Physics of the 21 cm hydrogen H1 line
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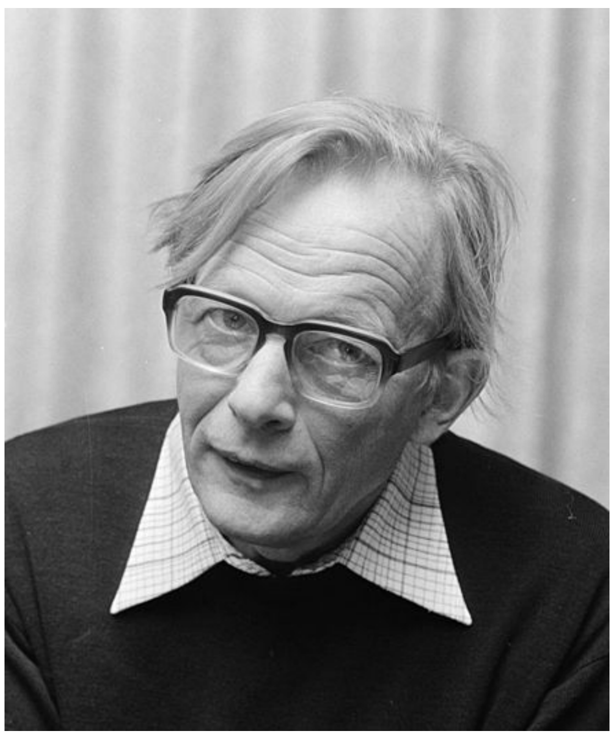
Here we discuss the hyperfine interaction between the proton and electron in the ground state hydrogen atom (1 $^2S_{1/2}$), where both the proton and electron have spin 1/2. Two particle states consists of $\left|+,+\right\rangle$, $\left|+,-\right\rangle$, $\left|-,+\right\rangle$ and $\left|-,-\right\rangle$. Because of the hyperfine interactions, these degenerate states are split into the singlet state (antisymmetric state) and the triplet state (symmetric state). The energy difference corresponds to 1042 MHz, or the wavelength 21 cm wave length. The transition between these states is due to the radiation through magnetic dipole, but not an electric dipole. The relaxation time is extremely long. We discuss the properties of the 21 cm hydrogen H1 line based on the numerical calculations. The magnitude of the hyperfine interaction depends only on the probability of the ground state hydrogen at the origin. So the s- electron with zero orbital angular momentum. Thanks to this fact, numerical calculations are so exact and can be compared with the experimental values. In this sense, the physic of 21 cm H1 line may provide a piece of evidence for the quantum mechanics.



Hendrik Christoffel "Henk" van de Hulst (19 November 1918 – 31 July 2000) was a Dutch astronomer and mathematician.

In 1944, while a student in Utrecht, he predicted the existence of the 21 cm hyperfine line of neutral interstellar hydrogen. After this line was discovered, he participated, with Jan Oort and C.A. Muller, in the effort to use radio astronomy to map out the neutral hydrogen in our galaxy, which first revealed its spiral structure. He spent most of his career at the University of Leiden, retiring in 1984. He published widely in astronomy, and dealt with the solar corona, and interstellar clouds. After 1960 he was a leader in international space research projects. In 1956 he became member of the Royal Netherlands Academy of Arts and Sciences.

1. Introduction

It is well known in optics that as the wavelength of light increases, attenuation through a given material decreases. It is no wonder, then, that radio waves reach us from parts of deep space that optical light does not. In particular, clouds of interstellar gas that would other- wise hide optical features in our galaxy can reveal much about general galactic structure and dynamics if we look in a frequency band around 1420.4 MHz. This is known as the 21 cm line. Emitted by neutral hydrogen atoms, this line can be seen with varying intensity coming from all directions in the sky, and due to its extremely sharp nature (very little dispersion in energy), is used widely in astronomy for spectroscopic velocity measurements.

