Derivation of the Solution for Dirac equation of a free particle
Eigenvalue problem

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Here we show the solution of Dirac equation for free particle by using two methods: (a) simultaneous eigenket of the Hamiltomian and the helicity, and (b) the Foldy-Wouthoeau transformation.

1. Hamiltonian and helicity

The Hamiltonian is given by

$$H = c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2 = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}$$

The helicity is defined by

$$\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p} = \begin{pmatrix} \frac{1}{p}\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & \frac{1}{p}\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}$$

The commutation relation between H and $\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}$

$$H\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p} = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix} \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & 0 \\ 0 & \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{mc^2}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & \frac{c}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 \\ \frac{c}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 & -\frac{mc^2}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \end{pmatrix}$$

$$\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}H = \begin{pmatrix} \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & 0 \\ 0 & \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \end{pmatrix} \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{mc^2}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & \frac{c}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 \\ \frac{c}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 & -\frac{mc^2}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \end{pmatrix}$$

leading to the commutation relation, $[H, \frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}] = 0$.

So we have a simultaneous eigenket

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

with

$$\psi_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \psi_2 = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}$$

The eigenvalue problem is now given by

$$H\psi = E\psi\ , \qquad \frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}\psi = h\psi$$

We note that

$$(\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p})^2 = \begin{pmatrix} \frac{1}{p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 & 0 \\ 0 & \frac{1}{p^2}(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{1} & 0 \\ 0 & \boldsymbol{1} \end{pmatrix}$$

Thus the eigenvalue of $\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}$ is $h = \pm 1$.

Since $H^2 = E_R{}^2$, the eigenvalue of H is $\pm E_R$.

(a) Eigenvalue problem of $\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}$

$$\frac{\boldsymbol{\Sigma}\cdot\boldsymbol{p}}{p}\psi = \begin{pmatrix} \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & 0 \\ 0 & \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \lambda\psi = \lambda\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$

leading to

$$\frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_1 = \lambda\psi_1, \qquad \frac{1}{p}(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_2 = \lambda\psi_2 \tag{1}$$

where $\lambda = \pm 1$.

(b) Eigenvalue problem of H

$$H\psi = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = E\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \tag{2}$$

or

$$H\psi = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = E\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$

or

$$mc^2\psi_1 + c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_2 = E\psi_1, \qquad c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_1 - mc^2\psi_2 = E\psi_2$$

Note that the energy eigenvalues are

$$E = E_R, E_R, -E_R, -E_R$$

where

$$E_R = c\sqrt{p^2 + m^2c^2}$$

Suppose that

$$c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_1 = cp\lambda\psi_1 \qquad (1)$$

$$c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_2 = cp\lambda\psi_2 \qquad (2)$$

$$mc^2\psi_1 + c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_2 = E\psi_1 \qquad (3)$$

$$c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi_1 - mc^2\psi_2 = E\psi_2 \qquad (4)$$

(the same E and λ are required)

where

$$\lambda^2 = 1, \qquad E^2 = {E_R}^2 = m^2c^4 + c^2p^2$$

Using Eqs.(2) and (3), we have

$$mc^2\psi_1 + cp\lambda\psi_2 = E\psi_1 \qquad (3')$$

Using Eqs.(1) and (4), we have

$$cp\lambda\psi_1 - mc^2\psi_2 = E\psi_2 \qquad (4')$$

From Eq.(4'),

$$\psi_1 = \frac{E + mc^2}{cp\lambda}\psi_2 \qquad (5)$$

From Eq.(3'),

$$\psi_2 = \frac{(E - mc^2)}{cp\lambda}\psi_1 \qquad (6)$$

In Eq.(5),

$$\psi_1 = \frac{(E+mc^2)(E-mc^2)}{cp\lambda(E-mc^2)}\psi_2$$
$$= \frac{c^2p^2}{cp\lambda(E-mc^2)}\psi_2$$
$$= \frac{cp\lambda}{E-mc^2}\psi_2$$

where we use $\lambda^2 = 1$. So this equation is the same as Eq.(6). Eq.(6) and Eq.(6) are essentially the same equation.

We use two equations

(i)

$$\psi_2 = \frac{(E-mc^2)}{cp\lambda}\psi_1, \qquad (\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{p})\psi_1 = \lambda\psi_1$$

or

(ii)

$$\psi_1 = \frac{E+mc^2}{cp\lambda}\psi_2, \qquad (\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{p})\psi_2 = \lambda\psi_2$$

