beta}\hat{p}] = \frac{\beta}{\sqrt{2}}\frac{1}{\sqrt{2}\hbar\beta}[\hat{x},\hat{p}] = \frac{1}{2\hbar}[\hat{x},\hat{p}] = \frac{1}{2}i\hat{1},$$

$$\begin{aligned}\hat{D}(\gamma) &= \exp(\gamma\hat{a}^{+} - \gamma^{*}\hat{a}) \\ &= \exp[\gamma(\hat{X} - i\hat{Y}) - \gamma^{*}(\hat{X} + i\hat{Y})] \\ &= \exp[(\gamma - \gamma^{*})\hat{X} - i(\gamma + \gamma^{*})\hat{Y}] \\ &= \exp[2i\gamma''\hat{X} - 2i\gamma'\hat{Y}] \\ &= \exp(-i\gamma'\gamma'')\exp(2i\gamma''\hat{X})\exp(-2i\gamma'\hat{Y})\end{aligned}$$

This operator can be rewritten as

$$\begin{aligned}\hat{D}(\gamma) &= \exp(-i\gamma'\gamma'')\exp(2i\gamma''\frac{\beta}{\sqrt{2}}\hat{x})\exp(-2i\gamma'\frac{1}{\sqrt{2}\hbar\beta}\hat{p}) \\ &= \exp(-i\gamma'\gamma'')\exp(i\sqrt{2}\beta\gamma''\hat{x})\exp(-\frac{\sqrt{2}\gamma'}{\beta}i\frac{\hat{p}}{\hbar})\end{aligned}$$

Then we get

$$\begin{aligned}\hat{D}(\gamma)\left|x'\right\rangle &= \exp(-i\gamma'\gamma'')\exp(i\sqrt{2}\beta\gamma''\hat{x})\exp(-\frac{\sqrt{2}\gamma'}{\beta}i\frac{\hat{p}}{\hbar})\left|x'\right\rangle \\ &= \exp(-i\gamma'\gamma'')\exp(i\sqrt{2}\beta\gamma''\hat{x})\left|x'+\frac{\sqrt{2}\gamma'}{\beta}\right\rangle \\ &= \exp(-i\gamma'\gamma'')\exp(i\sqrt{2}\beta\gamma''(x'+\frac{\sqrt{2}\gamma'}{\beta})\left|x'+\frac{\sqrt{2}\gamma'}{\beta}\right\rangle \\ &= \exp(i\gamma'\gamma'')\exp(i\sqrt{2}\beta\gamma''x')\left|x'+\frac{\sqrt{2}\gamma'}{\beta}\right\rangle\end{aligned}$$

Wigner function

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma\ \chi_S(\gamma)e^{\gamma^*\alpha-\gamma\alpha^*}$$
$$= \frac{1}{\pi^2}\int d^2\gamma\ e^{\gamma^*\alpha-\gamma\alpha^*}Tr[\hat{\rho}\hat{D}(\gamma)]$$
$$= \frac{1}{\pi^2}\int d^2\gamma\ e^{\gamma^*\alpha-\gamma\alpha^*}\int dx'\langle x'|\hat{\rho}\hat{D}(\gamma)|x'\rangle$$
$$= \frac{1}{\pi^2}\int d^2\gamma\ e^{\gamma^*\alpha-\gamma\alpha^*}\int dx'\exp(i\gamma'\gamma''+i\sqrt{2}\beta\gamma''x')\left\langle x'\middle|\hat{\rho}\middle|x'+\frac{\sqrt{2}\gamma'}{\beta}\right\rangle$$

Note that

$$\gamma\alpha^* - \gamma^*\alpha = 2i\gamma'\alpha'' - 2i\gamma''\alpha'$$

$$\gamma\alpha^* - \gamma^*\alpha + i\gamma'\gamma'' + i\sqrt{2}\beta\gamma''x' = 2i\gamma'\alpha'' - 2i\gamma''\alpha' + i\gamma'\gamma'' + i\sqrt{2}\beta\gamma''x'$$
$$= 2i\gamma'\alpha'' + i(-2\alpha' + \gamma' + \sqrt{2}\beta x')\gamma''$$

We use the Dirac delta function as

$$\int d\gamma''\exp[i(-2\alpha'+\gamma'+\sqrt{2}\beta x')\gamma''] = 2\pi\delta(-2\alpha'+\gamma'+\sqrt{2}\beta x')$$

Then we have

$$W(\alpha) = \frac{2}{\pi}\int dx'\int d\gamma'\exp(2i\gamma'\alpha'')\delta(-2\alpha'+\gamma'+\sqrt{2}\beta x')\left\langle x'\middle|\hat{\rho}\middle|x'+\frac{\sqrt{2}\gamma'}{\beta}\right\rangle$$
$$= \frac{2}{\pi}\int dx'\exp[2i\alpha''(2\alpha'-\sqrt{2}\beta x')]\left\langle x'\middle|\hat{\rho}\middle|x'+\frac{\sqrt{2}}{\beta}(2\alpha'-\sqrt{2}\beta x')\right\rangle$$
$$= \frac{2}{\pi}\int dx'\exp[4i\alpha''(\alpha'-\frac{\beta}{\sqrt{2}}x')]\left\langle x'\middle|\hat{\rho}\middle|\frac{2\sqrt{2}}{\beta}\alpha'-x'\right\rangle$$

Here we define

$$x' = x + \frac{u}{2}, \qquad \frac{2\sqrt{2}}{\beta}\alpha' - x' = x - \frac{u}{2}$$

$$\frac{\sqrt{2}}{\beta}\alpha' = x, \qquad \frac{\beta}{\sqrt{2}}\alpha'' = p$$

$$dx' = \frac{1}{2} du$$

$$\begin{aligned} 4\alpha''(\alpha' - \frac{\beta}{\sqrt{2}} x') &= 4\alpha''(\alpha' - \frac{\beta}{\sqrt{2}} x') \\ &= -4\frac{\beta}{\sqrt{2}} \alpha''(x' - \frac{\sqrt{2}}{\beta} \alpha') \\ &= -4\frac{\beta}{\sqrt{2}} \alpha''(x + \frac{u}{2} - x) \\ &= -2\frac{\beta}{\sqrt{2}} \alpha'' u \\ &= -2pu \end{aligned}$$

$$W(\alpha) = W(x,p) = \frac{1}{\pi} \int du\, e^{-2ipu} \left\langle x + \frac{u}{2} \middle| \hat{\rho} \middle| x - \frac{u}{2} \right\rangle$$

which is the Fourier transform of non-diagonal density matrix element in the position eigenstates basis. Using the inverse Fourier transform, the non-diagonal matrix element of the demsity operator is

$$\left\langle x + \frac{u}{2} \middle| \hat{\rho} \middle| x - \frac{u}{2} \right\rangle = \int dp\, e^{2ipu} W(x,p)$$

13. Displacement operator

With help of the 2D Dirac delta function, we introduce a new notation for the quantum states in phase space,

$$\begin{aligned} \delta^{(2)}(\hat{a} - \alpha) &= \frac{1}{\pi^2} \int d^2\gamma \exp[(\hat{a}^+ - \alpha^*)\gamma - (\hat{a} - \alpha)\gamma^*] \\ &= \frac{1}{\pi^2} \int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) \exp(\gamma\hat{a}^+ - \gamma^*\hat{a}) \\ &= \frac{1}{\pi^2} \int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) \hat{D}(\gamma) \end{aligned}$$

where $\hat{D}(\gamma)$ is the displacement operator,

$$\begin{aligned}\hat{D}(\gamma) &= \exp(\gamma\hat{a}^{+} - \gamma^{*}\hat{a}) \\ &= \exp(-\frac{1}{2}|\gamma|^{2})\exp(\gamma\hat{a}^{+})\exp(-\gamma^{*}\hat{a}) \\ &= \exp(\frac{1}{2}|\gamma|^{2})\exp(-\gamma^{*}\hat{a})\exp(\gamma\hat{a}^{+})\end{aligned}$$

