Eikonal approximation

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Of greater antiquity, but closely related to the WKB approximation, is the eikonal approximation. It is by means of the eikonal approximation that geometrical optics is obtained as a limiting approximation to physical optics. Here we describe a somewhat simplified from the eikonal which has been quite useful in discussing the scattering of very fast particles by smooth potential. (Goldberger and Watson, Collision Theory, John Wiley & Sons, 1964)).

The potential *V* itself does not need to be weak as long as $E >> V$. The domain of validity is different from the Born approximation. Note that eikon means "image" in Greek.

1. Introduction

The eikonal approximation in quantum mechanics works for processes involving the scattering of particles with large incoming momentum and when the scattering angle is very small. In the language of differential equations, the main advantage the eikonal approximation offers is that the equations reduce to a differential equation in a single variable. This reduction into a single variable is the result of the straight line approximation or the eikonal approximation which allows us to choose the straight line as a special direction. The early steps involved in the eikonal approximation in quantum mechanics are very closely related to the WKB approximation in quantum mechanics. The WKB approximation involves an expansion in terms of Planck's constant $\hbar$. The WKB approximation also reduces the equations into a differential equation in a single variable. But the complexity involved in the WKB approximation is that this variable is described by the trajectory of the particle which in general is complicated. The advantage of the eikonal approximation is in the classical trajectory being a straight line. Thus in this manner the eikonal approximation is a very stringent semi-classical limit.

Here we show that the eikonal approximation is valid for processes involving small angle scattering and very large incoming momentum. More rigorously the conditions are

$$\frac{V_0}{E} << 1, \text{ and } \quad 1 << ka\frac{V_0}{E} << \frac{1}{\frac{V_0}{E}} \qquad \text{(see the detail later).}$$

where a is the range of potential, E and k are the energy and wave number of incident particle. The scattering amplitude is given by

$$f(\theta) = \frac{k}{2\pi i}\int d^2\boldsymbol{b} e^{i\boldsymbol{Q}\cdot\boldsymbol{b}}[e^{i\chi(k,b)} - 1]$$

or

$$f(\theta) = -ik\int_0^\infty bdbJ_0(Qb)[e^{i\chi(k,b)} - 1]$$

where

$$\chi(k,b) = -\frac{1}{2k}\int_{-\infty}^{\infty} dz' U(\boldsymbol{b} + z'\boldsymbol{n}),$$

$\boldsymbol{Q}$ is the scattering vector,

$$Q = 2k\sin\frac{\theta}{2}.$$

2. Eikonal approximation for scattering

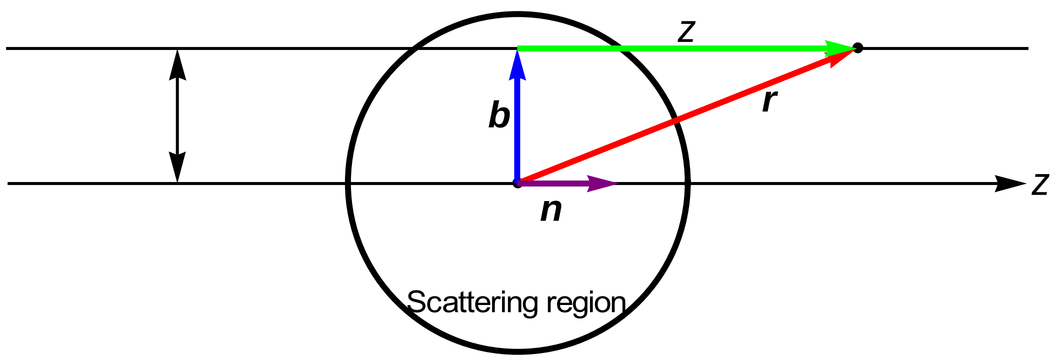


Fig. Schematic diagram of eikonal approximation scattering. The classical straight-line trajectory is along the *z* axis. $b\,(=|\boldsymbol{b}|)$ is the impact parameter.

