Notation of Tensor and Vector used in the relativistic quantum mechanics

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Our conventions for relativity follow Landau and Lifshitz, Ohanian, and nearly all recent field theory text books. We use the matrix tensor $g_{\mu\nu}$ with Greek indices running over 0, 1, 2, 3 or t, x, y, z. Roman indices – i, j, k – denote only the three spatial components. Repeated indices are summed in all cases. Four-vectors, like ordinary numbers, are denoted by light italic type; three-vectors are denoted by boldface type. For example,

$$x^{\mu} = (x^0, \boldsymbol{x}), \qquad x_{\mu} = g_{\mu\nu} x^{\nu} = (x^0, -\boldsymbol{x})$$

$$p \cdot x = g_{\mu\nu} p^{\mu} x^{\nu} = p^{\mu} x_{\mu} = p_{\mu} x^{\mu} = p^0 x^0 - \boldsymbol{p} \cdot \boldsymbol{x}$$

1. Contravariant and covariant vectors

The metric tensor is given by

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$g_{\mu\nu}$: covariant metric tensor

$g^{\mu\nu}$: contra-variant metric tensor

We note that

$$g_{\mu\nu} = g_{\nu\mu} \qquad \text{(symmetric)}$$

$$g^{\mu\nu} = g^{\nu\mu} \qquad \text{(symmetric)}$$

Here we define the contra-variant vector x^{μ} and the co-variant vector x_{μ}

(a) Space-time position vector

$$x^{\mu} = (ct, \boldsymbol{r}), \qquad x_{\mu} = (ct, -\boldsymbol{r})$$

(b) Energy-momentum vector

$$E = i\hbar\frac{\partial}{\partial t}, \qquad \boldsymbol{p} = \frac{\hbar}{i}\nabla = -i\hbar\nabla$$

(c) Momentum four-vector

$$p^{\mu} = i\hbar\partial^{\mu} = i\hbar\frac{\partial}{\partial x_{\mu}} = i\hbar(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla) = (\frac{E}{c}, \boldsymbol{p})$$

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$$

(d) Momentum four-vector

$$p_{\mu} = i\hbar\partial_{\mu} = i\hbar\frac{\partial}{\partial x^{\mu}} = i\hbar(\frac{1}{c}\frac{\partial}{\partial t}, \nabla) = (\frac{E}{c}, -\boldsymbol{p})$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = (\frac{1}{c}\frac{\partial}{\partial t}, \nabla)$$

(e)

$$\partial^2 = \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \qquad \text{(D'Alembertian)}$$

$$p^{\mu}p_{\mu} = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2$$

$$p^{\mu}x_{\mu} = p_{\mu}x^{\mu} = \frac{E}{c}(ct) - \boldsymbol{p}\cdot\boldsymbol{r} = Et - \boldsymbol{p}\cdot\boldsymbol{r}$$

(f) The probability current density four-vector

$$j^{\mu} = (c\rho, \boldsymbol{j})$$

$$\partial_\mu j^\mu = \frac{\partial j^\mu}{\partial x^\mu}$$

$$= \frac{\partial j^0}{\partial x^0} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^2}{\partial x^2} + \frac{\partial j^3}{\partial x^3} \qquad \text{(equation of continuity).}$$

$$= \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J}$$

(g) Vector potential and scalar potential

$$A^\mu = (\Phi, \boldsymbol{A})$$

$$\partial_\mu A^\mu = \frac{\partial A^\mu}{\partial x^\mu}$$

$$= \frac{\partial A^0}{\partial x^0} + \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3}$$

$$= \frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \boldsymbol{A}$$

(h)

$$A^\mu = (A^0, \boldsymbol{A}), \qquad A_\mu = (A^0, -\boldsymbol{A})$$

$$B^\mu = (B^0, \boldsymbol{B}), \qquad B_\mu = (B^0, -\boldsymbol{B})$$

$$A \cdot B = A^\mu B_\mu = A_\mu B^\mu = A^0 B^0 - \boldsymbol{A} \cdot \boldsymbol{B}$$