We note that

$$\begin{aligned}\left\langle \delta^{(2)}(\hat{a}-\alpha)\right\rangle &= Tr[\hat{\rho}\delta^{(2)}(\hat{a}-\alpha)] \\ &= \frac{1}{\pi^{2}}\int d^{2}\gamma \exp(\gamma^{*}\alpha - \gamma\alpha^{*})Tr[\hat{\rho}\hat{D}(\gamma)] \\ &= W(\alpha)\end{aligned}$$

15. Wigner representation for a pure coherent state

For $\hat{\rho} = |\beta\rangle\langle\beta|$

$$\begin{aligned}Tr[\hat{\rho}\hat{D}(\gamma)] &= \left\langle \hat{D}(\gamma)\right\rangle \\ &= Tr[|\beta\rangle\langle\beta|\hat{D}(\gamma)] \\ &= \langle\beta|\hat{D}(\gamma)|\beta\rangle \\ &= \exp(-\frac{1}{2}|\gamma|^{2})\exp(\gamma\beta^{*} - \gamma^{*}\beta)\end{aligned}$$

where

$$\begin{aligned}\langle\beta|\hat{D}(\gamma)|\beta\rangle &= \exp(-\frac{1}{2}|\gamma|^{2})\langle\beta|\exp(\gamma\hat{a}^{+})\exp(-\gamma^{*}\hat{a})|\beta\rangle \\ &= \exp(-\frac{1}{2}|\gamma|^{2})\exp(\gamma\beta^{*} - \gamma^{*}\beta)\end{aligned}$$

Thus we get the Wigner representation

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) Tr[\hat{\rho}\hat{D}(\gamma)]$$
$$= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\exp(-\frac{1}{2}|\gamma|^2)\exp(\gamma\beta^* - \gamma^*\beta)$$
$$= \frac{1}{\pi^2}\int \exp(-\frac{1}{2}|\gamma|^2) d^2\gamma \exp[\gamma^*(\alpha-\beta) - \gamma(\alpha^*-\beta^*)]$$
$$= \frac{2}{\pi}\exp[-2|\alpha-\beta|^2]$$
$$= \frac{2}{\pi}\exp\{-2[(x-x_0)^2 + (y-y_0)^2]\}$$

or

$$W(\alpha) = \frac{2}{\pi}\exp\{-2[(x-x_0)^2 + (y-y_0)^2]\}$$

We use $\gamma = x'+iy'$, $d^2\gamma = dx'dy'$, $\alpha - \beta = x - x_0 + i(y-y_0)$ (x, y, p, p_0, q and q_0 are real). Using the Mathematica, we get

$$\int \exp(-\frac{1}{2}|\gamma|^2) d^2\gamma \exp[\gamma^*(\alpha-\beta) - \gamma(\alpha^*-\beta^*)] = \iint dx'dy' \exp[-\frac{1}{2}(x'^2+y'^2) + 2i(y-y_0)x' - 2i(x-x_0)y']$$
$$= 2\pi \exp[-2(x-x_0)^2 - 2(y-y_0)^2)]$$

16. Wigner representation for a pure squeezed state

For $\hat{\rho} = |\varsigma\rangle\langle\varsigma|$

$$Tr[\hat{\rho}\hat{D}(\gamma)] = \langle \hat{D}(\gamma)\rangle$$
$$= Tr[|\varsigma\rangle\langle\varsigma|\hat{D}(\gamma)]$$
$$= \langle\varsigma|\hat{D}(\gamma)|\varsigma\rangle$$
$$= \langle 0|\hat{S}_\varsigma^{+}\hat{D}(\gamma)\hat{S}_\varsigma|0\rangle$$

Then the Wigner function is obtained as

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) Tr[\hat{\rho}\hat{D}(\gamma)]$$
$$= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\langle 0|\hat{S}_\varsigma^{+}\hat{D}(\gamma)\hat{S}_\varsigma|0\rangle$$

where

$$\hat{D}(\gamma) = \exp(\gamma \hat{a}^+ - \gamma^* \hat{a})$$

We use the Bogoliubov transformation

$$\hat{S}_\varsigma \hat{a} \hat{S}_\varsigma{}^+ = \hat{b} = \lambda \hat{a} + \mu \hat{a}^+ \quad , \qquad \hat{S}_\varsigma \hat{a}^+ \hat{S}_\varsigma{}^+ = \hat{b}^+ = \lambda^* \hat{a}^+ + \mu^* \hat{a}$$

where

$$\hat{S}_\varsigma = \exp[\frac{1}{2}\varsigma^* \hat{a}^2 - \frac{1}{2}\varsigma (\hat{a}^+)^2], \qquad \hat{S}_\varsigma{}^+ = \exp[\frac{1}{2}\varsigma \hat{a}^{+2} - \frac{1}{2}\varsigma^* (\hat{a})^2] = \hat{S}_{-\varsigma}$$

$$|\lambda|^2 - |\mu|^2 = 1, \qquad \varsigma = s e^{i\vartheta}$$

$$\lambda = \cosh s, \qquad \mu = e^{i\vartheta} \sinh s$$

$$\hat{b}|0\rangle = 0$$

Thus we have

$$\begin{aligned}\hat{S}_\varsigma (\gamma \hat{a}^+ - \gamma^* \hat{a}) \hat{S}_\varsigma{}^+ &= \gamma \hat{b}^+ - \gamma^* \hat{b} \\ &= \gamma (\lambda^* \hat{a}^+ + \mu^* \hat{a}) - \gamma^* (\lambda \hat{a} + \mu \hat{a}^+) \\ &= (\gamma \lambda^* - \gamma^* \mu) \hat{a}^+ - (\gamma^* \lambda - \gamma \mu^*) \hat{a}\end{aligned}$$

Here we switch the variable as $\varsigma \to -\varsigma$, leading to the change of parameters ($\mu \to -\mu$ and $\hat{S}_{-\varsigma} \to \hat{S}_\varsigma{}^+$).

$$\hat{S}_{-\varsigma} (\gamma \hat{a}^+ - \gamma^* \hat{a}) \hat{S}_{-\varsigma}{}^+ = (\gamma \lambda^* + \gamma^* \mu) \hat{a}^+ - (\gamma^* \lambda + \gamma \mu^*) \hat{a}$$

or

$$\begin{aligned}\hat{S}_\varsigma{}^+ (\gamma \hat{a}^+ - \gamma^* \hat{a}) \hat{S}_\varsigma &= (\gamma \lambda^* + \gamma^* \mu) \hat{a}^+ - (\gamma^* \lambda + \gamma \mu^*) \hat{a} \\ &= \gamma_1 \hat{a}^+ - \gamma_1{}^* \hat{a}\end{aligned}$$

where $\gamma_1 = \gamma\lambda^* + \gamma^*\mu$

By using the relation $\hat{S}_\varsigma^{\ +}\hat{S}_\varsigma = \hat{1}$ we get

$$\hat{S}_\varsigma^{\ +}(\gamma\hat{a}^+ - \gamma^*\hat{a})^2\hat{S}_\varsigma = \hat{S}_\varsigma^{\ +}(\gamma\hat{a}^+ - \gamma^*\hat{a})\hat{S}_\varsigma\hat{S}_\varsigma^{\ +}(\gamma\hat{a}^+ - \gamma^*\hat{a})\hat{S}_\varsigma$$
$$= (\gamma_1\hat{a}^+ - \gamma_1^{\ *}\hat{a})^2$$

which leads to the relation

$$\hat{S}_\varsigma^{\ +}\hat{D}(\gamma)\hat{S}_\varsigma = \hat{D}(\gamma_1) = \hat{D}(\gamma\lambda^* + \gamma^*\mu)$$

