Zitterbewegung

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Zitterbewegung is a hypothetical rapid motion of elementary particles, in particular electrons, that obey the Dirac equation. The existence of such motion was first proposed by Erwin Schrödinger in 1930 as a result of his analysis of the wave packet solutions of the Dirac equation for relativistic electrons in free space, in which an interference between positive and negative energy states produces what appears to be a fluctuation (at the speed of light) of the position of an electron around the median, with an angular frequency of

$$\omega = \frac{2mc^2}{\hbar} = 1.55268 \text{ x } 10^{21} \text{ rad/s}$$

A reexamination of Dirac theory, however, shows that interference between positive and negative energy states may not be a necessary criterion for observing zitterbewegung.
https://en.wikipedia.org/wiki/Zitterbewegung

1. Standard velocity (c) of a free particle

The operator for velocity in the x direction can be computed from the commutator with the Hamiltonian. Using the Heisenberg's equation of motion we have

$$\begin{aligned}
\dot{x} &= \frac{i}{\hbar}[H, x] \\
&= \frac{i}{\hbar}[c(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta mc^2, x] \\
&= \frac{i}{\hbar}[c\alpha_x p_x, x] \\
&= c\alpha_x
\end{aligned}$$

Similarly, we have

$$\dot{y} = c\alpha_y, \qquad \dot{z} = c\alpha_z$$

or

$$v_i = c\alpha_i$$

Using the Mathematica, we solve the eigenvalue for the velocity for

$$v_x = c\alpha_x$$

For $\lambda_1 = c$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -c$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

In the Dirac theory, the velocity is $\pm c$, while in the non-relativistic Pauli theory the velocity operator is

$$\boldsymbol{v} = \frac{1}{m}\boldsymbol{p}$$

Such a contradiction may be a good reason to suspect that the whole theory is up the creek. However, Foldy and Wouthuysen come to rescue our rescue again. The essential point is that these apparently contradictory operators do not actually represent quite the same observable.

((**Note**)) **Sakurai's comment** (Advanced Quantum Mechanics)

The plane-wave solutions which are eigenfunctions of $\boldsymbol{p}$ are not the eigenfunction of $c\boldsymbol{\alpha}$. Since $c\boldsymbol{\alpha}$ fails to commute with the Hamiltonian, no energy eigenfunctions are expected to be simultaneous eigenfunction of $c\boldsymbol{\alpha}$. In the second quantization, the quantum field operator $\psi(\boldsymbol{r})$ can be expressed by the superposition of positive and negative energy plane wave solutions.

$$\langle \boldsymbol{\alpha} \rangle = \int d\boldsymbol{r}\psi^+(\boldsymbol{r})\alpha_k\psi(\boldsymbol{r}) = \frac{c^2}{E_R}p = \frac{c^2 p}{c\sqrt{m^2c^2+p^2}} = \frac{cp}{\sqrt{m^2c^2+p^2}}$$

which is time independent, represents the group velocity of the wave packet made of exclusively of positive- (negative-) energy plane-wave components.

2. Classical relativistic velocity of a free particle (FW component)

We continue to use the Heisenberg's equation of motion for the acceleration operator such that

$$\begin{aligned}
\dot{v}_i &= \frac{i}{\hbar}[H, v_i] \\
&= \frac{i}{\hbar}[H, c\alpha_i] \\
&= \frac{ic}{\hbar}\{[H, \alpha_i]_+ - 2\alpha_i H\}
\end{aligned}$$

We note that

$$\begin{aligned}
[H, \alpha_i]_+ &= [\sum_j cp_j\alpha_j + \beta mc^2, \alpha_i]_+ \\
&= \sum_j cp_j\{\alpha_i, \alpha_j\} + mc^2\{\alpha_i \beta\} \\
&= \sum_j 2cp_j\delta_{i,j} \\
&= 2cp_i
\end{aligned}$$

where

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} I_4, \qquad \{\alpha_i, \beta\} = 0, \qquad \beta^2 = I_4$$

where $i = x$, y, and z, the curly bracket denotes an anti-commutator,

$$\{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i$$

Thus we get

$$\begin{aligned}
\dot{v}_i &= \frac{ic}{\hbar}(2cp_i - 2\alpha_i H) \\
&= -\frac{ic}{\hbar}2H(\alpha_i - \frac{1}{H}cp_i)
\end{aligned}$$