2. Relations between co-variant and contra-variant vectors

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x^\mu = g^{\mu\nu} x_\nu$$

$$x^\mu = g^\mu{}_\nu x^\nu, \qquad x_\mu = g_\mu{}^\nu x_\nu$$

$g^\mu{}_\nu$ and $g_\mu{}^\nu$ are the Kronecker delta ($=\delta_{\mu,\nu}$)

$$x'^{\alpha}\, x'_{\alpha} = x^{\mu} x_{\mu}\,,$$

or

$$x'^{\alpha}\, g_{\alpha\beta} x'^{\beta} = x^{\mu} g_{\mu\nu} x^{\nu}$$

Since

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}\,, \quad x'^{\beta} = \Lambda^{\beta}{}_{\nu} x^{\nu} \quad \text{(Lorentz transformation)}$$

we get

$$x'^{\alpha}\, g_{\alpha\beta} x'^{\beta} = x^{\mu} g_{\mu\nu} x^{\nu}$$

or

$$\Lambda^{\alpha}{}_{\mu} x^{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} x^{\nu} = x^{\mu} g_{\mu\nu} x^{\nu}$$

or

$$(\Lambda^{\alpha}{}_{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} - g_{\mu\nu}) x^{\mu} x^{\nu} = 0$$

For any $x^{\mu} x^{\nu}$, we have

$$\Lambda^{\alpha}{}_{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = g_{\mu\nu}$$

$$A^{\mu}{}' = \Lambda^{\mu}{}_{\alpha} A^{\alpha}$$

$$A^{\mu} = g^{\mu\nu} A_{\nu}$$

$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}, \qquad A^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} A_{\alpha\beta}$$

$$x'_{\alpha}\, x'^{\alpha} = x_{\mu} x^{\mu},$$

$$\Lambda_{\alpha}{}^{\mu} x_{\mu} \Lambda^{\alpha}{}_{\nu} x^{\nu} = \Lambda_{\alpha}{}^{\mu} \Lambda^{\alpha}{}_{\nu} x_{\mu} x^{\nu} = g^{\mu}{}_{\nu} x_{\mu} x^{\nu} = x_{\mu} x^{\mu}$$

$$\Lambda_{\alpha}{}^{\mu}\Lambda^{\alpha}{}_{\nu} = g^{\mu}{}_{\nu}$$ (Kronecker delta)

Kronecker delta

$g^{\alpha\beta}g_{\beta}{}^{\gamma} = g^{\alpha\gamma}$, leading to the relation the Kronecker delta, $g_{\beta}{}^{\gamma} = \delta_{\beta,\gamma}$

$g_{\alpha\beta}g^{\beta}{}_{\gamma} = g_{\alpha\gamma}$, leading to the relation the Kronecker delta, $g^{\beta}{}_{\gamma} = \delta_{\beta,\gamma}$

$$g^{\beta}{}_{\gamma} = g_{\beta}{}^{\gamma} = \delta_{\beta,\gamma}$$

We consider the expression

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$$

Multiplication of $\Lambda_{\mu}{}^{\beta}$ from the right side:

$$\Lambda_{\mu}{}^{\beta}x'^{\mu} = \Lambda_{\mu}{}^{\beta}\Lambda^{\mu}{}_{\nu}x^{\nu} = g^{\beta}{}_{\nu}x^{\nu} = x^{\beta}$$

or

$$x^{\beta} = \Lambda_{\mu}{}^{\beta}x'^{\mu}$$

In summary, we have

$$x^{\mu} = \Lambda_{\nu}{}^{\mu}x'^{\nu}$$

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu}x_{\nu}$$

Multiplication of $\Lambda^{\mu}{}_{\beta}$ from the right side:

$$\Lambda^{\mu}{}_{\beta}x'_{\mu} = \Lambda^{\mu}{}_{\beta}\Lambda_{\mu}{}^{\nu}x_{\nu} = g_{\beta}{}^{\nu}x_{\nu} = x_{\beta}$$

or

$$x_{\beta} = \Lambda^{\mu}{}_{\beta}x'_{\mu}$$

or

$$x_\mu = \Lambda^\nu{}_\mu x'_\nu$$

((**Expression of tensor**))