We start with the wave function $\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r})$ given by

$$\begin{aligned}
\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) &= \langle \boldsymbol{r}|\boldsymbol{k}\rangle + \int d\boldsymbol{r}'\langle \boldsymbol{r}|(E_k - \hat{H}_0 + i\varepsilon)^{-1}|\boldsymbol{r}'\rangle V(\boldsymbol{r}')\langle \boldsymbol{r}'|\psi^{(+)}\rangle \\
&= \langle \boldsymbol{r}|\boldsymbol{k}\rangle - \int d\boldsymbol{r}' G_0^{(+)}(\boldsymbol{r},\boldsymbol{r}')U(\boldsymbol{r}')\langle \boldsymbol{r}'|\psi^{(+)}\rangle \\
&= \frac{1}{(2\pi)^{3/2}}e^{ikz} - \int d\boldsymbol{r}'\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}U(\boldsymbol{r}')\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r})
\end{aligned}$$

Here we have

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{ikz} .$$

The Green's function is defined by

$$\begin{aligned}
G_0^{(+)}(\boldsymbol{r},\boldsymbol{r}') &= -\frac{\hbar^2}{2\mu}\langle \boldsymbol{r}|(E_k - \hat{H}_0 + i\varepsilon)^{-1}|\boldsymbol{r}'\rangle \\
&= -\frac{\hbar^2}{2\mu}\int \frac{d\boldsymbol{k}'}{(2\pi)^3} \frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{E_k - \frac{\hbar^2}{2\mu}\boldsymbol{k}'^2 + i\varepsilon} \\
&= -\int \frac{d\boldsymbol{k}'}{(2\pi)^3} \frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{\boldsymbol{k}^2 - \boldsymbol{k}'^2 + i\varepsilon} \\
&= \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}
\end{aligned}$$

When the incident particle enters into the effective range of potential, the wave function is supposed to take the form

$$\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \varphi(\boldsymbol{r}) e^{ikz} .$$

Substituting this form of $\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r})$ into the equation

$$\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} e^{ikz} - \int d\boldsymbol{r}' G_0^{(+)}(\boldsymbol{r},\boldsymbol{r}') U(\boldsymbol{r}') \psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}')$$

leads to the equation for $\varphi(\boldsymbol{r})$,

$$\begin{aligned}
\varphi(\boldsymbol{r}) &= 1 + \frac{1}{(2\pi)^3}\int d\boldsymbol{r}' \int d\boldsymbol{k}' \frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{\boldsymbol{k}^2 - \boldsymbol{k}'^2 + i\varepsilon} U(\boldsymbol{r}') \varphi(\boldsymbol{r}') e^{i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})} \\
&= 1 + \frac{1}{(2\pi)^3}\int d\boldsymbol{r}'' \int d\boldsymbol{k}' \frac{e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}''}}{\boldsymbol{k}^2 - \boldsymbol{k}'^2 + i\varepsilon} U(\boldsymbol{r}-\boldsymbol{r}'') \varphi(\boldsymbol{r}-\boldsymbol{r}'')
\end{aligned}$$

where

$$\boldsymbol{r} - \boldsymbol{r}' = \boldsymbol{r}''$$

$$G_0^{(+)}(\boldsymbol{r},\boldsymbol{r}') = -\int \frac{d\boldsymbol{k}'}{(2\pi)^3} \frac{e^{i\boldsymbol{k}'\cdot(\boldsymbol{r}-\boldsymbol{r}')}}{\boldsymbol{k}^2 - \boldsymbol{k}'^2 + i\varepsilon}$$

We introduce $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \boldsymbol{k}' - \boldsymbol{k} \qquad \text{(scattering vector)}$$

or

$$\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{\kappa}$$

Then we get

$$k'^2 = \kappa^2 + k^2 + 2\boldsymbol{k}\cdot\boldsymbol{\kappa} \approx k^2 + 2\boldsymbol{k}\cdot\boldsymbol{\kappa}$$

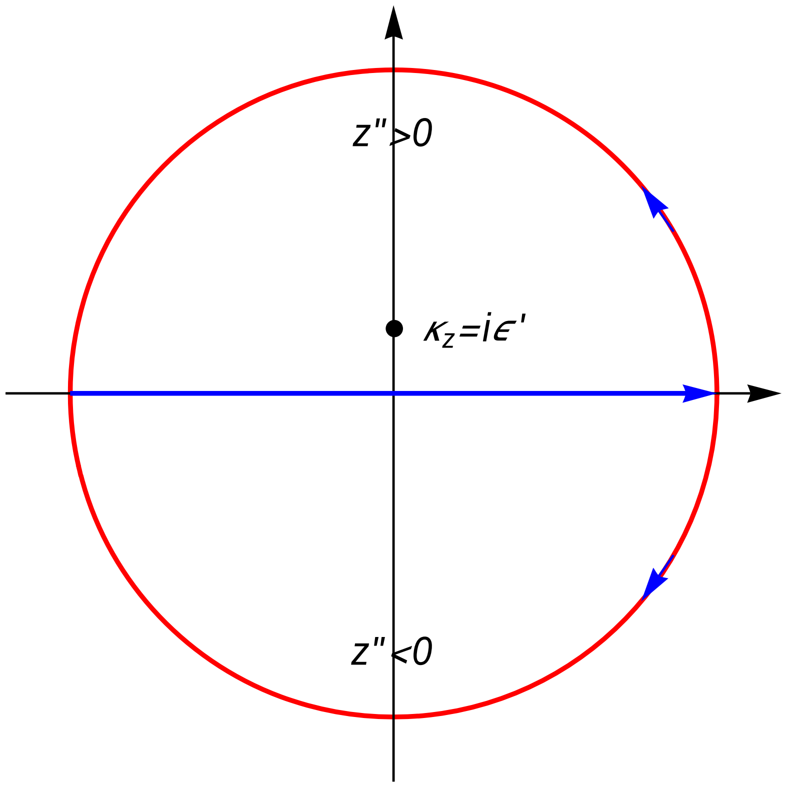
$$k^2 - k'^2 = -2\boldsymbol{k}\cdot\boldsymbol{\kappa}$$