2. Eigenvalue of $\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{p}$

$$(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{p})\psi = \lambda\psi$$

For $\lambda = 1$

$$\psi = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

For $\lambda = -1$

$$\psi = \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

3. Energy and helicity for the particle and antiparticle

(a) $E = E_R$, $\lambda = 1$,

$$\psi_2 = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix},$$

$$\psi_1 = \frac{(E_R + mc^2)}{cp}\psi_2 = \frac{(E_R + mc^2)}{cp}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\psi^{(+\uparrow)} = \frac{cp}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} \frac{(E_R + mc^2)}{cp}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

$$= \sqrt{\frac{E_R + mc^2}{2E_R}}\begin{pmatrix} \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \\ \frac{cp}{(E_R + mc^2)}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

(b) $E = E_R$, $\lambda = -1$

$$\psi_2 = \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix},$$

$$\psi_1 = -\frac{(E_R + mc^2)}{cp}\psi_2 = -\frac{(E_R + mc^2)}{cp}\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix}$$

$$\psi^{(+\downarrow)} = \frac{cp}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} -\frac{(E_R + mc^2)}{cp}\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

$$= \sqrt{\frac{E_R + mc^2}{2E_R}}\begin{pmatrix} \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \\ \frac{-cp}{E_R + mc^2}\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

(c) $E = -E_R, \qquad \lambda = 1,$

$$\psi_1 = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix},$$

$$\psi_2 = -\frac{(E_R + mc^2)}{cp}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\psi^{(-\uparrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{-cp}{E_R + mc^2} \begin{pmatrix} e^{-i\phi/2} \cos\dfrac{\theta}{2} \\ e^{i\phi/2} \sin\dfrac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} e^{-i\phi/2} \cos\dfrac{\theta}{2} \\ e^{i\phi/2} \sin\dfrac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

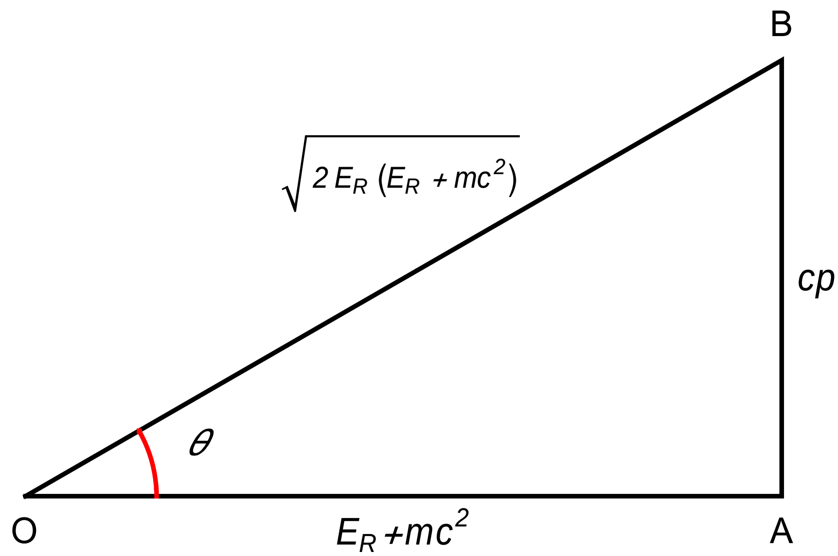
(d) $E = -E_R$, $\lambda = -1$

$$\psi_1 = \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix}$$

$$\psi_2 = \frac{(E_R + mc^2)}{cp}\psi_1 = \frac{E_R + mc^2}{cp} \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix}$$

$$\psi^{(-\downarrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{cp}{E_R + mc^2} \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

4. Normalization factor *A*



The normalization factor A is determined as

$$A^2[1+\frac{(E_R+mc^2)^2}{c^2p^2}]=A^2[\frac{(E_R+mc^2)^2+c^2p^2}{c^2p^2}]=1$$

In the above triangle, we have the relation

$$(E_R+mc^2)^2+c^2p^2=2E_R(E_R+mc^2).$$

Then the normalization factor A can be evaluated as

$$A=\frac{cp}{\sqrt{(E_R+mc^2)^2+c^2p^2}}=\frac{cp}{\sqrt{2E_R(E_R+mc^2)}}$$

5. The special case for $\boldsymbol{p}=(0,0,p)$

For the case of $\boldsymbol{p}=(0,0,p)$, we have $\theta=0$ and $\phi=0$.

(a) $E=E_R$, $\lambda=1$,

$$\psi^{(+\uparrow)}=\sqrt{\frac{E_R+mc^2}{2E_R}}\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ \frac{cp}{E_R+mc^2}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$$

(b) $E = E_R$, $\lambda = -1$

$$\psi^{(+\downarrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \dfrac{-cp}{E_R + mc^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

(c) $E = -E_R$, $\lambda = 1$,

$$\psi^{(-\uparrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{-cp}{E_R + mc^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

(d) $E = -E_R$, $\lambda = -1$

$$\psi^{(-\downarrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{cp}{E_R + mc^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