Using the relation

$$\langle 0|\hat{D}(\gamma_1)|0\rangle = \langle 0|\exp(-\frac{1}{2}|\gamma_1|^2)\exp(\gamma_1\hat{a}^+)\exp(-\gamma_1^{\ *}\hat{a})|0\rangle$$
$$= \exp(-\frac{1}{2}|\gamma_1|^2)\langle 0|\exp(\gamma_1\hat{a}^+)\exp(-\gamma_1^{\ *}\hat{a})|0\rangle$$
$$= \exp(-\frac{1}{2}|\gamma_1|^2)$$

we get

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\langle 0|\hat{D}(\gamma_1)|0\rangle$$
$$= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\exp(-\frac{1}{2}|\gamma_1|^2)$$
$$= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\exp(-\frac{1}{2}|\gamma\lambda^* + \gamma^*\mu|^2)$$
$$= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^* - \frac{1}{2}|\gamma\lambda^* + \gamma^*\mu|^2)$$

where $|\lambda|^2 - |\mu|^2 = 1$. This integral can be calculated by using the Mathematica with $d^2\gamma = dx'dy'$ and $\gamma = x' + iy'$.

$$W(\alpha) = \frac{2}{\pi}\exp[-2|\alpha\lambda^* + \alpha^*\mu|^2)$$

When $\alpha = x + iy$

$$W(x, y) = \frac{2}{\pi}\exp\{-2F(x, y)]$$

where

$$F(x, y) = (x^2 + y^2)\cosh(2s) + [(x - y)(x + y)\cos\vartheta + 2xy\sin\vartheta]\sinh(2s)]$$

When $\vartheta = 0$

$$\begin{aligned} F(x, y) &= (x^2 + y^2)\cosh(2s) + (x - y)(x + y)]\sinh(2s)] \\ &= x^2 e^{2s} + y^2 e^{-2s} \end{aligned}$$

((**Note**))

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} e^s\cos\frac{\vartheta}{2}x + e^s\sin\frac{\vartheta}{2}y \\ -e^{-s}\sin\frac{\vartheta}{2}x + e^{-s}\cos\frac{\vartheta}{2}y \end{pmatrix} \end{aligned}$$

under the rotation and squeezed processes. Thus we have

$$\begin{aligned} x'^2 + y'^2 &= [\cosh(2s) + \sinh(2s)\cos\theta]x^2 + [\cosh(2s) - \sinh(2s)\cos\theta]y^2 \\ &\quad + 2\sinh(2s)\sin\theta\, xy \end{aligned}$$

17. Wigner representation for a pure squeezed coherent state

For $\hat{\rho} = |\beta, \varsigma\rangle\langle\beta, \varsigma| = \hat{D}(\beta)\hat{S}_\varsigma|0\rangle\langle 0|\hat{S}_\varsigma^{+}\hat{D}(\beta)^{+}$

$$
\begin{aligned}
Tr[\hat{\rho}\hat{D}(\gamma)] = \left\langle \hat{D}(\gamma) \right\rangle \\
&= Tr[|\beta,\varsigma\rangle\langle\beta,\varsigma|\hat{D}(\gamma)] \\
&= \langle\beta,\varsigma|\hat{D}(\gamma)|\beta,\varsigma\rangle \\
&= \langle 0|\hat{S}_{\varsigma}{}^{+}\hat{D}^{+}(\beta)\hat{D}(\gamma)\hat{D}(\beta)\hat{S}_{\varsigma}|0\rangle
\end{aligned}
$$

Then the Wigner function is obtained as

$$
\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) Tr[\hat{\rho}\hat{D}(\gamma)] \\
&= \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*) \langle 0|\hat{S}_{\varsigma}{}^{+}\hat{D}^{+}(\beta)\hat{D}(\gamma)\hat{D}(\beta)\hat{S}_{\varsigma}|0\rangle
\end{aligned}
$$

where

$$
\hat{D}(\gamma) = \exp(\gamma\hat{a}^{+} - \gamma^*\hat{a})
$$

Using the formula

$$
\hat{D}_{\alpha}\hat{D}_{\beta} = \exp(\frac{\alpha\beta^* - \alpha^*\beta}{2})\hat{D}_{\alpha+\beta}
$$

We calculate

$$
\begin{aligned}
\hat{D}_{\beta}{}^{+}\hat{D}_{\gamma}\hat{D}_{\beta} &= \exp(\frac{\gamma\beta^* - \gamma^*\beta}{2})\hat{D}_{-\beta}\hat{D}_{\gamma+\beta} \\
&= \exp(\frac{\gamma\beta^* - \gamma^*\beta}{2})\exp(\frac{\gamma\beta^* - \gamma^*\beta}{2})\hat{D}_{\gamma} \\
&= \exp(\gamma\beta^* - \gamma^*\beta)\hat{D}_{\gamma}
\end{aligned}
$$

$$
W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp[\gamma^*(\alpha-\beta) - \gamma(\alpha^* - \beta^*)]\langle 0|\hat{S}_{\varsigma}{}^{+}\hat{D}(\gamma)\hat{S}_{\varsigma}|0\rangle
$$

where

$$
\hat{S}_{\varsigma}{}^{+}\hat{D}(\gamma)\hat{S}_{\varsigma} = \hat{D}(\gamma_1) = \hat{D}(\gamma\lambda^* + \gamma^*\mu)
$$

with

$$\gamma_1 = \gamma\lambda^* + \gamma^*\mu$$

Using the relation

$$\begin{aligned}\langle 0|\hat{D}(\gamma_1)|0\rangle &= \langle 0|\exp(-\frac{1}{2}|\gamma_1|^2)\exp(\gamma_1\hat{a}^+)\exp(-\gamma_1^*\hat{a})|0\rangle \\ &= \exp(-\frac{1}{2}|\gamma_1|^2)\langle 0|\exp(\gamma_1\hat{a}^+)\exp(-\gamma_1^*\hat{a})|0\rangle \\ &= \exp(-\frac{1}{2}|\gamma_1|^2)\end{aligned}$$

Thus we have

$$W(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp[\gamma^*(\alpha-\beta) - \gamma(\alpha^* - \beta^*) - \frac{1}{2}|\gamma\lambda^* + \gamma^*\mu|^2]$$

or

$$W(\alpha) = \frac{2}{\pi}\exp[-2|(\alpha-\beta)\lambda^* + (\alpha-\beta)^*\mu|^2)$$

When $\alpha = x + iy$ and $\beta = x_0 + iy_0$

$$W(x,y) = \frac{2}{\pi}\exp\{-2F(x,y)]$$

where

$$F(\Delta x = x - x_0, \Delta y = y - y_0) = (\Delta x^2 + \Delta y^2)\cosh(2s) + [(\Delta x - \Delta y)(\Delta x + \Delta y)\cos\vartheta + 2\Delta x\Delta y\sin\vartheta]\sinh(2s)]$$

18. Wigner function for the pure number state

We have the Wigner function for the density operator $\hat{\rho} = |n\rangle\langle n|$ as

$$
\begin{aligned}
W(\alpha) &= Tr[\hat{\rho}\delta^{(2)}(\hat{a}-\alpha)] \\
&= \frac{1}{\pi^2}\int d^2\beta \exp(\beta^*\alpha - \beta\alpha^*) Tr[\hat{\rho}\hat{D}(\beta)] \\
&= \frac{1}{\pi^2}\int d^2\beta \exp(\beta^*\alpha - \beta\alpha^*) Tr[|n\rangle\langle n|\hat{D}(\beta)] \\
&= \frac{1}{\pi^2}\int d^2\beta \exp(\beta^*\alpha - \beta\alpha^*) \langle n|\hat{D}(\beta)|n\rangle
\end{aligned}
$$