$$\varphi(\boldsymbol{r}) = 1 + \frac{1}{(2\pi)^3}\int d\boldsymbol{r}''\int d\boldsymbol{\kappa}\frac{e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}''}}{-2\boldsymbol{k}\cdot\boldsymbol{\kappa} + i\varepsilon}U(\boldsymbol{r}-\boldsymbol{r}'')\varphi(\boldsymbol{r}-\boldsymbol{r}'')$$

We note that

$$\int d\boldsymbol{\kappa}\frac{e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}''}}{-2\boldsymbol{k}\cdot\boldsymbol{\kappa} + i\varepsilon} = \int d\kappa_x e^{i\kappa_x x''}\int d\kappa_y e^{i\kappa_y y''}\int d\kappa_z \frac{e^{i\kappa_z z''}}{-2k\kappa_z + i\varepsilon}$$
$$= -\frac{i}{2k}(2\pi)^3\delta(x'')\delta(y'')\Theta(z'')$$

where $\boldsymbol{k} = (0,0,k)$ and $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. Here we have



$$I = \int d\kappa_z \frac{e^{i\kappa_z z''}}{-2k\kappa_z + i\varepsilon} = -\frac{1}{2k}\int d\kappa_z \frac{e^{i\kappa_z z''}}{\kappa_z - i\dfrac{\varepsilon}{2k}} = -\frac{1}{2k}\int d\kappa_z \frac{e^{i\kappa_z z''}}{\kappa_z - i\varepsilon'}$$

or

$$I = -\frac{1}{2k}(2\pi i) \text{ for } z''>0, \qquad \text{and} \qquad I = 0 \text{ for } z''<0,$$

For $z''>0$, we use the upper half plane of the complex plane. There is a single pole at $\kappa_z = i\varepsilon'$. Using the residue theorem, we get

$$I = -\frac{1}{2k}(2\pi i)\,\mathrm{Re}\,s[\kappa_z = i\varepsilon'] = -\frac{1}{2k}(2\pi i)\exp(-\varepsilon' z'') = -\frac{1}{2k}(2\pi i).$$

with $\varepsilon' \to 0$ For $z''<0$, we use the lower half plane of the complex plane. There is no single pole. Using the residue theorem, we get

$$I = 0$$

Then we have

$$\varphi(\boldsymbol{r}) = 1 - \frac{i}{2k}\int dx''dy''dz''U(\boldsymbol{r}-\boldsymbol{r}'')\varphi(\boldsymbol{r}-\boldsymbol{r}'')\delta(x'')\delta(y'')\Theta(z'')$$
$$= 1 - \frac{i}{2k}\int_0^\infty dz''U(x,y,z-z'')\varphi(x,y,z-z'')$$

For simplicity we use the relation

$$z - z'' = z', \qquad -dz'' = dz'$$

$$\varphi(x,y,z) = 1 - \frac{i}{2k}\int_{-\infty}^{z} dz'U(x,y,z')\varphi(x,y,z')$$

The solution of this integral equation is given by

$$\varphi(x,y,z) = \exp[-\frac{i}{2k}\int_{-\infty}^{z} dz'U(x,y,z')].$$

Using this relation we get

$$\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}}\varphi(\boldsymbol{r})e^{ikz}$$
$$= \frac{1}{(2\pi)^{3/2}}\exp[ikz - \frac{i}{2k}\int_{-\infty}^{z} dz'U(x,y,z')]$$
$$= \frac{1}{(2\pi)^{3/2}}\exp[ikz - \frac{i}{\hbar v}\int_{-\infty}^{z} dz'V(x,y,z')]$$