6. Approach based on FW transformation

(a) Energy eigenkets

Hamiltonian: $H = c(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta mc^2$

$$\hat{H} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \pm E_R \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

with the eigenvalue $\pm E_R$ since

$$\hat{H}^2 = E_R{}^2$$

Using the FW transformation, we have the new eigenket

$$\psi_{FW} = \hat{U}\psi ,$$

with

$$\hat{U} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + \beta H) ,$$

$$\hat{U}^+ = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + H\beta)$$

(a) For the energy eigenvalue $+E_R$, (particle)

$$\begin{aligned}
\psi_{FW}{}^{(+)} &= \hat{U}\psi \\
&= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + \beta H)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R + mc^2)}}(1 + \beta)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\
&= \sqrt{\frac{2E_R}{E_R + mc^2}}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

with the normalization condition

$$|u_1|^2 + |u_2|^2 = \frac{E_R + mc^2}{2E_R}$$

(b) For the energy eigenvalue $-E_R$ (anti-particle)

$$
\begin{aligned}
\psi_{FW}{}^{(-)} &= \hat{U}\psi \\
&= \frac{1}{\sqrt{2E_R(E_R+mc^2)}}(E_R+\beta H)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R+mc^2)}}(1-\beta)\psi \\
&= \frac{E_R}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \\
&= \sqrt{\frac{2E_R}{E_R+mc^2}}\begin{pmatrix} 0 \\ 0 \\ u_3 \\ u_4 \end{pmatrix}
\end{aligned}
$$

From the normalization condition, we have

$$
|u_3|^2+|u_4|^2=\frac{E_R+mc^2}{2E_R}
$$

Helicity:

$$
\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{|p|}=\begin{pmatrix} \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} \end{pmatrix}
$$

$$
\boldsymbol{\Sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}=\begin{pmatrix} \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}
$$

with

$$
\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\pm\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad \boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|p|}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}=\pm\begin{pmatrix} u_3 \\ u_4 \end{pmatrix}
$$

with the eigenvalues ± 1 since $(\boldsymbol{\sigma}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|})^2 = 1$. When we define $\frac{\boldsymbol{p}}{|\boldsymbol{p}|} = \boldsymbol{n}$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}|\pm\boldsymbol{n}\rangle = \pm|\pm\boldsymbol{n}\rangle$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix} \text{ for the helicity 1}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi/2} \\ -\cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix} \text{ for the helicity -1}$$

We note that

$$\hat{U}^{+} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + H\beta)$$

$$H = c(\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta mc^2 = \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}$$

$$\begin{aligned}(E_R + H\beta) &= \begin{pmatrix} E_R & 0 \\ 0 & E_R \end{pmatrix} + \begin{pmatrix} mc^2 & c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & -mc^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} E_R & 0 \\ 0 & E_R \end{pmatrix} + \begin{pmatrix} mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & mc^2 \end{pmatrix} \\ &= \begin{pmatrix} E_R + mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & E_R + mc^2 \end{pmatrix}\end{aligned}$$

(a) The particle

$$\hat{U}^{+}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2E_R(E_R + mc^2)}}(E_R + H\beta)\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} E_R + mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & E_R + mc^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} (E_R + mc^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} (E_R + mc^2)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \pm cp\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

or

$$\hat{U}^{+}\begin{pmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{pmatrix} == \frac{(E_R + mc^2)}{\sqrt{2E_R(E_R + mc^2)}}\begin{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \pm\frac{cp}{E_R + mc^2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{pmatrix}$$

This leads to

$$\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \pm\frac{cp}{E_R + mc^2}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

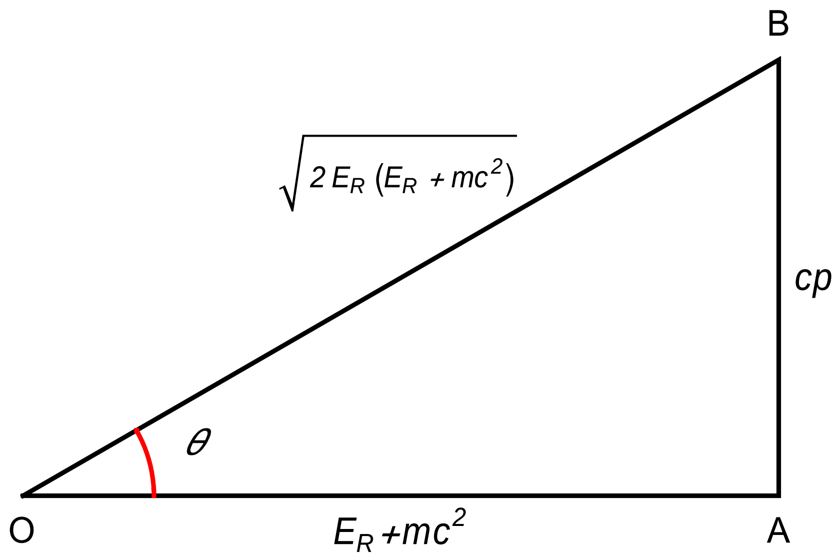
The normalization condition is satisfied,