$$A_\mu{}^\nu = g_{\mu\alpha} A^{\alpha\nu}$$

$$A^\mu{}_\nu = g_{\nu\alpha} A^{\mu\alpha}$$

$$A^{\mu\nu} = g^{\nu\lambda} A^\mu{}_\lambda$$

$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

(a)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu$$

(b)

$$x^\mu = \Lambda_\nu{}^\mu x'^\nu$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = \Lambda_\nu{}^\mu$$

$$\partial_\mu' = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu$$

(c) Trace

$\Lambda^{\mu}{}_{\mu}$; trace of the tensor

$$\Lambda^{\mu}{}_{\mu} = \Lambda^{1}{}_{1} + \Lambda^{2}{}_{2} + \Lambda^{3}{}_{3} + \Lambda^{4}{}_{4}$$

(d) Feynman-dagger

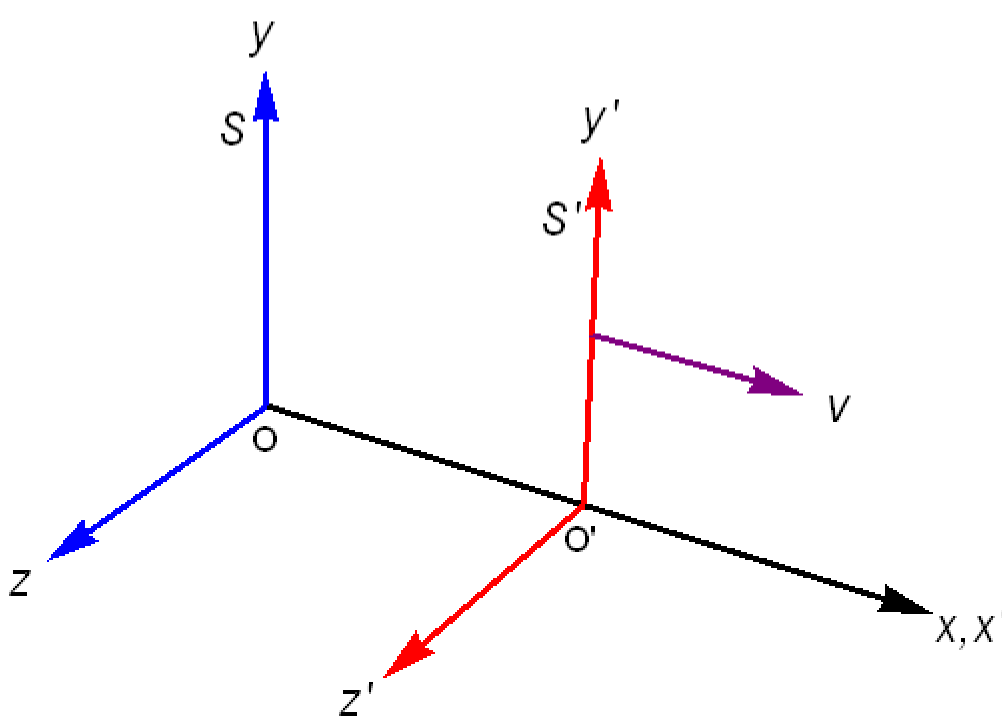
$$\gamma^{\mu} A_{\mu}$$

where γ^{μ} is one of the Dirac matrices.

2. Lorentz transformation

We consider the transformation from an inertial reference frame K to a system K' moving relative to K with velocity v along the x axis.

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\mu} x^{\mu}, \qquad x^{\mu} = \Lambda_{\nu}{}^{\mu} x'^{\nu}$$



$$x' = \gamma(x - vt), \qquad t' = \gamma(t - \frac{v}{c^2} x), \qquad y' = y, \qquad z' = z$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}.$$