The condition:

$$\frac{|V_0|}{\hbar v} << k,$$

and

$ka >> 1$ (high energy)

where a is the scattering region of potential. Note that $\frac{a}{v}$ is the time for the particle to pass the scattering region. $\frac{\hbar}{|V_0|}$ is the time during which the potential sufficiently exerts the influence on the particle. In the eikonal approximation, there is no limitation for $(a/v)/(\hbar/|V_0|) = \frac{|V_0|a}{\hbar v}$. In this sense, the approximation is applicable to the case when $|V_0|$ is large.

$$\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \exp[ikz - \frac{i}{\hbar v}\int_{-\infty}^{z} dz' V(x, y, z')]$$

and

$$\psi^{(+)}(r,\theta) = \frac{1}{(2\pi)^{3/2}}[e^{ikz} + \frac{1}{r}e^{ikr} f(\theta)]$$

From these two equations, we try to get the correct asymptotic form of $f(\theta)$.

$$\begin{aligned}
\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}) &= \frac{1}{(2\pi)^{3/2}} e^{ikz} - \int d\boldsymbol{r}' \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} U(\boldsymbol{r}')\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}') \\
&= \frac{1}{(2\pi)^{3/2}} e^{ikz} - \frac{1}{4\pi r} e^{ikr} \int d\boldsymbol{r}' e^{-i\boldsymbol{k}\cdot\boldsymbol{r}'} U(\boldsymbol{r}')\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{r}') \\
&= \frac{1}{(2\pi)^{3/2}} e^{ikz} - \frac{1}{4\pi r} e^{ikr} \int d\boldsymbol{r}' e^{-i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}'} U(\boldsymbol{r}') \frac{1}{(2\pi)^{3/2}} \exp(-\frac{i}{2k}\int_{-\infty}^{z'} dz'' U(x', y', z'')) \\
&= \frac{1}{(2\pi)^{3/2}} [e^{ikz} - \frac{1}{4\pi}\frac{1}{r} e^{ikr} \int d\boldsymbol{r}' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}'} U(\boldsymbol{r}') \exp[-\frac{i}{2k}\int_{-\infty}^{z'} dz'' U(x', y', z'')]
\end{aligned}$$

$$f(\theta) = -\frac{1}{4\pi}\int d\boldsymbol{r}' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}'} U(\boldsymbol{r}') \exp[-\frac{i}{2k}\int_{-\infty}^{z'} dz'' U(x', y', z'')]$$

For simplicity, we have replacement of $\boldsymbol{r}' \to \boldsymbol{r}$, $\boldsymbol{r}'' \to \boldsymbol{r}'$, leading to

$$f(\theta) = -\frac{1}{4\pi}\int d\boldsymbol{r} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} U(\boldsymbol{r}) \exp[-\frac{i}{2k}\int_{-\infty}^{z} dz' U(x,y,z')].$$

We consider the scattering amplitude using the cylindrical co-ordinate.

$$\boldsymbol{r} = \boldsymbol{b} + z\boldsymbol{n}$$

We define the scattering vector as

$$\boldsymbol{Q} = \boldsymbol{k} - \boldsymbol{k}'$$

$$\boldsymbol{Q}\cdot\boldsymbol{r} = \boldsymbol{Q}\cdot\boldsymbol{b} + z\boldsymbol{Q}\cdot\boldsymbol{n} \approx \boldsymbol{Q}\cdot\boldsymbol{b}$$

since $\boldsymbol{Q}$ is nearly perpendicular to $\boldsymbol{n}$; $\boldsymbol{Q}\cdot\boldsymbol{n} \approx 0$. Then we get

$$f(\theta) = -\frac{1}{4\pi}\int d^2\boldsymbol{b} e^{i\boldsymbol{Q}\cdot\boldsymbol{b}} \int_{-\infty}^{\infty} dz U(\boldsymbol{b}+z\boldsymbol{n}) \exp[-\frac{i}{2k}\int_{-\infty}^{z} dz' U(\boldsymbol{b}+z'\boldsymbol{n})]$$

Here we note

$$\begin{aligned}
\int_{-\infty}^{\infty} dz U(\boldsymbol{b}+z\boldsymbol{n}) \exp[-\frac{i}{2k}\int_{-\infty}^{z} dz' U(\boldsymbol{b}+z'\boldsymbol{n})] &= 2ki\int_{-\infty}^{\infty} dz \frac{d}{dz}\exp[-\frac{i}{2k}\int_{-\infty}^{z} dz' U(\boldsymbol{b}+z'\boldsymbol{n})] \\
&= 2ki\left[\exp[-\frac{i}{2k}\int_{-\infty}^{z} dz' U(\boldsymbol{b}+z'\boldsymbol{n})]\right]_{-\infty}^{\infty} \\
&= 2ki\{\exp[-\frac{i}{2k}\int_{-\infty}^{\infty} dz' U(\boldsymbol{b}+z'\boldsymbol{n})] - 1\} \\
&= 2ki[e^{i\chi(k,b)} - 1]
\end{aligned}$$