A neutral hydrogen (H1) atom has one proton and one electron. Both particles have spin angular momentum (spin 1/2). These have the corresponding magnetic moments. There is a hyperfine interaction between these magnetic moments. The ground state of the neutral hydrogen (H1) is non-degenerate. When the spins of proton and electron are taken into account, the ground state is no longer non-degenerate. There are four two-spin states such as $\left|++\right\rangle$, $\left|+-\right\rangle$, $\left|-+\right\rangle$, and $\left|--\right\rangle$. In the state $\left|+-\right\rangle$, the proton is in the $\left|+z\right\rangle$ and the electron is in the $\left|-z\right\rangle$ state. In the presence of the hyperfine interaction between the spins of proton and electron, the ground state (4 states) splits into the F = 1 states (symmetric state) and F = 0 state (antisymmetric state). The energy level of F = 1 state is higher than that of F = 0 state, where $D_{1/2}\times D_{1/2}=D_1+D_0$. A photon is emitted during the transition from the F = 1 state to F = 0 state. The wavelength of the photon is 21 cm. Such a transition is a highly forbidden process, with a mean-life of approximately 10^7 years. The Heisenberg's principle of uncertainty leads one to expect a very sharp emission line with small energy dispersion, or line width, in frequency. This feature allows for highly accurate determinations of H1 source velocity by simple measurements of the Doppler shift of the 21 cm line.

2. Properties of proton and electron

Here we put the physical data of magnetic moments of electron and proton.

(a) Electron magnetic moment:

$$\boldsymbol{\mu}_e = -\frac{g_2\mu_B}{\hbar}\boldsymbol{S} = -\mu_B\sigma \qquad \text{with } g_e = 2.00$$

with

$$\mu_B = \frac{e\hbar}{2mc} = 9.274009994(57) \times 10^{-21} \text{ emu (= erg/Gauss)}$$

(b) Proton magnetic moment:

$$\mu_p = g_p\mu_N I = 2.79284730509 \ \mu_N$$

with

$$\mu_N = \frac{e\hbar}{2m_p c} = 5.050783699(31) \times 10^{-21} \text{ emu (emu=erg/G).}$$

The effective g-factor for proton:

$$g_p = 5.586947136$$

Angular momentum: $\hbar I$

$$I = \frac{1}{2} \text{ (nuclear spin)}$$

Gyromagnetic ratio:

$$\gamma = \frac{\mu_p}{\hbar I} = \frac{g_p\mu_N I}{\hbar I} = \frac{g_p\mu_N}{\hbar} = \frac{eg_p}{2m_p c} = 2.675222005(63) \times 10^4 \text{ rad/(s Oe).}$$

with

$$\mu_N = 5.050783699 \times 10^{-24} \text{ emu (emu= erg/Gauss)}$$

Rydberg constant $R = \frac{me^4}{2\hbar^2}$

Bohr radius $a = \frac{\hbar^2}{me}$

Fine structure constant $\alpha = \frac{e^2}{\hbar c}$

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3. Eigenvalue problem

$$E_{10} = \frac{4g_p\hbar^4}{3m_p m^2 c^2 a_B{}^4} = 5.87746 \text{ x } 10^{-6} \text{ eV}$$

$$\nu_{10} = \frac{E_{10}}{h} = 1412.16 \text{ MHz}$$

$$\lambda_{10} = \frac{c}{\nu_{10}} = 21.0948 \text{ cm}$$

The Hamiltonian of the hyperfine interaction between the electron and proton is

$$\hat{H}_0 = \frac{g_p e^2\hbar^2}{3m_e m_p a_B{}^3 c^2}\hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e = \frac{g_p\hbar^4}{3m_p m_e{}^2 c^2 a_B{}^4}\hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e = \frac{1}{4}E_{10}\hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e$$

We introduce the Dirac exchange operator which is defined by

$$\hat{P}_{pe} = \frac{1}{2}(\hat{1} + \hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e)$$

The Hamiltonian can be rewritten as

$$\hat{H}_0 = \frac{1}{4}E_{10}\hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e = \frac{1}{4}E_{10}(2\hat{P}_{pe} - \hat{1})$$

Then we have

$$\hat{H}_0\left|++\right\rangle = \frac{1}{4}E_{10}(2\hat{P}_{pe} - \hat{1})\left|++\right\rangle = \frac{1}{4}E_{10}\left|++\right\rangle$$

$$\hat{H}_0 \left| - - \right\rangle = \frac{1}{4} E_{10} (2\hat{P}_{pe} - \hat{1}) \left| - - \right\rangle = \frac{1}{4} E_{10} \left| - - \right\rangle$$

So $\left| + + \right\rangle$ and $\left| - - \right\rangle$ are the eigenkets of $\hat{H}_0$ with the energy eigenvalue $\frac{1}{4} E_{10}$.