Here we note that

$$
\begin{aligned}
\langle n|\hat{D}(\beta)|n\rangle &= \exp(-\frac{1}{2}|\beta|^2)\langle n|\exp(\beta\hat{a}^+)\exp(-\beta^*\hat{a})|n\rangle \\
&- \exp(-\frac{1}{2}|\beta|^2)\sum_{m=0}^{n}\frac{(-|\beta|^2)^m}{(m!)^2}\langle n|(\hat{a}^+)^m\hat{a}^m)|n\rangle \\
&= \exp(-\frac{1}{2}|\beta|^2)\sum_{m=0}^{n}\frac{(-|\beta|^2)^m}{m!}\binom{n}{m} \\
&= \exp(-\frac{1}{2}|\beta|^2)L_n(|\beta|^2)
\end{aligned}
$$

where $L_n(x)$ is a Laguerre polynomial. Thus we have

$$
\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int d^2\beta \exp(\beta^*\alpha - \beta\alpha^*)\exp(-\frac{1}{2}|\beta|^2)L_n(|\beta|^2) \\
&= \frac{2}{\pi}(-1)^n \exp(-2|\alpha|^2)L_n(4|\alpha|^2)
\end{aligned}
$$

where

$$
|\alpha|^2 = x^2 + y^2
$$

$$
L_n(x) = \sum_{k=0}^{n}\frac{(-x)^k}{k!}\binom{n}{k}
$$

19. Relations between the Wigner, Q and P representations

(a) Relation between *W* and *P*

Using the expression for $\hat{\rho}$

$$\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$$

we have

$$W(\alpha) = \int d^2\beta P(\beta)\frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*\alpha-\gamma\alpha^*} Tr[|\beta\rangle\langle\beta|\hat{D}(\gamma)]$$
$$= \int d^2\beta P(\beta)\frac{1}{\pi^2}\int d^2\gamma\, e^{\gamma^*\alpha-\gamma\alpha^*}\langle\beta|\hat{D}(\gamma)|\beta\rangle]$$
$$= \int d^2\beta P(\beta)\frac{1}{\pi^2}\int d^2\gamma\, \exp(-\frac{1}{2}|\gamma|^2)\exp[\gamma(\beta^* - \alpha^*) - \gamma^*(\beta-\alpha)]]$$
$$= \frac{2}{\pi}\int d^2\beta P(\beta)\exp[-2|\alpha-\beta|^2]$$

where

$$\langle\beta|\hat{D}(\gamma)|\beta\rangle = \exp(-\frac{1}{2}|\gamma|^2)\langle\beta|\exp(\gamma\hat{a}^+)\exp(-\gamma^*\hat{a})|\beta\rangle$$
$$= \exp(-\frac{1}{2}|\gamma|^2)\exp(\gamma\beta^* - \gamma^*\beta)$$

(b) Relation between *Q* and *P*

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle$$
$$= \frac{1}{\pi}\int d^2\beta P(\beta)|\langle\alpha|\beta\rangle|^2$$
$$= \frac{1}{\pi}\int d^2\beta\, P(\beta)\exp[-|\alpha-\beta|^2]$$

where

$$\langle\beta|\alpha\rangle = \exp(-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha\beta^*)$$

$$|\langle\beta|\alpha\rangle|^2 = \exp(-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha\beta^*)\exp(-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha^*\beta)$$
$$= \exp[-(|\alpha|^2 + |\beta|^2) + \alpha\beta^* + \alpha^*\beta]$$
$$= \exp[-|\alpha-\beta|^2]$$

(c) Fourier transform

$$Q(\beta) = \frac{1}{\pi}\int d^2\alpha\, P(\alpha)\exp[-|\alpha-\beta|^2]$$
$$= \frac{1}{\pi}\int d^2\alpha\, P(\alpha)\exp(-|\alpha|^2 - |\beta|^2)\exp(\alpha\beta^* + \alpha^*\beta)$$
$$= \exp(-|\beta|^2)\frac{1}{\pi}\int d^2\alpha\, P(\alpha)\exp(-|\alpha|^2)\exp(\alpha\beta^* + \alpha^*\beta)$$

or

$$\exp(|\beta|^2)Q(\beta) = \frac{1}{\pi}\int d^2\alpha\, P(\alpha)\exp(-|\alpha|^2)\exp(\alpha\beta^* + \alpha^*\beta)$$

Thus $\exp(|\beta|^2)Q(\beta)$ is the Fourier transform of the function $P(\alpha)\exp(-|\alpha|^2)$. On taking the Inverse Fourier transform, we have

$$P(\alpha)\exp(-|\alpha|^2) = \frac{1}{\pi}\int d^2\beta\, \exp(|\beta|^2)Q(\beta)\exp(-\alpha\beta^* - \alpha^*\beta)$$

or

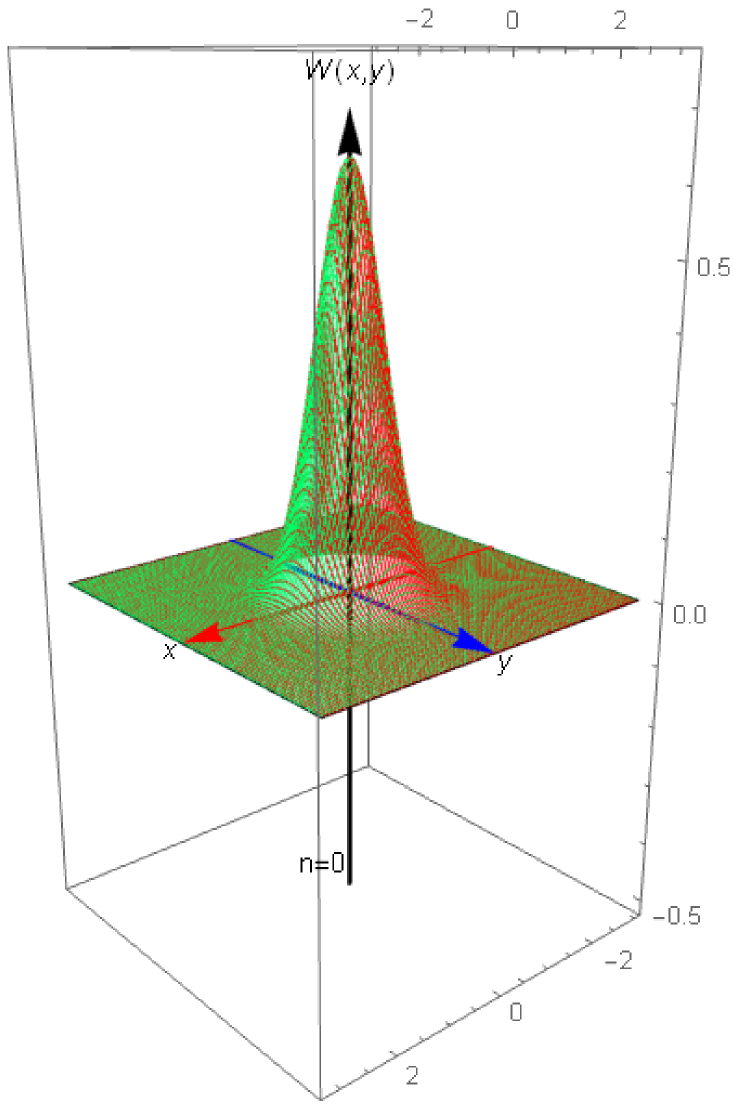
$$P(\alpha) = \frac{1}{\pi}\exp(|\alpha|^2)\int d^2\beta\, [\exp(|\beta|^2)Q(\beta)]\exp(-\alpha\beta^* - \alpha^*\beta)$$