where

$$\chi(k,b) = -\frac{1}{2k}\int_{-\infty}^{\infty} dz' U(\boldsymbol{b}+z'\boldsymbol{n})$$

$$f(\theta) = \frac{k}{2\pi i}\int d^2\boldsymbol{b} e^{i\boldsymbol{Q}\cdot\boldsymbol{b}} [e^{i\chi(k,b)} - 1]$$

$$\frac{1}{2\pi}\int d^2\boldsymbol{b}e^{i\boldsymbol{Q}\cdot\boldsymbol{b}} = \frac{1}{2\pi}\int_0^\infty bdb\int_0^{2\pi} d\phi \exp(iQb\cos\phi)$$
$$= \int_0^\infty bdbJ_0(Qb)$$

$$f(\theta) = -ik\int_0^\infty bdbJ_0(Qb)[e^{i\chi(k,b)} - 1]$$

where

$$Q = 2k\sin\frac{\theta}{2}, \qquad \boldsymbol{Q}\cdot\boldsymbol{b} = Qb\cos\phi$$

and ϕ is the angle between $\boldsymbol{Q}$ and $\boldsymbol{b}$. We note that the scattering amplitude is the Hankel transform of

$$f(b) \equiv e^{i\chi(k,b)} - 1$$

In the book of Sakurai and Napolitano (Quantum Mechanics),

$$\chi(k,b) = 2\Delta(b)$$

with

$$\Delta(b) = -\frac{1}{4k}\int_{-\infty}^{\infty} dz' U(\mathbf{b} + z'\mathbf{n}) = -\frac{\mu}{2k\hbar^2}\int_{-\infty}^{\infty} dz' V(\mathbf{b} + z'\mathbf{n})$$

((**Definition of the Hankel transform**))

$$g(\alpha) = \int_0^\infty bdbJ_0(\alpha b)f(b)$$

where

$$f(b) \equiv e^{i\chi(k,b)} - 1.$$

3. Partial waves and eikonal approximation

The eikonal approximation is valid at high energies. Therefore, many partial waves contribute.

$$l = kb, \qquad l_{\max} = kR$$

The scattering amplitude is given by

$$f(\theta) = \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$$

We have the following substitutions.

$$\sum_{l=0}^{l_{\max}} \to k\int db,$$

$$P_l(\cos\theta) \approx J_0[(l+\frac{1}{2})\theta] \approx J_0(l\theta) = J_0(kb\theta) \quad \text{for large } l\text{, small } \theta$$

where

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{ix\cos\phi}$$

We use the approximations given by

$$\delta_l \to \Delta(b)|_{b=\frac{l}{k}}$$

and

$$e^{i\delta_l}\sin\delta_l = \frac{1}{2i}(e^{2i\delta_l}-1) \to \frac{1}{2i}(e^{2i\Delta(b)}-1)$$

Note that the latter is equal to zero for $l > l_{\max}$. Then we get

$$f(\theta) \to -ik\int db b J_0(kb\theta)[e^{2i\Delta(b)}-1].$$

This agrees with the equation derived above,

$$f(\theta) = -ik\int_0^\infty bdbJ_0(Qb)[e^{i\chi(k,b)} - 1]$$

where

$$Q = 2k\sin\frac{\theta}{2} \approx k\theta, \qquad \text{and} \qquad \chi(k,b) = 2\Delta(b).$$

4. Born approximation

Here we show that the expression of $f(\theta)$ in the Born approximation can be derived from the eikonal approximation.

$$f(\theta) = \frac{k}{2\pi i}\int d^2\boldsymbol{b}e^{i\boldsymbol{Q}\cdot\boldsymbol{b}}[e^{i\chi(k,b)} - 1]$$

with

$$\chi(k,b) = -\frac{1}{2k}\int_{-\infty}^{\infty} dz' U(\boldsymbol{b} + z'\boldsymbol{n})$$