$$\hat{H}_0 \left| + - \right\rangle = \frac{1}{4} E_{10} (2\hat{P}_{pe} - \hat{1}) \left| + - \right\rangle = \frac{1}{2} E_{10} \left| - + \right\rangle - \frac{1}{4} E_{10} \left| + - \right\rangle$$

$$\hat{H}_0 \left| - + \right\rangle = \frac{1}{4} E_{10} (2\hat{P}_{pe} - \hat{1}) \left| - + \right\rangle = \frac{1}{2} E_{10} \left| + - \right\rangle - \frac{1}{4} E_{10} \left| - + \right\rangle$$

The Hamiltonian is expressed by the matrix

$$\hat{H}_0 = \frac{E_0}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

under the basis of { $\left| + + \right\rangle$, $\left| + - \right\rangle$, $\left| - + \right\rangle$, $\left| - - \right\rangle$ }. In the subspace of { $\left| + - \right\rangle$, $\left| - + \right\rangle$ }, the Hamiltonian can be simplified by

$$\hat{H}_0 = \frac{E_0}{4} \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = -\frac{1}{4} E_0 \hat{1} + \frac{1}{2} E_0 \hat{\sigma}_x$$

$$\hat{H}_0 \left| + x \right\rangle = (-\frac{1}{4} E_0 \hat{1} + \frac{1}{2} E_0 \hat{\sigma}_x) \left| + x \right\rangle = \frac{1}{4} E_0 \left| + x \right\rangle$$

$$\hat{H}_0 \left| - x \right\rangle = (-\frac{1}{4} E_0 \hat{1} + \frac{1}{2} E_0 \hat{\sigma}_x) \left| - x \right\rangle = -\frac{3}{4} E_0 \left| + x \right\rangle$$

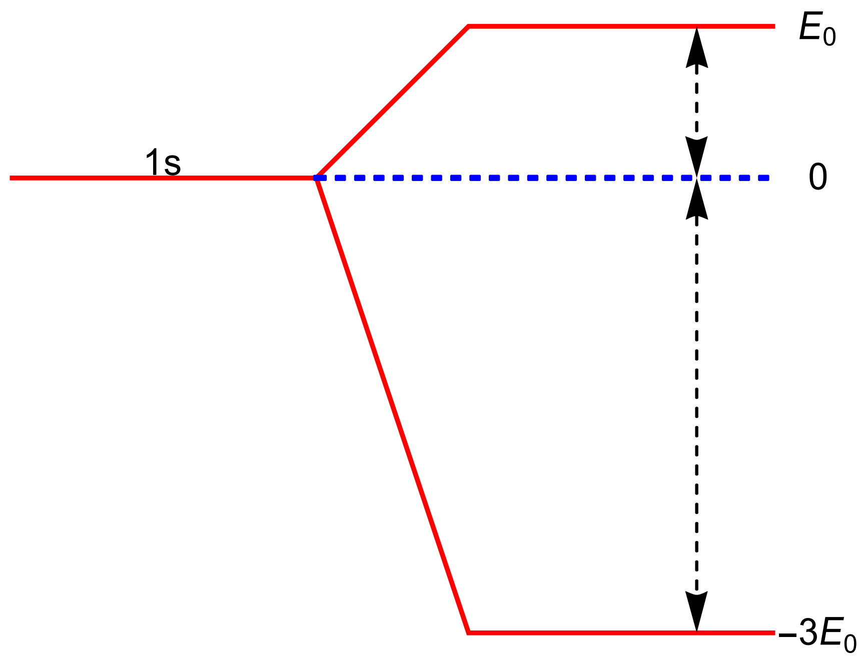


Fig. Hyperfine splitting in the ground state of hydrogen. Triplet state with energy E_0 ($|++\rangle$, $\frac{1}{\sqrt{2}}[|+-\rangle + |-+\rangle]$, $|--\rangle$ and the singlet with the energy $(-3E_0)$ ($\frac{1}{\sqrt{2}}[|+-\rangle - |-+\rangle]$