The inverse relations are obtained by solving the four equations with respect to the variables *x*, *y*, *z*, and *t*;

$$x = \gamma(x' + vt'), \qquad t = \gamma(t' + \frac{v}{c^2}x')$$

(a) The Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ and $\Lambda^{\mu\nu}$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \text{(Lorentz transformation)}$$

$$\begin{pmatrix} x^{0}{}' \\ x^{1}{}' \\ x^{2}{}' \\ x^{3}{}' \end{pmatrix} = \begin{pmatrix} \Lambda^{0}{}_{0} & \Lambda^{0}{}_{1} & \Lambda^{0}{}_{2} & \Lambda^{0}{}_{3} \\ \Lambda^{1}{}_{0} & \Lambda^{1}{}_{1} & \Lambda^{1}{}_{2} & \Lambda^{1}{}_{3} \\ \Lambda^{2}{}_{0} & \Lambda^{2}{}_{1} & \Lambda^{2}{}_{2} & \Lambda^{2}{}_{3} \\ \Lambda^{3}{}_{0} & \Lambda^{3}{}_{1} & \Lambda^{3}{}_{2} & \Lambda^{3}{}_{3} \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

or

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that

$$\cosh\chi = \gamma, \qquad \sinh\chi = \gamma\beta, \qquad \tanh\chi = \beta.$$

(b) Lorentz transformation: $\Lambda^{\nu}{}_{\mu}$

Since the matrix $\Lambda^{\nu}{}_{\mu}$ is the transpose of the matrix $\Lambda^{\nu}{}_{\mu}$, $\Lambda^{\nu}{}_{\mu}$ can be expressed by

$$\Lambda^{\nu}{}_{\mu} = (\Lambda^{T})^{\mu}{}_{\nu}$$

$$= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that in this case, $\Lambda^{\nu}{}_{\mu} = \Lambda^{\mu}{}_{\nu}$ (symmetric)

(c) Lorentz transformation: $\Lambda^{\mu\nu}$

From the definition we have

$$\Lambda^{\mu\nu} = g^{\nu\lambda}\Lambda^{\mu}{}_{\lambda}$$

$$= \Lambda^{\mu}{}_{\lambda} g^{\lambda\nu}$$

$$= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ -\sinh\chi & -\cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Lorentz transformation: $\Lambda_{\nu}{}^{\mu}$

The inverse Lorentz transformation is defined by

$$x^{\mu} = \Lambda_{\nu}{}^{\mu} x'^{\nu},$$

$\Lambda_{\nu}{}^{\mu}$ is the inverse matrix of $\Lambda^{\mu}{}_{\nu}$, $\Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}$

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \Lambda_0{}^0 & \Lambda_1{}^0 & \Lambda_2{}^0 & \Lambda_3{}^0 \\ \Lambda_0{}^1 & \Lambda_1{}^1 & \Lambda_2{}^1 & \Lambda_3{}^1 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} x^0{}' \\ x^1{}' \\ x^2{}' \\ x^3{}' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0{}' \\ x^1{}' \\ x^2{}' \\ x^3{}' \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0{}' \\ x^1{}' \\ x^2{}' \\ x^3{}' \end{pmatrix}$$

or

$$\Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We note that

$$\Lambda_{\nu}{}^{\mu} = g_{\nu\alpha} g^{\mu\beta} \Lambda^{\alpha}{}_{\beta}$$

or

$$\Lambda_{\nu}{}^{\mu} \Lambda^{\nu}{}_{\lambda} = g_{\nu\alpha} g^{\mu\beta} \Lambda^{\alpha}{}_{\beta} \Lambda^{\nu}{}_{\lambda}$$

We note that x'_{μ} is given by either

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu}$$

or

$$x'_\mu = g_{\mu\lambda} x'^\lambda = g_{\mu\lambda} \Lambda^\lambda{}_\rho x^\rho = g_{\mu\lambda} \Lambda^\lambda{}_\rho g^{\rho\nu} x_\nu = g_{\mu\lambda} g^{\rho\nu} \Lambda^\lambda{}_\rho x_\nu$$

Then we get

$$\Lambda_\mu{}^\nu = g_{\mu\lambda} g^{\rho\nu} \Lambda^\lambda{}_\rho = g_{\mu\lambda} g^{\nu\rho} \Lambda^\lambda{}_\rho = g_{\mu\lambda} \Lambda^\lambda{}_\rho g^{\rho\nu}$$