Suppose that

$$e^{i\chi(k,b)} - 1 \approx i\chi(k,b)$$

Then we have

$$\begin{aligned} f(\theta) &= \frac{k}{2\pi i}\int d^2\boldsymbol{b}e^{i\boldsymbol{Q}\cdot\boldsymbol{b}}\chi(k,b) \\ &= -\frac{1}{4\pi}\int d^2\boldsymbol{b}\int_{-\infty}^{\infty} dze^{i\boldsymbol{Q}\cdot\boldsymbol{b}}U(\boldsymbol{b} + z\boldsymbol{n}) \\ &= -\frac{1}{4\pi}\int d^2\boldsymbol{b}\int_{-\infty}^{\infty} dze^{i\boldsymbol{Q}\cdot\boldsymbol{b}}U(\boldsymbol{b} + z\boldsymbol{n}) \end{aligned}$$

where

$$\boldsymbol{r} = \boldsymbol{b} + z\boldsymbol{n}, \qquad \boldsymbol{Q} = \boldsymbol{k} - \boldsymbol{k}'$$

$$\boldsymbol{Q}\cdot\boldsymbol{r}=\boldsymbol{Q}\cdot\boldsymbol{b}+z\boldsymbol{Q}\cdot\boldsymbol{n}\approx\boldsymbol{Q}\cdot\boldsymbol{b}$$

since $\boldsymbol{Q}$ is nearly perpendicular to $\boldsymbol{n}$; $\boldsymbol{Q}\cdot\boldsymbol{n}\approx 0$.

Thus the scattering amplitude can be rewritten as

$$f(\theta)=-\frac{1}{4\pi}\int d\boldsymbol{r}e^{i\boldsymbol{Q}\cdot\boldsymbol{r}}U(\boldsymbol{r})$$

which is the expression for the Born approximation.

5. Condition for the validity for the eikonal equation

Landau and Lifshitz: Quantum Mechanics, p.160

Here we consider the scattering potential as a perturbation. What is the condition for the validity of the Born approximation. We start with a differential equation

$$\nabla^2\psi+k^2\psi=\frac{2\mu}{\hbar^2}\lambda V\psi$$

where V is a scattering potential as a perturbation and λ is a small real parameter. The energy of the incident particle is given by

$$E=\frac{p^2}{2m}=\frac{\hbar^2k^2}{2m}, \qquad p=\hbar k.$$

We assume that

$$\psi=\psi_0+\lambda\psi_1+\lambda^2\psi_2+...$$

The substitution of this equation for ψ into the original differential equation yields to

$$\nabla^2(\psi_0+\lambda\psi_1+\lambda^2\psi_2+...)+k^2(\psi_0+\lambda\psi_1+\lambda^2\psi_2+...)$$
$$=\frac{2\mu}{\hbar^2}\lambda V(\psi_0+\lambda\psi_1+\lambda^2\psi_2+...)$$

For the order of λ^0 (the 0-th approximation)

$$\nabla^2\psi_0+k^2\psi_0=0$$

For the order of λ^1 (the first approximation)

$$\nabla^2\psi_1 + k^2\psi_1 = \frac{2\mu}{\hbar^2}V\psi_0$$

For the order of λ^2 (the second approximation)

$$\nabla^2\psi_2 + k^2\psi_2 = \frac{2\mu}{\hbar^2}V\psi_1$$

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(a) Green function

$$\nabla^2\psi_1 + k^2\psi_1 = \frac{2\mu}{\hbar^2}V\psi_0$$

$$(\nabla^2 + k^2)G(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2),$$

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp(ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

The solution of $\nabla^2\psi_1 + k^2\psi_1 = \frac{2\mu}{\hbar^2}V\psi_0$ is given by

$$\begin{aligned}\psi_1(\boldsymbol{r}) &= -\frac{2\mu}{\hbar^2}\int \frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi_0(\boldsymbol{r}')d\boldsymbol{r}' \\ &= -\frac{\mu}{2\pi\hbar^2}\int \frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi_0(\boldsymbol{r}')d\boldsymbol{r}'\end{aligned}.$$

Let us find what conditions must be satisfied by the field V in order that it may be regarded as a perturbation. It is required that $|\psi^{(1)}| << |\psi^{(0)}|$. Let a be the order of magnitude of the dimensions of the region of space in which the field is noticeably different from zero. We shall first suppose that the energy of the particle is so low that ka is at most of the order of unity. Then the factor $\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)$ in the integrand is unimportant in an order-of-magnitude estimate. Then we have

$$|\psi_1| \approx \frac{\mu}{\hbar^2}|V||\psi_0|a^2$$

From the condition that $|\psi_1| \approx \frac{\mu}{\hbar^2}|V||\psi_0|a^2 << |\psi_0|$, we get

$$|V| << \frac{\hbar^2}{\mu a^2}$$

We notice that the expression on the right has a simple physical meaning; it is the order of the kinetic energy which the particle would have if enclosed in a volume of linear dimension *a* (since, by the Heisenberg's principle of uncertainty relation, its momentum would be of the order of $\frac{\hbar}{a}$.