Here we introduce a new variable,

$$\eta_i = \alpha_i - \frac{1}{H} c p_i$$

Since

$$v_i = c\alpha_i$$

$$\dot{v}_i = c\dot{\alpha}_i = c\dot{\eta}_i = \frac{ic}{\hbar} 2H\eta_i$$

Thus we have the first-order differential equation for η_i as

$$\dot{\eta}_i = -\frac{i}{\hbar} 2H\eta_i .$$

The solution for this equation is

$$\eta_i(t) = \eta_i(0)\exp(-\frac{2i}{\hbar} Ht)$$

or

$$\begin{aligned}\frac{dx_i}{dt} &= c\alpha_i = c\eta_i + \frac{1}{H} c^2 p_i \\ &= c\eta_i(0)\exp(-\frac{2i}{\hbar} Ht) + \frac{1}{H} c^2 p_i\end{aligned}$$

or

$$\begin{aligned}x_i(t) &= x_i(0) + \frac{1}{H} c^2 p_i t + \frac{c\hbar}{2iH} \eta_i(0)\exp(-\frac{2i}{\hbar} Ht) \\ &= x_i(0) + \frac{1}{H} c^2 p_i t + \frac{c\hbar}{2iH} [\alpha_i(0) - \frac{1}{H} c p_i]\exp(-\frac{2i}{\hbar} Ht)\end{aligned}$$

where

$$\eta_i(0) = \alpha_i(0) - \frac{1}{H} c p_i$$

where $x_i(t)$ is the position operator at time *t*. The *x*-component of the velocity has two parts. The second term is the FW term and is associated with the average motion of the particle (classical relativistic formula). The first term is the Zitterbewegung which oscillates extremely rapidly.

3. Numerical calculation of zitterbewegung

The resulting expression consists of an initial position, a motion proportional to time, and an unexpected oscillation term with an amplitude equal to the Compton wavelength. That oscillation term is the so-called zitterbewegung. Interestingly, the zitterbewegung term vanishes on taking expectation values for wave-packets that are made up entirely of positive- (or entirely of negative-) energy waves. This can be achieved by taking a Foldy–Wouthuysen transformation. Thus, we arrive at the interpretation of the zitterbewegung as being caused by interference between positive- and negative-energy wave components.