4. Zeeman energy

In the presence of a magnetic field along the z axis,

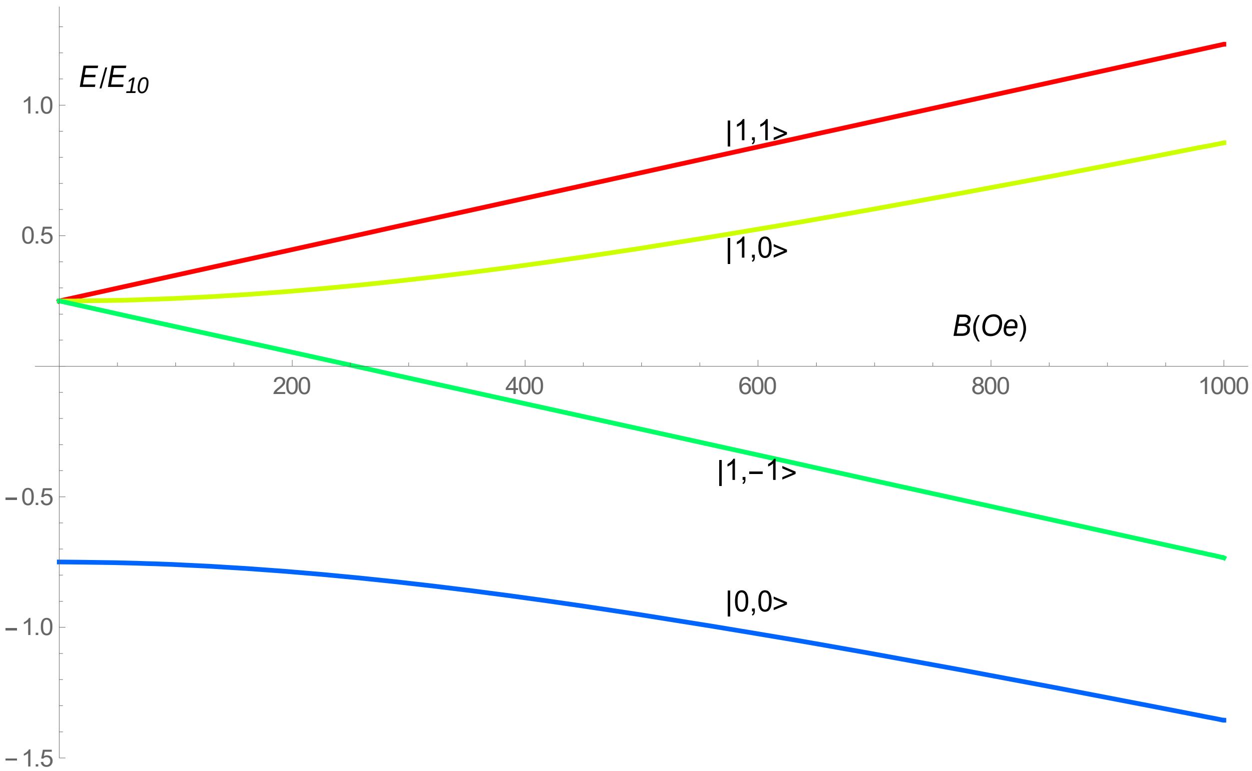
$$\hat{H} = J\hat{\boldsymbol{\sigma}}_p \cdot \hat{\boldsymbol{\sigma}}_e + \mu_B B\hat{\sigma}_{ez} - \frac{g_p \mu_N B}{2}\hat{\sigma}_{pz}$$