20. Example of Wigner representation for the density operator with a pure number state.

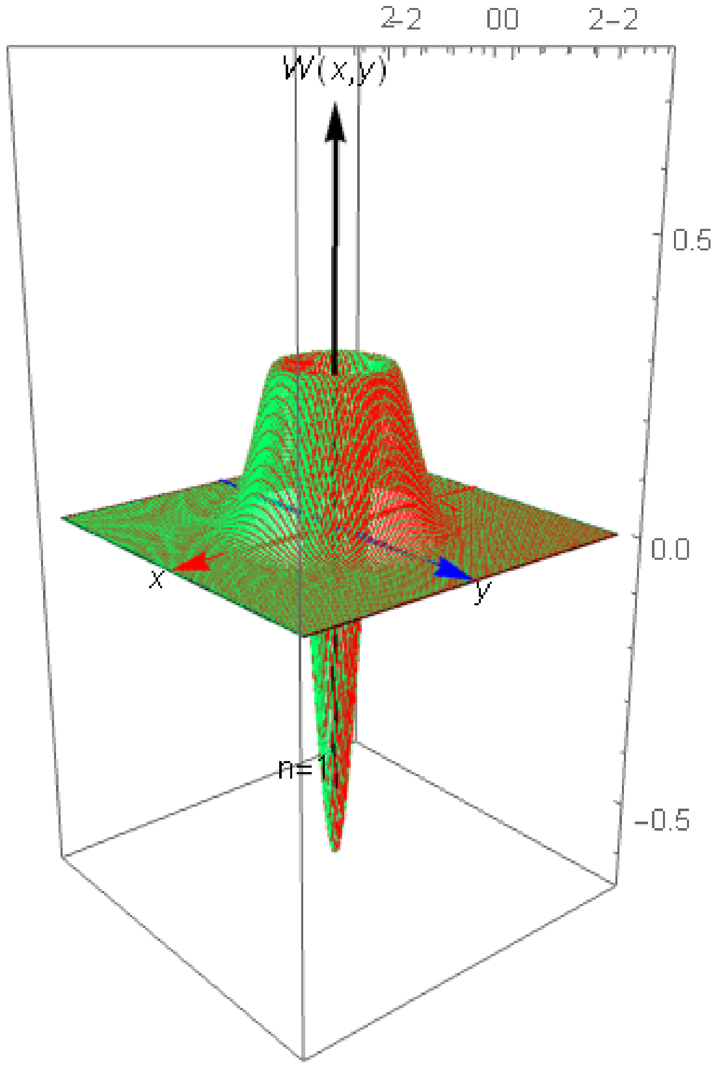
Wigner function for the pure number state $|n\rangle$ ($n = 1, 2, 3, \ldots$).

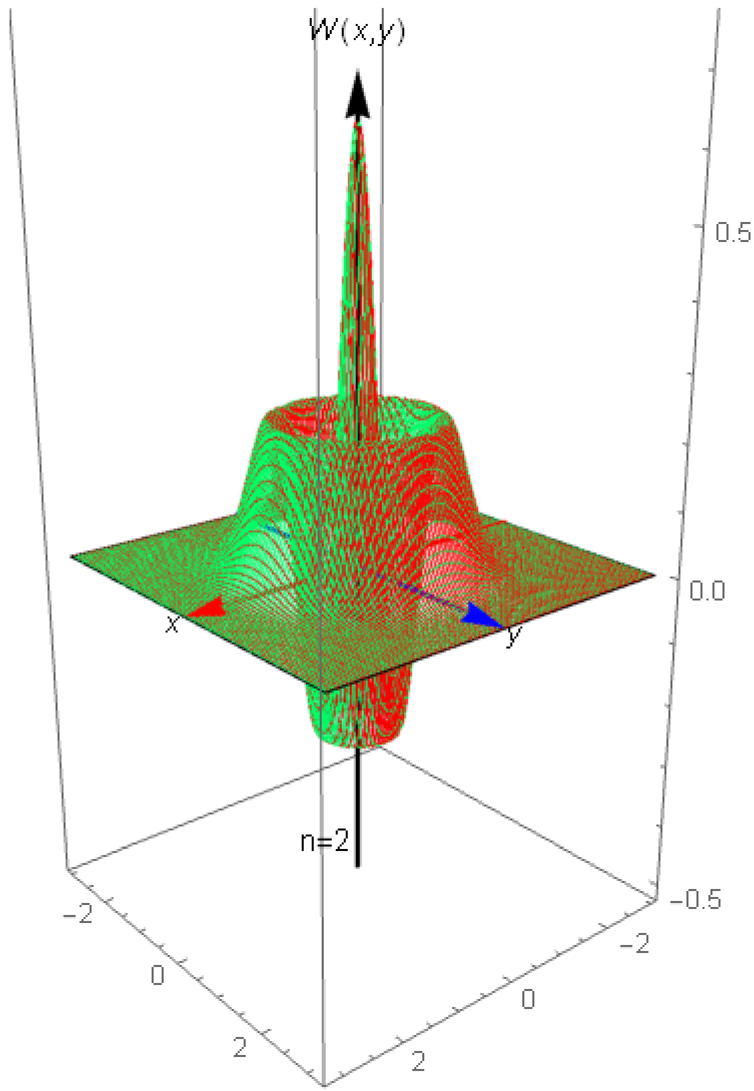
$$W(\alpha) = \frac{2}{\pi}(-1)^n \exp(-2|\alpha|^2)L_n(4|\alpha|^2)$$