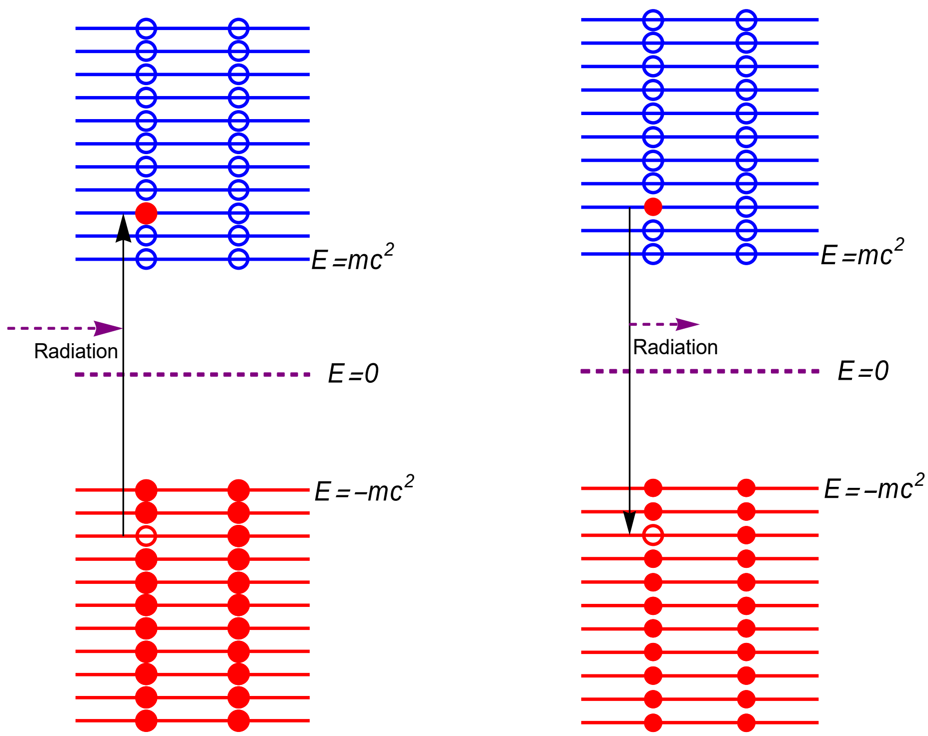

Fig. Dirac sea. The energy gap is $\Delta E = 2mc^2$. The zitterbewegung is related to the transition between the upper level with the positive energy mc^2 and the lower level with the negative energy $-mc^2$.

The angular frequency of zitterbewegung

$$\omega = \frac{2mc^2}{\hbar} = 1.55268 \text{ x } 10^{21} \text{ rad/s}$$

where mc^2 =0.510997 MeV. The Compton wavelength is defined by

$$\lambda_C = \frac{2\pi\hbar}{mc} = 2.4263102367 \text{ x } 10^{-10} \text{ cm} = 0.024263 \text{ Å}$$

Note that the amplitude of the oscillation of the zitterbewegung

$$\frac{\hbar}{2mc} = \lambda_c \frac{1}{4\pi} = 1.9308 \text{ x } 10^{-11} \text{ cm}$$

3. Foldy–Wouthuysen term of the velocity (classical relativistic velocity)

We note that the Heisenberg's equation of motion is given by

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}[\hat{H}, \hat{A}] \tag{1}$$

where

$$\hat{H} = c(\alpha \cdot p) + \beta mc^2$$

So we have

$$\frac{d}{dt}\hat{\boldsymbol{r}} = \frac{i}{\hbar}[\hat{H}, \hat{\boldsymbol{r}}] = c\boldsymbol{\alpha} \tag{2}$$

for $\boldsymbol{r}$ (velocity equal to the speed of light, c). We note that $\hat{\boldsymbol{r}}_{FW}$ for the position vector in the FW representation is defined by

$$\begin{aligned} \hat{\boldsymbol{r}}_{FW} &= \hat{U}\hat{\boldsymbol{r}}\hat{U}^+ \\ &= \boldsymbol{r} - \frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R} + \frac{\hbar c^2}{2E_R{}^2(E_R + mc^2)}\{ic\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})\boldsymbol{p} - E_R(\boldsymbol{\Sigma} \times \boldsymbol{p})]\} \end{aligned} \tag{3}$$

Here we introduce a new operator defined by

$$\hat{\boldsymbol{R}} = \hat{U}^+\hat{\boldsymbol{r}}\hat{U} = \boldsymbol{r} + \frac{i\hbar c\beta\boldsymbol{\alpha}}{2E_R} - \frac{\hbar c^2}{2E_R{}^2(E_R + mc^2)}\{ic\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p})\boldsymbol{p} + E_R(\boldsymbol{\Sigma} \times \boldsymbol{p})]\} \tag{4}$$

Corresponding to $\hat{\boldsymbol{R}}$, the operator $\hat{\boldsymbol{R}}_{FW}$ is defined by

$$\hat{\boldsymbol{R}}_{FW} = \hat{U}\hat{\boldsymbol{R}}\hat{U}^{+} = \hat{U}\hat{U}^{+}\hat{\boldsymbol{r}}\hat{U}\hat{U}^{+} = \hat{\boldsymbol{r}}$$

We also note that

$$\hat{\boldsymbol{r}}_{FW} = \hat{U}\boldsymbol{r}\hat{U}^{+}, \qquad \hat{\boldsymbol{R}} = \hat{U}^{+}\boldsymbol{r}\hat{U}$$

$$\hat{U}^{+}\hat{\boldsymbol{r}}_{FW}\hat{U} = \hat{U}^{+}\hat{U}\hat{\boldsymbol{r}}\hat{U}^{+}\hat{U} = \hat{\boldsymbol{r}}$$