The interexchange between μ and ν leads to

$$\Lambda_\nu{}^\mu = g_{\nu\lambda} \Lambda^\lambda{}_\rho g^{\rho\mu} = g_{\nu\lambda} \Lambda^\lambda{}_\rho g^{\mu\rho} = g^{\mu\rho} \Lambda^\lambda{}_\rho g_{\lambda\nu}$$

or

$$\begin{aligned}
\Lambda_\nu{}^\mu &= g^{\mu\rho} \Lambda^\lambda{}_\rho g_{\lambda\nu} \\
&= g^{\mu\rho} (\Lambda^T)^\rho{}_\lambda g_{\lambda\nu} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

where

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that

$$\Lambda_{\alpha}{}^{\mu}\Lambda^{\alpha}{}_{\nu} = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$\Lambda_{\alpha}{}^{\mu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\alpha}$$

(d) Lorentz transformation $\Lambda_{\mu\nu}$

$$\Lambda_{\mu\nu} = g_{\alpha\mu}\Lambda^{\alpha}{}_{\nu}$$

$$= g_{\mu\alpha}\Lambda^{\alpha}{}_{\nu}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ -\sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\chi & -\sinh\chi & 0 & 0 \\ \sinh\chi & -\cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Infinitesimal Lorentz transformation

For infinitesimal Lorentz transformation,; $\chi \approx 0$,

$$\cosh\chi \approx 1, \qquad \sinh\chi \approx \varepsilon\chi$$

The infinitesimal Lorentz transformation can be expressed by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & -\varepsilon\chi & 0 & 0 \\ -\varepsilon\chi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \varepsilon\chi \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= g^{\mu}{}_{\nu} + \varepsilon\omega^{\mu}{}_{\nu}$$

$$= \delta_{\mu,\nu} + \varepsilon\omega^{\mu}{}_{\nu}$$

where

$$g^{\mu}{}_{\nu} = \delta_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\varepsilon\omega^{\mu}{}_{\nu} = \varepsilon\chi \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \varepsilon\chi M^{\mu}{}_{\nu}$$

where

$$M^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that

$$
\begin{aligned}
M^{\mu\nu} &= g^{\nu\alpha} M^{\mu}{}_{\alpha} \\
&= M^{\mu}{}_{\alpha} g^{\alpha\nu} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
M_{\mu\nu} &= g_{\mu\alpha} M^{\alpha}{}_{\nu} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

4. Lorentz transformation (general case)

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\frac{\gamma}{c}v_x & -\frac{\gamma}{c}v_y & -\frac{\gamma}{c}v_z \\ -\frac{\gamma}{c}v_x & 1+(\gamma-1)\frac{{v_x}^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\ -\frac{\gamma}{c}v_y & (\gamma-1)\frac{v_x v_y}{v^2} & 1+(\gamma-1)\frac{{v_y}^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\ -\frac{\gamma}{c}v_z & (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} & 1+(\gamma-1)\frac{{v_z}^2}{v^2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

and

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} {\Lambda_0}^0 & {\Lambda_1}^0 & {\Lambda_2}^0 & {\Lambda_3}^0 \\ {\Lambda_0}^1 & {\Lambda_1}^1 & {\Lambda_2}^1 & {\Lambda_3}^1 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix}$$

with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \frac{\gamma}{c}v_x & \frac{\gamma}{c}v_y & \frac{\gamma}{c}v_z \\ \frac{\gamma}{c}v_x & 1+(\gamma-1)\frac{{v_x}^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\ \frac{\gamma}{c}v_y & (\gamma-1)\frac{v_x v_y}{v^2} & 1+(\gamma-1)\frac{{v_y}^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\ \frac{\gamma}{c}v_z & (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} & 1+(\gamma-1)\frac{{v_z}^2}{v^2} \end{pmatrix}$$

Infinitesimal Lorentz transformation (in the limit of $\gamma \to 1$)

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\beta_x{}^2 & (\gamma-1)\beta_x\beta_y & (\gamma-1)\beta_z\beta_x \\ -\gamma\beta_y & (\gamma-1)\beta_x\beta_y & 1+(\gamma-1)\beta_y{}^2 & (\gamma-1)\beta_y\beta_z \\ -\gamma\beta_z & (\gamma-1)\beta_z\beta_x & (\gamma-1)\beta_y\beta_z & 1+(\gamma-1)\beta_z{}^2 \end{pmatrix}$$
$$\approx \begin{pmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & 0 & 0 \\ -\beta_y & 0 & 1 & 0 \\ -\beta_z & 0 & 0 & 1 \end{pmatrix}$$