where $|\alpha|^2 = x^2 + y^2$, $\qquad W(\alpha) = W_n(x, y)$

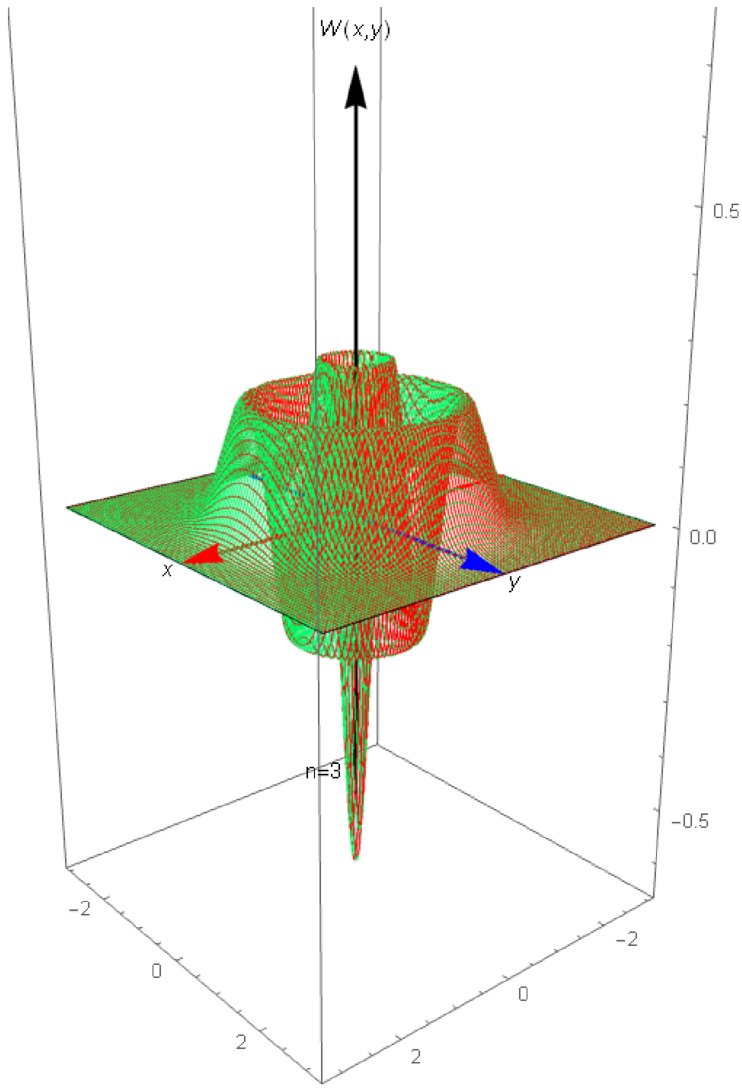


W(x,y)
x
y
n=0
−2
0
2
0.5
0.0
−0.5
−2
0
2


2-2
00
2-2
W(x,y)
0.5
0.0
x
y
n=1
-0.5



W(x,y)
0.5
0.0
−0.5
x
y
n=2
−2
0
2
−2
0
2


W(x,y)
0.5
0.0
−0.5
x
y
n=3
−2
0
2
−2
0
2

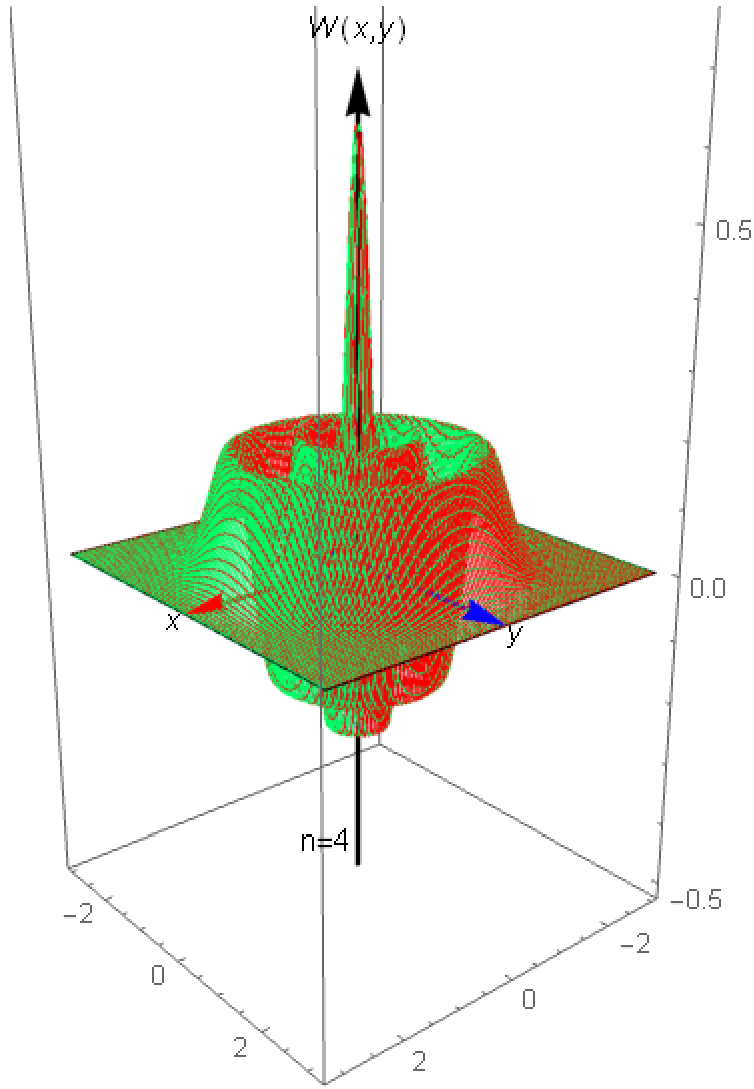


Fig. Wigner function for the density operator with the pure state $\left|n\right\rangle$ (n = 0, 1, 2, 3, 4).

21. Example of Wigner representation for the density operator with a coherent state

The Wigner function for the density operator with a coherent state is given by

$$W(\alpha-\beta)=\frac{2}{\pi}\exp\{-2[(x-x_0)^2+(y-y_0)^2]\},$$

where $\alpha = x + iy$ and $\beta = x_0 + iy_0$.

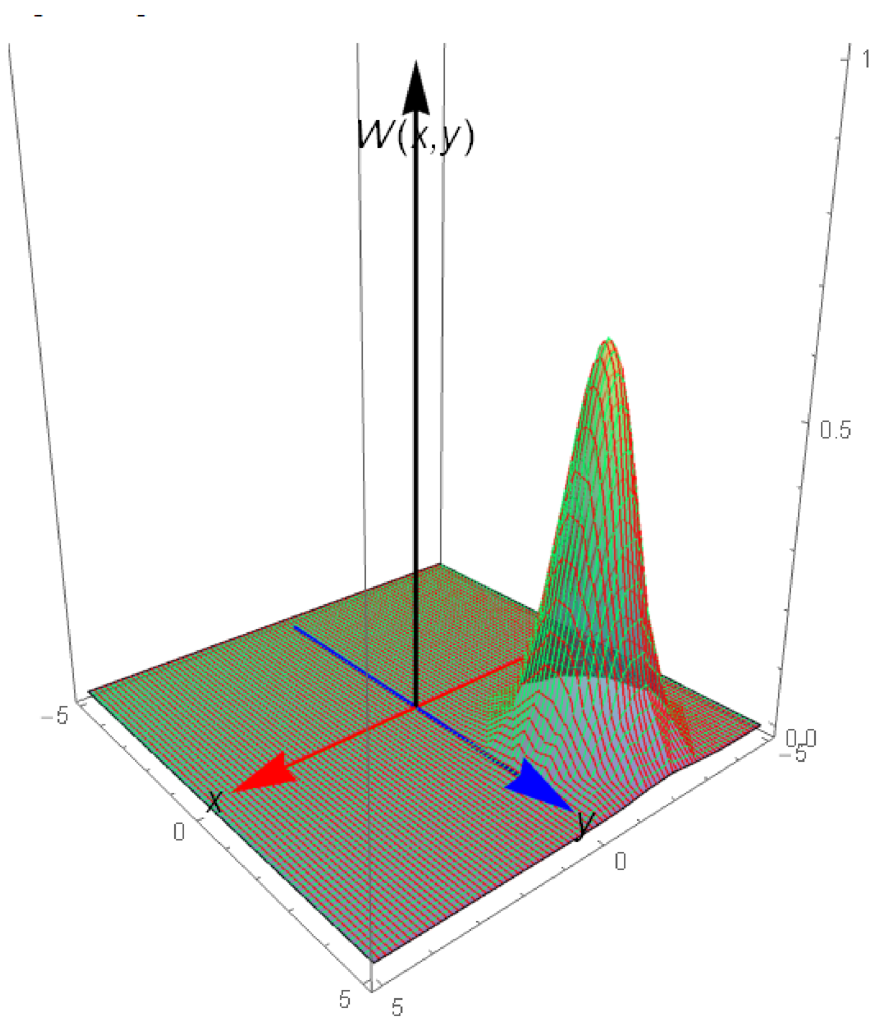


Fig. Plot of $W(x,y)$ with the density operator (a pure coherent state) with a fixed point (x_0, y_0).