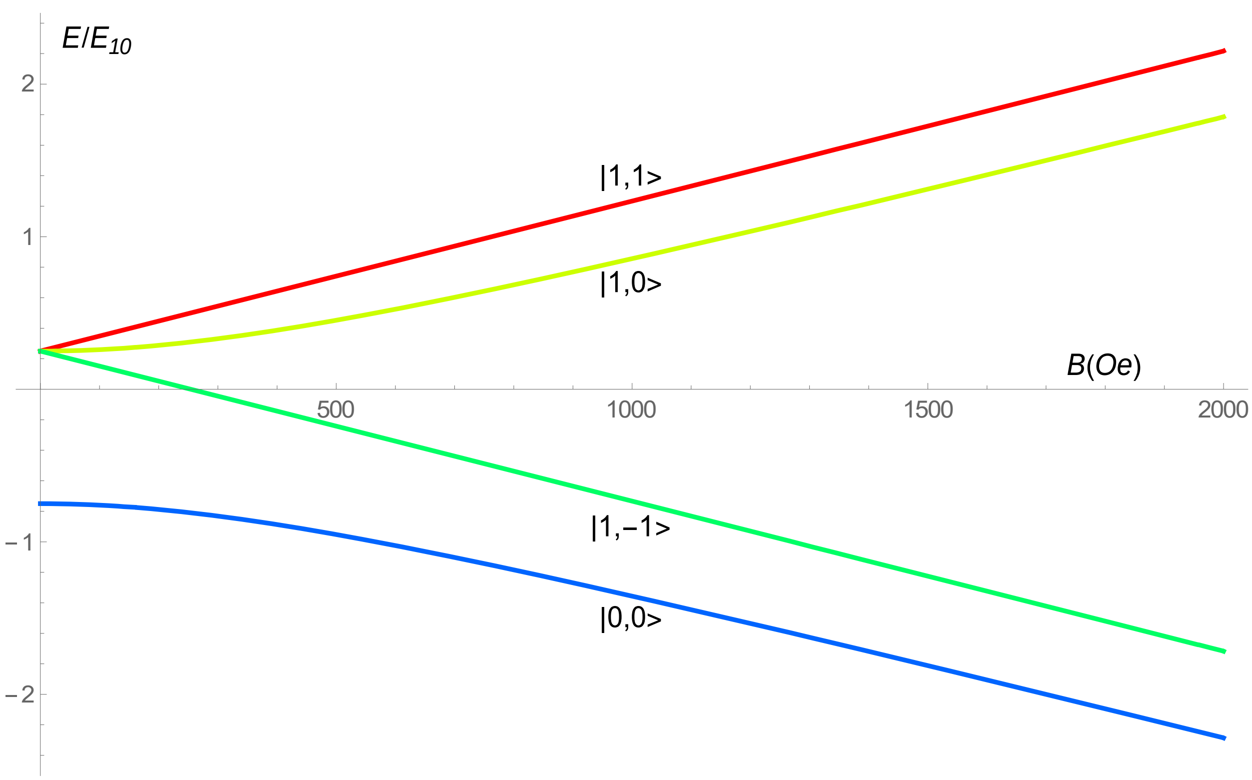
$$J = \frac{1}{4}E_{10} = 1.469365\ \mu\text{eV}$$

$$\mu_B = 5.78838 \text{ x } 10^{-3}\ \mu\,\text{eV/G}$$

$$\mu_N = 3.15245 \text{ x } 10^{-6}\ \mu\,\text{eV/G}$$

$$g_p = 5.58569$$





4. Canonical ensemble for the one particle

We consider the system with one particle. The partition function is given by

$$Z_1 = g_0 + g_1 e^{-\beta\varepsilon_0}$$

where g_0 (=1) and g_1 (=3) are the degeneracies of the singlet and triplet, respectively. The probability of finding the particle at $\varepsilon = 0$ is

$$P(\varepsilon = 0) = \frac{1}{Z_1} = \frac{g_0}{g_0 + g_1 e^{-\beta\varepsilon_0}} .$$

The probability of finding the particle at $\varepsilon = 0$ is

$$P(\varepsilon = \varepsilon_0) = \frac{g_1 e^{-\beta\varepsilon_0}}{g_0 + g_1 e^{-\beta\varepsilon_0}} .$$

$$\frac{N_1}{N_0} = \frac{P(\varepsilon = \hbar\nu_{10})}{P(\varepsilon = 0)} = \frac{g_1}{g_0} e^{-h\beta\nu_{10}} = 3e^{-h\beta\nu_{10}}$$

((Wien's displacement law))

The equilibrium temperature of cool interstellar HI is determined by the balance of heating and cooling. The primary heat sources are cosmic rays and ionizing photons from hot stars. The main coolant in the cool interstellar medium (ISM) is radiation from the fine-structure line of singly ionized carbon, CII, at $\lambda_{\max}$ 157.7 μm. This line is strongly only when the temperature is evaluated from the Wien's displacement law

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ (m K)},$$

So the temperature of the interstellar is

$$T = \frac{2.898 \times 10^{-3}}{\lambda_{\max}(\mu\text{m})} \times 10^6 = \frac{2.898 \times 10^{-3}}{157.7} \times 10^6 = 18.4 \text{ K}$$

Suppose that the spin temperature is T_s = 150 K,

$$\frac{h\nu_{10}}{k_B T_s} = 4.547 \text{ x} 10^{-4}.$$

Thus we have

$$\frac{N_1}{N_0} = \frac{g_1}{g_0} e^{-h\beta\nu_{10}} \approx 3$$

Why the magnetic dipole radiation is much important than the dipole radiation?
What is the reason?