Now we consider the Heisenberg's equation of motion for $\hat{\boldsymbol{R}}$:

$$\frac{d\hat{\boldsymbol{R}}}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{\boldsymbol{R}}]$$

or

$$\hat{U}\frac{d\hat{\boldsymbol{R}}}{dt}\hat{U}^{+} = \frac{d\hat{\boldsymbol{R}}_{FW}}{dt} = \frac{i}{\hbar}\hat{U}[\hat{H}, \hat{\boldsymbol{R}}]\hat{U}^{+} = \frac{i}{\hbar}[\hat{U}\hat{H}\hat{U}^{+}, \hat{U}\hat{\boldsymbol{R}}\hat{U}^{+}]$$

or

$$\begin{aligned}\frac{d\hat{\boldsymbol{R}}_{FW}}{dt} &= \frac{i}{\hbar}[\hat{H}_{FW}, \hat{\boldsymbol{R}}_{FW}] \\ &= \frac{i}{\hbar}[\hat{H}_{FW}, \hat{\boldsymbol{R}}_{FW}] \\ &= -\frac{i}{\hbar}[\hat{\boldsymbol{r}}, \beta E_R] \\ &= \beta\frac{c^2\boldsymbol{p}}{E_R}\end{aligned}$$

or

$$\boxed{\frac{d\hat{\boldsymbol{R}}_{FW}}{dt} = \beta\frac{c^2\boldsymbol{p}}{E_R}}$$

where

$$\hat{H}_{FW} = \beta E_R \,, \qquad\qquad \hat{\boldsymbol{R}}_{FW} = \hat{\boldsymbol{r}}$$

Since $\hat{\boldsymbol{R}}_{FW} = \hat{\boldsymbol{r}}$, we have

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \beta \frac{c^2 \boldsymbol{p}}{E_R}$$

This leads to

$$\frac{d\hat{\boldsymbol{r}}_{FW}}{dt} = \hat{U} \frac{d\hat{\boldsymbol{r}}}{dt} \hat{U}^+ = \hat{U}\beta\hat{U}^+ \frac{c^2 \boldsymbol{p}}{E_R} = \frac{c^2 \boldsymbol{p}}{E_R} \frac{\hat{H}}{E_R}$$

$$\frac{d\hat{\boldsymbol{R}}}{dt} = \hat{U}^+ \frac{d\hat{\boldsymbol{r}}}{dt} \hat{U} = \hat{U}^+ \beta \frac{c^2 \boldsymbol{p}}{E_R} \hat{U} = \frac{c^2 \boldsymbol{p}}{E_R} \frac{\hat{H}}{E_R}$$

Note that

$$\hat{U}^+ \beta E_R \hat{U} = \hat{U}^+ \hat{H}_{FW} \hat{U} = \hat{U}^+ \hat{U} \hat{H} \hat{U}^+ \hat{U} = \hat{H}$$

or

$$\hat{U}^+ \beta \hat{U} = \frac{\hat{H}}{E_R}$$

Now the term $\hat{H}/E_R$ has eigenvalue +1 for particle wavefunctions and -1 for antiparticle wavefunctions. So, $d\hat{\boldsymbol{R}}/dt = d\hat{\boldsymbol{r}}_{FW}/dt$ is just $c^2\boldsymbol{p}/E_R$ for particles, which is the classical relativistic velocity. We also note that $d\hat{\boldsymbol{r}}/dt = c\boldsymbol{\alpha}$ (standard velocity)

4. Summary

The standard velocity operator $c\boldsymbol{\alpha}$ gives the electron speed as equal to the speed of light, while $d\hat{\boldsymbol{r}}_{FW}/dt$ leads to a velocity that has a sensible non-relativistic limit. This means that the motion of the electron can be divided into two parts. First is the average velocity. Secondary there is very rapid oscillatory motion (Zitterbewegung)

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