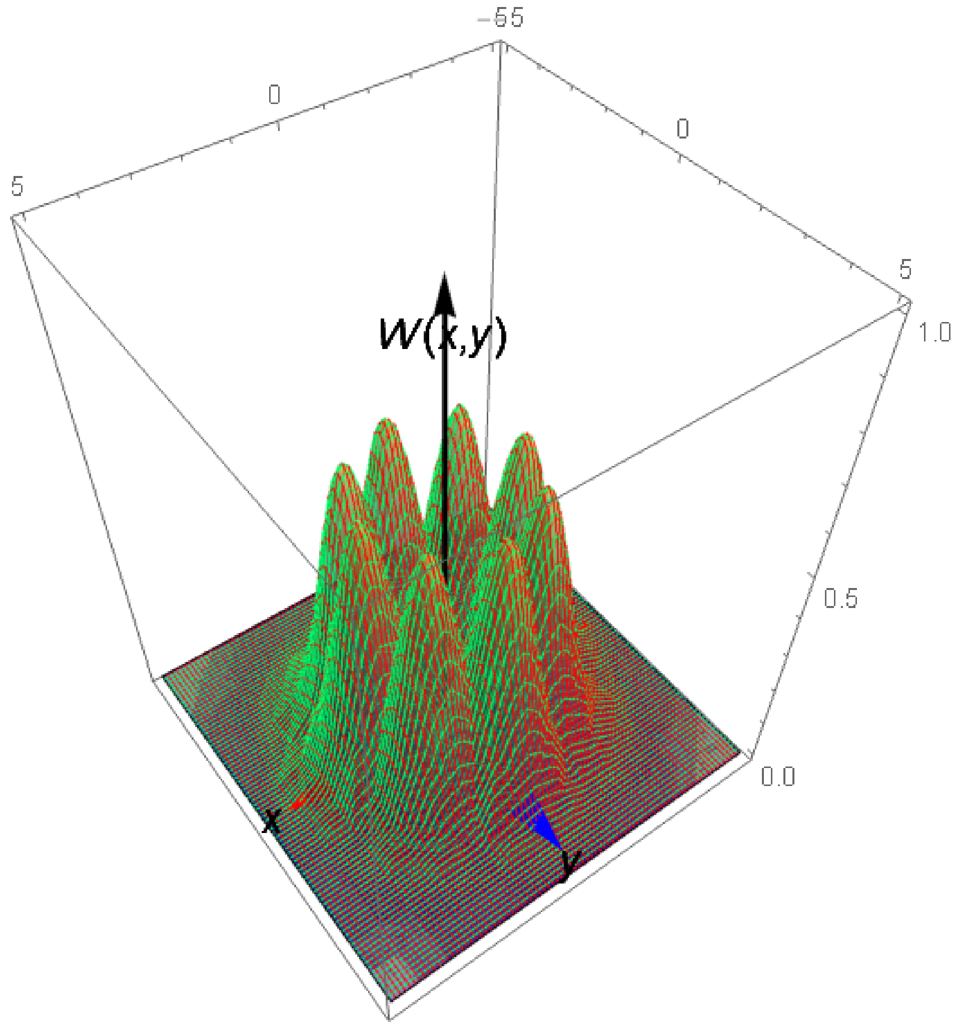


Fig. Plot of $W(x,y)$ with $x_0 = r\cos\theta$ and $y_0 = r\sin\theta$, where $\theta = \frac{\pi}{3}n$ (n = 0, 1, 2, 3, 4, 5).

22. Example of Wigner representation for the density operator with a squeezed coherent state

The wigner function for the density operator with a squeezed coherent state is given by

$$W(\alpha) = \frac{2}{\pi}\exp[-2\left|(\alpha-\beta)\lambda^* + (\alpha-\beta)^*\mu\right|^2)$$

where $\alpha = x + iy$ and $\beta = x_0 + iy_0$.

$$W(\Delta x = x - x_0, \Delta y = y - y_0) = \frac{2}{\pi} \exp\{-2F(\Delta x, \Delta y)]$$

where

$$F(\Delta x = x - x_0, \Delta y = y - y_0) = (\Delta x^2 + \Delta y^2)\cosh(2s) + [(\Delta x^2 - \Delta y^2)\cos\vartheta + 2\Delta x \Delta y \sin\vartheta]\sinh(2s)]$$

where $\lambda = \cosh(s)$ and $\mu = e^{i\vartheta}\sinh(s)$.

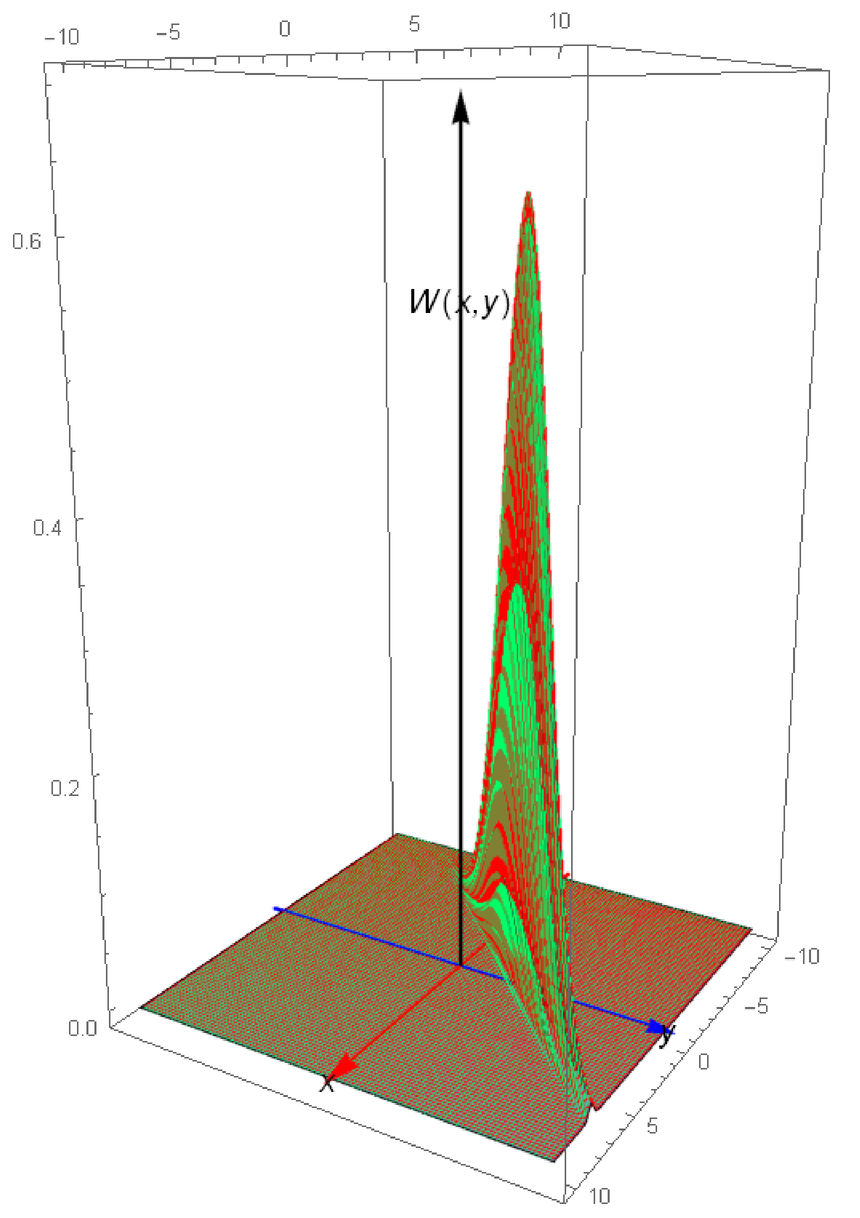


Fig. Plot of $W(x, y)$ for the density operator (a pure squeezed coherent state) with fixed values of s and ϑ.

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APPENDIX I Formula

The formula which are used in this chapter are listed below.

(a)

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = 1$$

(b)

$$\begin{aligned}\langle\beta|\alpha\rangle &= \exp[-\frac{1}{2}|\alpha-\beta|^2 \exp[\frac{1}{2}(\beta^*\alpha - \beta\alpha^*)] \\ &= \exp[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha\beta^*]\end{aligned}$$

(c)

$$\int d^2\beta \exp(\alpha\beta^* - \alpha^*\beta)\exp(-\frac{1}{2}|\beta|^2) = 2\pi\exp(-2|\alpha|^2)$$

APPENDIX II 2D Dirac delta function

$$\delta^{(2)}(\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp(\alpha\gamma^* - \alpha^*\gamma)$$

$$\delta^{(2)}(\hat{a}-\alpha) = \frac{1}{\pi^2}\int d^2\gamma \exp(\gamma^*\alpha - \gamma\alpha^*)\hat{D}(\gamma)$$

where $\hat{D}(\gamma)$ is the displacement operator.