5. M1 transition due to the magnetic moment: 21 cm H1

Here we introduce a full magnetic dipole operator which is defined as

$$-\frac{\mu_B}{\hbar}(\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}}) = -\frac{\mu_B}{\hbar}(\hat{\boldsymbol{J}} + \hat{\boldsymbol{S}})$$

The hyper transition in an atomic hydrogen from F = 1 to F = 0 state at 1420 MHz transition is an M1 transition. The M1 transition involves a change in a spin (a) from the state $\left|F = 1, m_f = 1,\right\rangle$ to $\left|F = 0, m_f = 0\right\rangle$, and (b) from the state $\left|F = 1, m_f = 0,\right\rangle$ to $\left|F =, m_f = 0\right\rangle$.

$$\hat{O}_{M1} = \frac{i}{2}(\boldsymbol{k} \times \boldsymbol{\varepsilon}) \cdot (\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}})$$

The matrix element is given by

$$\begin{aligned}\langle 1, m | \hat{O}_{M1} | 0,0 \rangle &= \frac{i}{2}(\boldsymbol{k} \times \boldsymbol{\varepsilon}) \cdot \langle 1, m | (\hat{\boldsymbol{L}} + 2\hat{\boldsymbol{S}}) | 0,0 \rangle \\ &\approx i(\boldsymbol{k} \times \boldsymbol{\varepsilon}) \cdot \langle 1, m | \hat{\boldsymbol{S}} | 0,0 \rangle\end{aligned}$$

Note that the magnetic moment of the proton is much smaller than that of the magnetic moment. Then $\hat{\boldsymbol{S}}$ is actually equal to the spin of electrons. Using the kronecker product, the spin operator of electron are given by

$$\hat{S}_{xe} = \frac{\hbar}{2}(\hat{\sigma}_x \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{S}_{ye} = \frac{\hbar}{2}(\hat{\sigma}_y \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\hat{S}_{ze} = \frac{\hbar}{2}(\hat{\sigma}_z \otimes \hat{1}_2) = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The matrix elements can be evaluated as

$$\langle 1,1|\hat{S}_{ze}|0,0\rangle = 0, \qquad \langle 1,0|\hat{S}_{ze}|0,0\rangle = \frac{\hbar}{2}, \qquad \langle 1,-1|\hat{S}_{ze}|0,0\rangle = 0$$

$$\langle 1,1|\hat{S}_{xe}|0,0\rangle = -\frac{\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{xe}|0,0\rangle = 0, \qquad \langle 1,-1|\hat{S}_{xe}|0,0\rangle = \frac{\hbar}{2\sqrt{2}}$$

$$\langle 1,1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{ye}|0,0\rangle = 0 \qquad \langle 1,-1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}$$

Then we have the transitions

S_{ze} component:	$\|1,0\rangle \to \|0,0\rangle$	allowed	(photon: linear polarization)
S_{xe}, S_{ye} component:	$\|1,1\rangle \to \|0,0\rangle$	allowed	(photon $\|R\rangle$ polarization wit $\hbar$)
	$\|1,-1\rangle \to \|0,0\rangle$	allowed	(photon $\|L\rangle$ polarization wit $-\hbar$)

((**Note**))
(a)

$$\langle 1,1|\hat{S}_{ze}|0,0\rangle = 0, \qquad \langle 1,0|\hat{S}_{ze}|0,0\rangle = \frac{\hbar}{2}, \qquad \langle 1,-1|\hat{S}_{ze}|0,0\rangle = 0$$

((**Proof**))

$$\hat{S}_{ze}|1,1\rangle = \frac{\hbar}{2}\hat{\sigma}_{ze}|+_e,+_p\rangle = \frac{\hbar}{2}|+_e,+_p\rangle = \frac{\hbar}{2}|1,1\rangle,$$

$$\begin{aligned}\hat{S}_{ze}|1,0\rangle &= \frac{\hbar}{2}\hat{\sigma}_{ze}\frac{|+_e,-_p\rangle + |-_e,+_p\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{|+_e,-_p\rangle - |-_e,+_p\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}|0,0\rangle\end{aligned}$$

$$\hat{S}_{ze}|1,-1\rangle = \frac{\hbar}{2}\hat{\sigma}_{ze}|-_e,-_p\rangle = -\frac{\hbar}{2}|-_e,-_p\rangle = -\frac{\hbar}{2}|1,-1\rangle$$

(b)

$$\langle 1,1|\hat{S}_{xe}|0,0\rangle = -\frac{\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{xe}|0,0\rangle = 0, \qquad \langle 1,-1|\hat{S}_{xe}|0,0\rangle = \frac{\hbar}{2\sqrt{2}}$$