Then we have

$$\Lambda^{\mu}{}_{\nu} == g^{\mu}{}_{\nu} + \varepsilon\omega^{\mu}{}_{\nu}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 0 & 0 & 0 \\ -\beta_y & 0 & 0 & 0 \\ -\beta_z & 0 & 0 & 0 \end{pmatrix}$$

with

$$\varepsilon\omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 0 & 0 & 0 \\ -\beta_y & 0 & 0 & 0 \\ -\beta_z & 0 & 0 & 0 \end{pmatrix}$$

5. Formula

$$x'_{\nu}\, x'^{\nu} = g_{\nu\mu} x'^{\mu}\, x'^{\nu} = g_{\mu\nu} x'^{\mu}\, x'^{\nu}$$

$$x_{\nu} x^{\nu} = g_{\nu\mu} x^{\mu} x^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu}$$

where $g_{\nu\mu} = g_{\mu\nu}$ (symmetric)

Then we have

$$g_{\mu\nu} x'^{\mu}\, x'^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu}$$

From the definition of x'^{μ} and x'^{ν}, we get

$$g_{\mu\nu}(\Lambda^{\mu}{}_{\rho}x^{\rho})(\Lambda^{\nu}{}_{\sigma}x^{\sigma}) = g_{\mu\nu}x^{\mu}x^{\nu} = g_{\rho\sigma}x^{\rho}x^{\sigma}$$

The comparison of the coefficient of $x^{\rho}x^{\sigma}$ for both sides:

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma},$$

or

$$\Lambda_{\nu\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}.$$

Switching between μ and ν;

$$\Lambda_{\mu\rho}\Lambda^{\mu}{}_{\sigma} = g_{\rho\sigma}$$

Multiplication of $g^{\rho\alpha}$ from the right-hand side:

$$g^{\rho\alpha}\Lambda_{\mu\rho}\Lambda^{\mu}{}_{\sigma} = g^{\rho\alpha}g_{\rho\sigma} = g^{\alpha}{}_{\sigma}$$

or

$$\Lambda_{\mu}{}^{\alpha}\Lambda^{\mu}{}_{\sigma} = g^{\rho\alpha}g_{\rho\sigma} = g_{\rho\sigma}g^{\rho\alpha} = g^{\alpha}{}_{\sigma} = g_{\sigma}{}^{\alpha} = \delta_{\sigma,\alpha} \qquad \text{(Kronecker delta)}$$

(a)

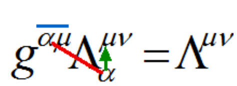
$$\Lambda_{\mu\rho}\Lambda^{\mu}{}_{\sigma} = g_{\rho\sigma}$$

$$\Lambda_{\mu}{}^{\alpha}\Lambda^{\mu}{}_{\sigma} = g^{\alpha}{}_{\sigma} = g_{\sigma}{}^{\alpha} = \delta_{\sigma,\alpha}$$

$$g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

(b)

$$g^{\alpha\mu}\Lambda_{\alpha}{}^{\nu} = \Lambda^{\mu\nu}$$



$$g_{\alpha\mu}\Lambda^{\alpha}{}_{\nu} = \Lambda_{\mu\nu}$$

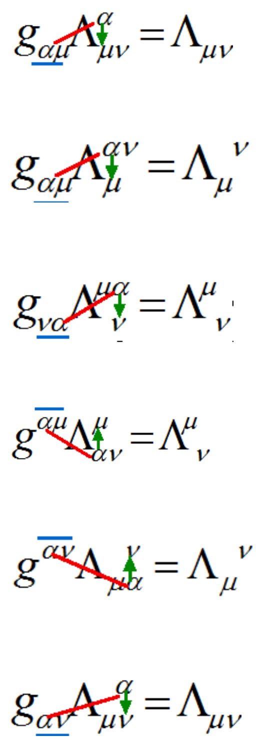
$$g_{\alpha\mu}\Lambda^{\alpha\nu} = \Lambda_{\mu}{}^{\nu}$$