APPENDIX III 2D Fourier Transform

The 2D Fourier transform is defined as

$$F(\beta) = \frac{1}{\pi}\int d^2\alpha\ f(\alpha)e^{\alpha\beta^* + \alpha^*\beta} \qquad \text{(Fourier Transform)}$$

The Inverse Fourier transform is given by

$$f(\alpha) = \frac{1}{\pi}\int d^2\beta\, F(\beta)e^{-(\alpha\beta^* + \alpha^*\beta)} \qquad \text{(Inverse Fourier transform)}$$

((**Proof**))

$$\begin{aligned}\frac{1}{\pi}\int d^2\beta\, F(\beta)e^{-(\alpha\beta^* + \alpha^*\beta)} &= \frac{1}{\pi}\int d^2\beta\, e^{-(\alpha\beta^* + \alpha^*\beta)}\frac{1}{\pi}\int d^2\gamma\ f(\gamma)e^{\gamma\beta^* + \gamma^*\beta} \\ &= \frac{1}{\pi^2}\int d^2\gamma\ f(\gamma)\int d^2\beta\, e^{\beta(\gamma^* - \alpha^*) - \beta^*(\gamma - \alpha)} \\ &= \int d^2\gamma\ f(\gamma)\delta^{(2)}(\gamma - \alpha) \\ &= f(\alpha)\end{aligned}$$

APPENDIX IV Formula related to displacement operator

$\hat{D}_\alpha$ is called the displacement operator,

$$\hat{D}_\alpha = \exp(\alpha\hat{a}^+ - \alpha^*\hat{a}).$$

$$\hat{D}_\alpha^{\ +} = \hat{D}_{-\alpha} = \exp(-\alpha\hat{a}^+ + \alpha^*\hat{a})$$

$$\hat{D}_\alpha^{\ +}\hat{a}\hat{D}_\alpha = \hat{a} + \alpha\hat{1}$$

$$\hat{D}_\alpha\hat{a}\hat{D}_\alpha^{\ +} = \hat{a}^+ - \alpha\hat{1}$$

$$\hat{D}_\alpha^{\ +}\hat{a}^+\hat{D}_\alpha = \hat{a}^+ + \alpha^*\hat{1}$$

$$\hat{D}_\alpha^{\ +}\hat{D}_\alpha = \hat{1}. \qquad \text{(Unitary operator).}$$

$$\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a}) = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a} + \frac{1}{2}|\alpha|^{2})$$
$$= \exp(\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a})$$

$$\hat{D}_{\alpha} = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}) = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a})$$

$$\hat{D}_{\alpha}^{+}\hat{a}\hat{a}\hat{D}_{\alpha} = \hat{a}\hat{a} + \alpha^{2}\hat{1} + 2\alpha\hat{a} = (\hat{a} + \alpha\hat{1})^{2}$$

$$\hat{D}_{\alpha}^{+}\hat{a}^{+}\hat{a}^{+}\hat{D}_{\alpha} = \hat{a}^{+}\hat{a}^{+} + (\alpha^{*})^{2}\hat{1} + 2\alpha^{*}\hat{a}^{+} = (\hat{a}^{+} + \alpha^{*}\hat{1})^{2}$$

$$\hat{D}_{\alpha}^{+} f(\hat{a},\hat{a}^{+})\hat{D}_{\alpha} = f(\hat{a} + \alpha\hat{1}, \hat{a}^{+} + \alpha^{*}\hat{1})$$

$$\hat{D}_{\alpha} = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}) = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a})$$
$$\hat{D}_{\beta} = \exp(\beta\hat{a}^{+} - \beta^{*}\hat{a}) = \exp(-\frac{1}{2}|\beta|^{2})\exp(\beta\hat{a}^{+})\exp(-\beta^{*}\hat{a})$$

$$\hat{D}_{\alpha+\beta} = \exp[(\alpha+\beta)\hat{a}^{+} - (\alpha+\beta)^{*}\hat{a})$$
$$= \exp(-\frac{1}{2}|\alpha+\beta|^{2})\exp(\alpha\hat{a}^{+})\exp(\beta\hat{a}^{+})\exp(-\alpha^{*}\hat{a})\exp(-\beta^{*}\hat{a})$$

$$\hat{D}_{\alpha}\hat{D}_{\beta} = \exp(-\frac{1}{2}|\alpha|^{2})\exp(-\frac{1}{2}|\beta|^{2})\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a})\exp(\beta\hat{a}^{+})\exp(-\beta^{*}\hat{a}).$$

$$\hat{D}_{\alpha}\hat{D}_{\beta} = \exp(\alpha\beta^{*} - \alpha^{*}\beta)\hat{D}_{\beta}\hat{D}_{\alpha}$$

$$\hat{D}_{\alpha}\hat{D}_{\beta} = \exp(\frac{\alpha\beta^{*} - \alpha^{*}\beta}{2})\hat{D}_{\alpha+\beta}$$

$$\exp(-\alpha^{*}\hat{a})\exp(\beta\hat{a}^{+}) = \exp(-\frac{1}{2}\alpha^{*}\beta)\exp(-\alpha^{*}\hat{a} + \beta\hat{a}^{+})$$

$$\exp(\beta\hat{a}^{+})\exp(-\alpha^{*}\hat{a}) = \exp(\beta\hat{a}^{+} - \alpha^{*}\hat{a})\exp(\frac{1}{2}\alpha^{*}\beta[\hat{a},\hat{a}^{+}])$$

$$= \exp(\frac{1}{2}\alpha^{*}\beta)\exp(-\alpha^{*}\hat{a} + \beta\hat{a}^{+})$$

$$\exp(-\alpha^{*}\hat{a})\exp(\beta\hat{a}^{+}) = \exp(-\alpha^{*}\beta)\exp(\beta\hat{a}^{+})\exp(-\alpha^{*}\hat{a})\,.$$

APPENDIX V Equivalence between $\exp[\frac{i(\sigma\hat{x} + \tau\hat{p})}{\hbar}]$ and displacement operator

Using the expressions of $\hat{a}$ and $\hat{a}^{+}$, we have

$$\exp[\frac{i(\sigma\hat{x} + \tau\hat{p})}{\hbar}] = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a})$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}$$

$$\hat{a} = \frac{\beta}{\sqrt{2}}(\hat{x} + \frac{i\hat{p}}{m\omega_0})\,,$$

$$\hat{a}^{+} = \frac{\beta}{\sqrt{2}}(\hat{x} - \frac{i\hat{p}}{m\omega_0})$$

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a} + \hat{a}^{+}\right) = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\hat{a} + \hat{a}^{+}\right)$$

$$\hat{p} = \frac{1}{\sqrt{2}\beta}\frac{m\omega_0}{i}\left(\hat{a} - \hat{a}^{+}\right) = \frac{1}{i}\sqrt{\frac{m\hbar\omega_0}{2}}\left(\hat{a} - \hat{a}^{+}\right)$$

The operator $\frac{i(\sigma\hat{x} + \tau\hat{p})}{\hbar}$ can be rewritten in terms of $\{\hat{a}, \hat{a}^{+}\}$,

$$\frac{i(\sigma\hat{x}+\tau\hat{p})}{\hbar}=\frac{i\sigma}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\left(\hat{a}+\hat{a}^+\right)+\frac{\tau}{\hbar}\sqrt{\frac{m\hbar\omega_0}{2}}\left(\hat{a}-\hat{a}^+\right)$$

$$=[\frac{i\sigma}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}-\frac{\tau}{\hbar}\sqrt{\frac{m\hbar\omega_0}{2}}]\hat{a}^+-[-\frac{i\sigma}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}-\frac{\tau}{\hbar}\sqrt{\frac{m\hbar\omega_0}{2}}]\hat{a}$$

$$=\alpha\hat{a}^+-\alpha^*\hat{a}$$

where α is a complex number,

$$\alpha=\frac{i\sigma}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}-\frac{\tau}{\hbar}\sqrt{\frac{m\hbar\omega_0}{2}}=\frac{1}{\sqrt{2}}(\frac{i\sigma}{\hbar\beta}-\tau\beta)$$

$$\alpha^*=-\frac{i\sigma}{\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}-\frac{\tau}{\hbar}\sqrt{\frac{m\hbar\omega_0}{2}}=\frac{1}{\sqrt{2}}(-\frac{i\sigma}{\hbar\beta}-\tau\beta)$$

Thus we have

$\exp[\frac{i(\sigma\hat{x}+\tau\hat{p})}{\hbar}]$ is the displacement operator,

$$\hat{D}_\alpha=\exp(\alpha\hat{a}^+-\alpha^*\hat{a})=\exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})$$