((**Proof**))

$$\hat{S}_{xe}|1,1\rangle = \frac{\hbar}{2}\hat{\sigma}_{xe}|+_e,+_p\rangle = \frac{\hbar}{2}|-_e,+_p\rangle = \frac{\hbar}{2}\frac{|1,0\rangle - |0,0\rangle}{\sqrt{2}},$$

$$\begin{aligned}\hat{S}_{xe}|1,0\rangle &= \frac{\hbar}{2}\hat{\sigma}_{xe}\frac{|+_e,-_p\rangle + |-_e,+_p\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{|-_e,-_p\rangle + |+_e,+_p\rangle}{\sqrt{2}} \\ &= \frac{\hbar}{2}\frac{|1,1\rangle + |1,-1\rangle}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\hat{S}_{xe}|1,-1\rangle &= \frac{\hbar}{2}\hat{\sigma}_{xe}|-_e,-_p\rangle \\ &= \frac{\hbar}{2}|+_e,-_p\rangle \\ &= \frac{\hbar}{2}\frac{|1,0\rangle + |0,0\rangle}{\sqrt{2}}\end{aligned}$$

(c)

$$\langle 1,1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}, \qquad \langle 1,0|\hat{S}_{ye}|0,0\rangle = 0 \qquad \langle 1,-1|\hat{S}_{ye}|0,0\rangle = \frac{i\hbar}{2\sqrt{2}}$$

((**Proof**))

$$\hat{S}_{ye}|1,1\rangle = \frac{\hbar}{2}\hat{\sigma}_{ye}|+_e,+_p\rangle = i\frac{\hbar}{2}|-_e,+_p\rangle = i\frac{\hbar}{2}\frac{|1,0\rangle - |0,0\rangle}{\sqrt{2}},$$

$$\hat{S}_{ye}\left|1,0\right\rangle = \frac{\hbar}{2}\hat{\sigma}_{ye}\frac{\left|+_e,-_p\right\rangle + \left|-_e,+_p\right\rangle}{\sqrt{2}}$$

$$= \frac{\hbar}{2}\frac{i\left|-_e,-_p\right\rangle - i\left|+_e,+_p\right\rangle}{\sqrt{2}}$$

$$= -i\frac{\hbar}{2}\frac{\left|1,1\right\rangle - \left|1,-1\right\rangle}{\sqrt{2}}$$

$$\hat{S}_{ye}\left|1,-1\right\rangle = \frac{\hbar}{2}\hat{\sigma}_{ye}\left|-_e,-_p\right\rangle$$

$$= -i\frac{\hbar}{2}\left|+_e,-_p\right\rangle$$

$$= -i\frac{\hbar}{2}\frac{\left|1,0\right\rangle + \left|0,0\right\rangle}{\sqrt{2}}$$