$$\frac{(E_R + mc^2)}{2E_R(E_R + mc^2)}(|u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^2) = \frac{(E_R + mc^2)}{2E_R(E_R + mc^2)}(|u_1|^2 + |u_2|^2)[1 + \frac{c^2 p^2}{(E_R + mc^2)^2}]$$

$$= \frac{(E_R + mc^2)}{2E_R(E_R + mc^2)}(|u_1|^2 + |u_2|^2)\frac{(E_R + mc^2)^2 + c^2 p^2}{(E_R + mc^2)^2}$$

$$= \frac{E_R + mc^2}{2E_R}[\frac{2E_R(E_R + mc^2)}{(E_R + mc^2)^2}]$$

$$= 1$$



(b) Anti-particle:

$$\hat{U}^{+}\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}(E_R+H\beta)\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}$$

$$=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}E_R+mc^2 & -c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\\ c(\boldsymbol{\sigma}\cdot\boldsymbol{p}) & E_R+mc^2\end{pmatrix}\begin{pmatrix}0\\0\\u_3\\u_4\end{pmatrix}$$

$$=\frac{1}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}-c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\begin{pmatrix}u_3\\u_4\end{pmatrix}\\(E_R+mc^2)\begin{pmatrix}u_3\\u_4\end{pmatrix}\end{pmatrix}$$

$$=\frac{E_R+mc^2}{\sqrt{2E_R(E_R+mc^2)}}\begin{pmatrix}\mp\frac{cp}{E_R+mc^2}\begin{pmatrix}u_3\\u_4\end{pmatrix}\\\begin{pmatrix}u_3\\u_4\end{pmatrix}\end{pmatrix}$$

This leads to

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=\mp\frac{cp}{E_R+mc^2}\begin{pmatrix}u_3\\u_4\end{pmatrix}$$

where

$$\begin{pmatrix}u_3\\u_4\end{pmatrix}=\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\\sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\quad\text{for the helicity +1}$$

$$\begin{pmatrix}u_3\\u_4\end{pmatrix}=\begin{pmatrix}\sin\frac{\theta}{2}e^{-i\phi/2}\\-\cos\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\quad\text{for the helicity -1}$$

The normalization condition is satisfied,

$$\frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_1|^2+|u_2|^2+|u_3|^2+|u_4|^2) = \frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_3|^2+|u_4|^2)[1+\frac{c^2p^2}{(E_R+mc^2)^2}]$$

$$= \frac{(E_R+mc^2)}{2E_R(E_R+mc^2)}(|u_3|^2+|u_4|^2)\frac{(E_R+mc^2)^2+c^2p^2}{(E_R+mc^2)^2}$$

$$= \frac{E_R+mc^2}{2E_R}[\frac{2E_R(E_R+mc^2)}{(E_R+mc^2)^2}]$$

$$=1$$

In summary we have the solution of Dirac equation

(a) The energy eigenvalue $+E_R$, helicity (+1)

$$\psi^{(+\uparrow)} = \sqrt{\frac{E_R+mc^2}{2E_R}}\begin{pmatrix} \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \\ \frac{cp}{E_R+mc^2}\begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

(b) The energy eigenvalue $+E_R$, helicity (-1)

$$\psi^{(+\downarrow)} = \sqrt{\frac{E_R+mc^2}{2E_R}}\begin{pmatrix} \begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \\ \frac{-cp}{E_R+mc^2}\begin{pmatrix} -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{i\phi/2}\cos\frac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

(c) The energy eigenvalue $-E_R$, helicity (+1)

$$\psi^{(-\uparrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{-cp}{E_R + mc^2} \begin{pmatrix} e^{-i\phi/2} \cos\dfrac{\theta}{2} \\ e^{i\phi/2} \sin\dfrac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} e^{-i\phi/2} \cos\dfrac{\theta}{2} \\ e^{i\phi/2} \sin\dfrac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

(d) The energy eigenvalue $-E_R$, helicity (-1)

$$\psi^{(-\downarrow)} = \sqrt{\frac{E_R + mc^2}{2E_R}} \begin{pmatrix} \dfrac{cp}{E_R + mc^2} \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix} \\ \begin{pmatrix} -e^{-i\phi/2} \sin\dfrac{\theta}{2} \\ e^{i\phi/2} \cos\dfrac{\theta}{2} \end{pmatrix} \end{pmatrix}$$

REFERENCES

L.L. Foldy and S.A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

S.S. Schweber, Introduction to Relativistic Quantum mechanics (Row, Peterson and Company, 1961).

P. Strange, Relativistic Quantum Mechanics (Cambridge, 1998).

APPENDIX

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{p}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$