$$g_{\nu\alpha}\Lambda^{\mu\alpha} = \Lambda^{\mu}{}_{\nu}$$

$$g^{\alpha\mu}\Lambda_{\alpha\nu} = \Lambda^{\mu}{}_{\nu}$$

$$g^{\alpha\nu}\Lambda_{\mu\alpha} = \Lambda_{\mu}{}^{\nu}$$

$$g_{\alpha\nu}\Lambda_{\mu}{}^{\alpha} = \Lambda_{\mu\nu}$$



(c)

$$\Lambda^{\mu\nu} = g^{\alpha\mu}\Lambda_{\alpha}{}^{\nu} = g^{\mu\alpha}\Lambda_{\alpha}{}^{\nu}$$

$$\Lambda_{\mu\nu} = g_{\alpha\mu}\Lambda^{\alpha}{}_{\nu} = g_{\mu\alpha}\Lambda^{\alpha}{}_{\nu}$$

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$$\Lambda^{\mu}{}_{\nu} = g_{\nu\alpha}\Lambda^{\mu\alpha}$$

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$$\Lambda_{\mu}{}^{\nu} = g^{\alpha\nu}\Lambda_{\mu\alpha} = g^{\nu\alpha}\Lambda_{\mu\alpha}$$

$$\Lambda_{\mu\nu} = g_{\alpha\nu}\Lambda_{\mu}{}^{\alpha} = g_{\nu\alpha}\Lambda_{\mu}{}^{\alpha}$$

(d)

$$\varepsilon^{\mu\nu} = g^{\mu\alpha}\varepsilon_{\alpha}{}^{\nu}$$

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$$\varepsilon_{\mu}{}^{\nu} = g_{\mu\alpha}\varepsilon^{\alpha\nu}$$

$$\varepsilon^{\mu}{}_{\nu} = g_{\nu\alpha}\varepsilon^{\mu\alpha}$$

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$$\varepsilon_{\mu}{}^{\nu} = g^{\nu\alpha}\varepsilon_{\mu\alpha}$$

$$\varepsilon_{\mu\nu} = g_{\nu\alpha}\varepsilon_{\mu}{}^{\alpha}$$

((**Note**))

$$\Lambda^{\mu}{}_{\nu} = g^{\alpha\mu}\Lambda_{\alpha\nu} = g^{\mu\alpha}\Lambda_{\alpha\nu}$$

Suppose that

$$\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}$$

Then we have

$$\begin{aligned} g^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu} &= g^{\mu\alpha}(g_{\alpha\nu} + \varepsilon_{\alpha\nu}) \\ &= g^{\mu}{}_{\nu} + g^{\mu\alpha}\varepsilon_{\alpha\nu} \end{aligned}$$

which leads to the expression

$$\varepsilon^{\mu}{}_{\nu} = g^{\mu\alpha}\varepsilon_{\alpha\nu}\,.$$

(e)

$$\Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}$$

$$\Lambda^{\nu}{}_{\mu} = (\Lambda^T)^{\mu}{}_{\nu}$$

$$\Lambda^{\mu}{}_{\lambda}(\Lambda^{-1})^{\lambda}{}_{\nu} = \Lambda^{\mu}{}_{\lambda}\Lambda_{\nu}{}^{\lambda} = g^{\mu}{}_{\nu}$$

$$(\Lambda^{-1})^{\mu}{}_{\lambda}\Lambda^{\lambda}{}_{\nu} = \Lambda_{\lambda}{}^{\mu}\Lambda^{\lambda}{}_{\nu} = g^{\mu}{}_{\nu}$$

$$\left(\Lambda^T\right)^{\mu}{}_{\lambda}(\left(\Lambda^T\right)^{-1})^{\lambda}{}_{\nu} = \Lambda^{\lambda}{}_{\mu}\left(\Lambda^{-1}\right)^{\nu}{}_{\lambda} = \left(\Lambda^{-1}\right)^{\nu}{}_{\lambda}\Lambda^{\lambda}{}_{\mu} = g^{\nu}{}_{\mu}$$

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