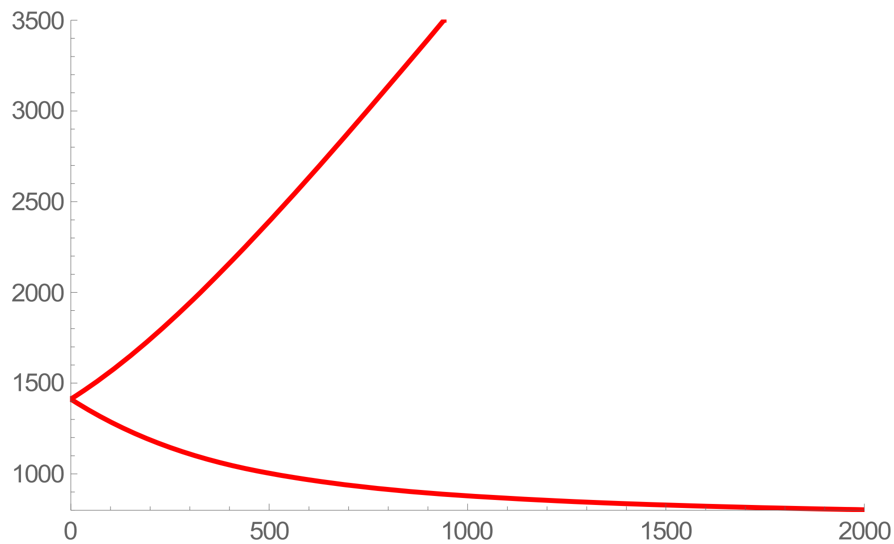
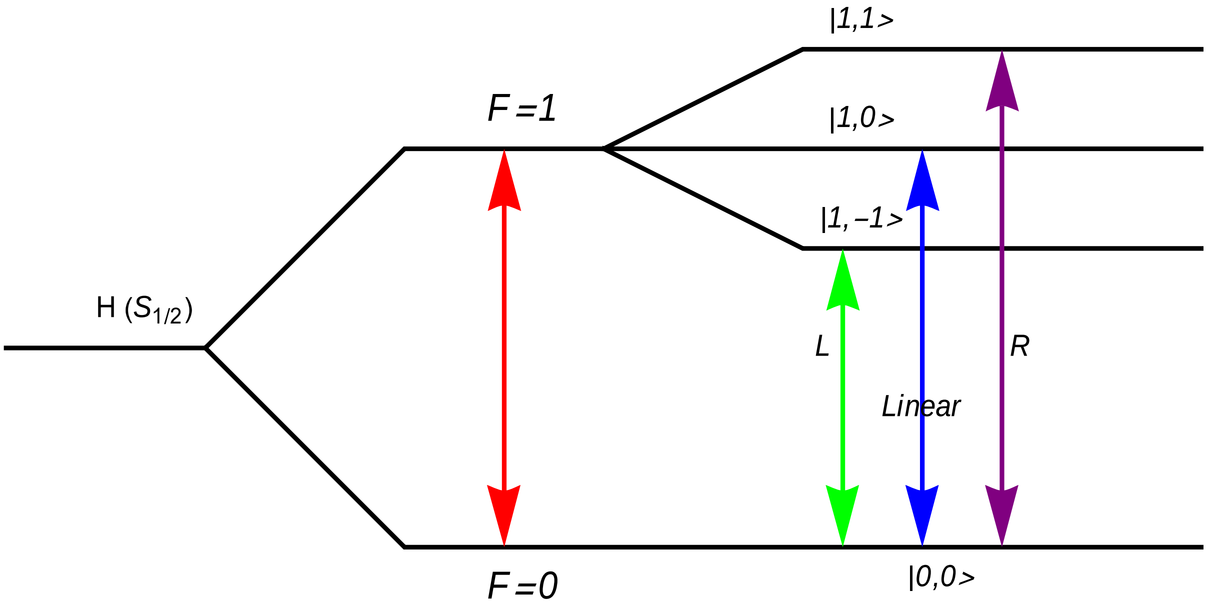


Fig. Plot of the frequencies for the transitions $\left|1,1\right\rangle \rightarrow \left|0,0\right\rangle$ and $\left|1,-1\right\rangle \rightarrow \left|0,0\right\rangle$. The *x*-axis: the magnetic field *B*(Oe). The *y* axis is the frequency (in units of MHz).



The enrgy level of the ground state 1 $S_{1/2}$ state in hydrogen (H) atom. The levels are depicted wuth the account for the proton-electron spin system (the total momentum F) and the Zeeman splitting. The energy levels resulting from the Zeeman effect are denoted by the states $\left|F=1, m_F=1,0,-1\right\rangle$ and $\left|F=0, m_F=0\right\rangle$. The photons with right hand circular polarization are generated during the transition from $\left|1,1\right\rangle$ to $\left|0,0\right\rangle$, with the angular momentum $\hbar$. The photons with left hand circular polarization are generated during the transition from $\left|1,-1\right\rangle$ to $\left|0,0\right\rangle$, with the angular momentum $-\hbar$. The photons with the ;linear polarization generated during the transition from $\left|1,0\right\rangle$ to $\left|0,0\right\rangle$, with the angular momentum 0.

6. Relaxation time

The spontaneous emission for the magnetic dipole is given by

$$A_{mag} = A_{10} = \frac{4\omega_0^{\ 3}}{3\hbar c^3}\left|\left\langle f\left|\hat{\boldsymbol{\mu}}\right|i\right\rangle\right|^2 = \frac{32\pi^3}{3\hbar c^3}\nu_{10}^{\ 3}\mu_{10}^{\ 2}$$

where

$$\mu_{10} = \left\langle f\left|\hat{\boldsymbol{\mu}}\right|i\right\rangle$$

$$A_{10} = \frac{32\pi^3}{3\hbar c^3}\nu_{10}^{\ 3}\mu_{10}^{\ 2} = 2.87349 \text{ x } 10^{-15}\ \text{s}^{-1}.$$

$$\tau_{10} = \frac{1}{A_{10}} = 3.48009 \text{ x } 10^{14} \text{ s} = 11.0278 \text{ million year.}$$

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