$$p\Delta x \geq \frac{\hbar}{2}$$

When $\Delta x \approx a$, we have $p \geq \frac{\hbar}{2a}$, leading to the condition

$$ka \geq 1$$

The kinetic energy is

$$E \approx \frac{p^2}{2\mu} \approx \frac{\hbar^2}{\mu a^2}$$

In summary, the condition for the validity of the above discussion is that

$$|V| << \frac{\hbar^2}{\mu a^2} = E, \qquad ka \geq 1 \tag{1}$$

For large energies, when $ka >> 1$, the factor $\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)$ in the integrant

$$\psi_1(\boldsymbol{r}) = -\frac{\mu}{2\pi\hbar^2}\int \frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')\psi_0(\boldsymbol{r}')d\boldsymbol{r}',$$

plays an important part, and markedly reduces the value of the integral. Now we return to

$$\nabla^2\psi_1 + k^2\psi_1 = \frac{2\mu}{\hbar^2}V\psi_0 \qquad (2)$$

We take as *x*-axis the direction of the unperturbed motion; the unperturbed function then has the form

$$\psi_0 = e^{ikx}$$

(the constant factor is arbitrarily taken as unity). In Eq.(1), we assume that

$$\psi_1 = e^{ikx} f$$

The substitution of the trial function to Eq.(2) yields

$$2ikf'(x) = \frac{2\mu}{\hbar^2}V$$

or

$$f(x) = \frac{\mu}{ik\hbar^2}\int V dx$$

or

$$\psi_1 = e^{ikx} f = -\frac{i\mu}{k\hbar^2}e^{ikx}\int V dx$$

An estimate of this integral gives

$$|\psi_1| \approx \frac{\mu}{k\hbar^2}|V|a$$

Since $|\psi_1| \approx \frac{\mu}{k\hbar^2}|V|a << |\psi_0| = 1$, we have the condition

$$|V| << \frac{\hbar^2 k}{\mu a} = \frac{\hbar^2 k}{\mu a(ka)}ka = \frac{\hbar^2}{\mu a^2}ka = \frac{\hbar v}{a} \qquad (ka >> 1). \qquad (3)$$

where $v = \frac{\hbar k}{m}$ is the velocity of the particle. The condition (3) is weaker than the condition (1). The time $\Delta\tau = \frac{a}{v}$ is the time when the particle passes through the range of potential. Equation (3) indicates that

$$|V| << \frac{\hbar}{\Delta\tau}.$$

6. Example-I; scattering of particle by black sphere with radius *a*

We start with

$$f(\theta) = -ik\int_0^\infty bdbJ_0(Qb)[e^{i\chi(k,b)} - 1]$$

Suppose that

$$e^{i\chi(k,b)} = 1 \text{ for } b > a, \text{ and } e^{i\chi(k,b)} = 0 \text{ for } b < a$$

corresponding to the assumption that all particles are absorbed by the black sphere with radius *a*, and that and there is no potential for *b*>*a* [$\chi(k,b) = 0$]. Then we get

$$f(\theta) = ik\int_0^a bdbJ_0(Qb) = ika^2 \frac{J_1(Qa)}{Qa}$$

The scattering cross section is

$$\sigma_s = |f(\theta)|^2 = (ka)^2 \left[\frac{J_1(Qa)}{Qa}\right]^2$$

with

$$Q = 2k\sin\frac{\theta}{2}$$

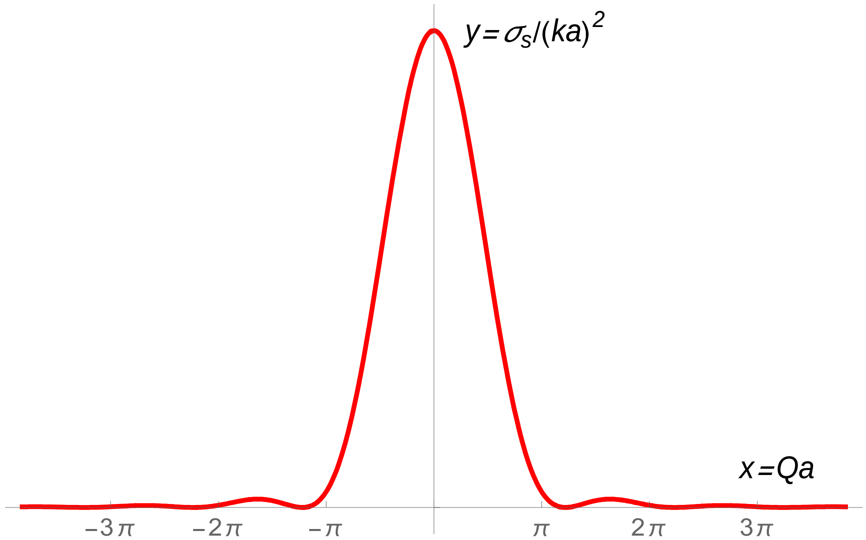


Fig. The Diffraction pattern. $\sigma_s/(ka)^2$ vs Qa.

7. Example-II Repulsive square potential

We consider the scattering amplitude due to the repulsive potential

$$V(r) = \begin{cases} V_0 & (r < a) \\ 0 & (r > a) \end{cases}$$

We note that

$$\chi(k,b) = -\frac{1}{2k}\int_{-\infty}^{\infty} dz' U(\boldsymbol{b} + z'\boldsymbol{n}) = \begin{cases} -\frac{U_0}{k}\sqrt{a^2 - b^2} & 0 < b < a \\ 0 & b > a \end{cases}$$

where

$$U_0 = \frac{2\mu}{\hbar^2} V_0$$

Then the scattering amplitude is

$$
\begin{aligned}
f(\theta) &= -ik\int_0^\infty bdbJ_0(Qb)[e^{i\chi(k,b)}-1] \\
&= -ik\int_0^a bdbJ_0(2kb\sin\frac{\theta}{2})\{\exp[-\frac{iU_0}{k}\sqrt{a^2-b^2}-1] \\
&\approx -ik\int_0^a bdbJ_0(kb\theta)\{\exp[-\frac{iU_0}{k}\sqrt{a^2-b^2}-1]
\end{aligned}
$$

where

$$Q = 2k\sin\frac{\theta}{2}$$

We introduce a new variable t as $b = at$. Then scattering amplitude can be rewritten as

$$f(\theta) = -ika^2\int_0^1 tdtJ_0(tka\theta)[\exp(-\frac{iU_0a}{k}\sqrt{1-t^2})-1]$$

Using the optical theorem, the total cross section can be obtained as

$$\sigma_{total} = \frac{4\pi}{k}\mathrm{Im}[f(0)] = 8\pi a^2\int_0^1 tdt\sin^2[\frac{U_0a}{k}\sqrt{1-t^2}]$$

When $\alpha = \frac{U_0a}{k}$

$$
\begin{aligned}
\frac{\sigma_{total}}{\pi a^2} &= 8\int_0^1 tdt\sin^2[\alpha\sqrt{1-t^2}] \\
&= \frac{1}{\alpha^2}[1+2\alpha^2-\cos(2\alpha)-2\alpha\sin(2\alpha)]
\end{aligned}
$$

where

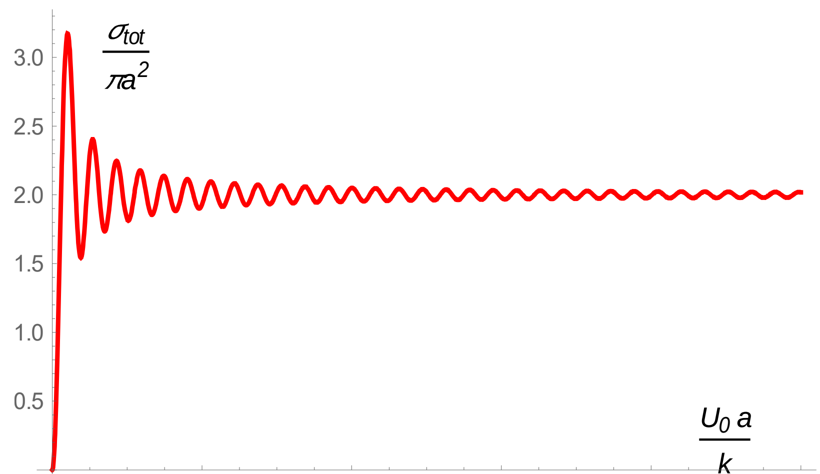
$$\mathrm{Im}[f(\theta=0)] = 2ka^2\int_0^1 tdtJ_0(0)\sin^2[\frac{U_0 a}{k}\sqrt{1-t^2}]$$
$$= 2ka^2\int_0^1 tdt\sin^2[\frac{U_0 a}{k}\sqrt{1-t^2}]$$

with

$$J_0(0) = 1$$

((**Mathematica**))

Plot of $\frac{\sigma_{total}}{\pi a^2}$ as a function of $\frac{U_0 a}{k}$ (eikonal equation for the repulsive potential)



REFERENCES

M.L. Goldberger and K.M. Watson, Collision (John Wiley & Sons, 1964).

J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, second edition (Addison-Wesley, 2011).

R.G. Newton, Scattering Theory of Waves and Particles, second edition (Springer, 1982).

L.D. Landau and E.M. Lifshitz, Quantum Mechanics (Pergamon Press, 1977).