Landau level of conduction electron in the presence of magnetic field

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(Date: May 02, 2013)

1. Landau gauge

$$\boldsymbol{\pi} = m\boldsymbol{v} = \boldsymbol{p} - \frac{q}{c}\boldsymbol{A} = \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}$$

Hamiltonian H [$q = -e$ ($e>0$)]

$$H = \frac{1}{2m}(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})^2 + q\phi = \frac{1}{2m}(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A})^2 - e\phi$$

In the presence of the magnetic field $\boldsymbol{B}$ (constant), we can choose the vector potential as

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B}\times\boldsymbol{r}) = \frac{1}{2}\begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 0 & 0 & B \\ x & y & z \end{vmatrix} = \frac{1}{2}(-By, Bx, 0)$$

(symmetric gauge)

Gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi$$

with

We choose $\chi = \frac{1}{2}Bxy$

$$\nabla\chi = \frac{1}{2}B(y, x, 0)$$

Therefore the new vector potential A' is obtained as

$$\boldsymbol{A}' = (0, Bx, 0) \qquad \text{(Landau gauge)}$$

Gauge transformation: $q = -e$ ($e>0$)

$$\psi'(\boldsymbol{r}) = \exp(\frac{iq\chi}{c\hbar})\psi(\boldsymbol{r}) = \exp(\frac{-ieB}{2c\hbar}xy)\psi(r)$$

2. Qunatum mechanics

$$\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}$$

$$[\hat{\pi}_x, \hat{\pi}_y] = [\hat{p}_x + \frac{e}{c}A_x, \hat{p}_y + \frac{e}{c}A_y] = \frac{e}{c}[\hat{p}_x, A_y] - \frac{e}{c}[\hat{p}_y, A_x] = \frac{e\hbar}{ic}\frac{\partial A_y}{\partial \hat{x}} - \frac{e\hbar}{ic}\frac{\partial A_x}{\partial \hat{y}} = \frac{e\hbar}{ic}B_z$$

or

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar}{ic}B_z$$

where

$$\frac{\partial A_y}{\partial \hat{x}} - \frac{\partial A_x}{\partial \hat{y}} = B_z .$$

Similarly we have

$$[\hat{\pi}_y, \hat{\pi}_z] = \frac{e\hbar}{ic}B_x , \qquad \text{and} \qquad [\hat{\pi}_z, \hat{\pi}_x] = \frac{e\hbar}{ic}B_y$$

Since $\boldsymbol{A}$ commute with $\hat{\mathbf{r}}$,

$$[\hat{x}, \hat{\pi}_x] = [\hat{x}, \hat{p}_x] = i\hbar , \qquad [\hat{y}, \hat{\pi}_y] = [\hat{y}, \hat{p}_y] = i\hbar , \qquad [\hat{z}, \hat{\pi}_z] = [\hat{z}, \hat{p}_z] = i\hbar .$$

When $\boldsymbol{B} = (0,0,B)$ or $B_z = B$,

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar B}{ic} , \qquad [\hat{\pi}_y, \hat{\pi}_z] = 0 , \qquad [\hat{\pi}_z, \hat{\pi}_x] = 0$$

Note that

$$[\hat{\pi}_x, \hat{\pi}_y] = \frac{e\hbar^2 B}{ic\hbar} = -i\frac{\hbar^2}{\ell^2}$$

where

$$\ell^2 = \frac{c\hbar}{eB}$$

The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{2m}(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A})^2 = \frac{1}{2m}(\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2)$$

We define the creation and annihilation operators,

$$\hat{a} = \frac{\ell}{\sqrt{2}\hbar}(\hat{\pi}_x - i\hat{\pi}_y), \qquad \hat{a}^+ = \frac{\ell}{\sqrt{2}\hbar}(\hat{\pi}_x + i\hat{\pi}_y)$$

or

$$\hat{\pi}_x = \frac{\hbar}{\sqrt{2}\ell}(\hat{a} + \hat{a}^+), \qquad \hat{\pi}_y = \frac{\hbar}{i\sqrt{2}\ell}(\hat{a}^+ - \hat{a})$$

$$[\hat{a}, \hat{a}^+] = \frac{\ell^2}{2\hbar^2}[\hat{\pi}_x - i\hat{\pi}_y, \hat{\pi}_x + i\hat{\pi}_y] = \frac{\ell^2}{\hbar^2}i[\hat{\pi}_x, \hat{\pi}_x] = \frac{\ell^2}{\hbar^2}i(-i\frac{\hbar^2}{\ell^2}) = 1$$

$$\hat{\pi}_x{}^2 + \hat{\pi}_y{}^2 = \frac{\hbar^2}{2\ell^2}[(\hat{a} + \hat{a}^+)^2 - (\hat{a}^+ - \hat{a})^2] = \frac{\hbar^2}{\ell^2}(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}) = \frac{\hbar^2}{\ell^2}(2\hat{a}^+\hat{a} + 1)$$

Thus we have

$$\hat{H} = \frac{\hbar^2}{m\ell^2}(\hat{a}^+\hat{a} + \frac{1}{2}) = \hbar\omega_c(\hat{a}^+\hat{a} + \frac{1}{2})$$

where

$$\hbar\omega_c = \frac{\hbar^2}{m\ell^2} = \frac{\hbar^2}{m\frac{c\hbar}{eB}} = \frac{\hbar eB}{mc}$$

When $\hat{a}^+\hat{a} = \hat{N}$, the Hamiltonian is described by

$$\hat{H} = \hbar\omega_c(\hat{N} + \frac{1}{2})$$

We have thus find the energy levels for the free electrons in a homogeneous magnetic field- also known as Landau levels.

3. Schrödinger equation

In the absence of an electric field

$$\hat{H} = \frac{1}{2m}[\hat{p}_x{}^2 + (\hat{p}_y + \frac{e}{c}B\hat{x})^2 + \hat{p}_z{}^2]$$

This Hamiltonian $\hat{H}$ commutes with $\hat{p}_y$ and $\hat{p}_z$.

$[\hat{H}, \hat{p}_y] = 0$ and $[\hat{H}, \hat{p}_z] = 0$

$$\hat{H}\left|n, k_y, k_z\right\rangle = E_n\left|n, k_y, k_z\right\rangle$$

and

$$\hat{p}_y\left|n, k_y, k_z\right\rangle = \hbar k_y\left|n, k_y, k_z\right\rangle,$$

and

$$\hat{p}_z\left|n, k_y, k_z\right\rangle = \hbar k_z\left|n, k_y, k_z\right\rangle$$

$$\left\langle y\right|\hat{p}_y\left|n, k_y, k_z\right\rangle = \hbar k_y\left\langle y\middle|n, k_y, k_z\right\rangle,$$

$$\left\langle z\right|\hat{p}_y\left|n, k_y, k_z\right\rangle = \hbar k_y\left\langle y\middle|n, k_y, k_z\right\rangle$$

or

$$\frac{\hbar}{i}\frac{\partial}{\partial y}\left\langle y\middle|n, k_y, k_z\right\rangle = \hbar k_y\left\langle y\middle|n, k_y, k_z\right\rangle,$$

$$\frac{\hbar}{i}\frac{\partial}{\partial z}\left\langle z\middle|n, k_y, k_z\right\rangle = \hbar k_z\left\langle z\middle|n, k_y, k_z\right\rangle$$

Schrödinger equation

$$\frac{1}{2m}[(\frac{\hbar}{i}\frac{\partial}{\partial x})^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y} + \frac{e}{c}Bx)^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y})^2]\psi(x, y, z) = \varepsilon\psi(x, y, z)$$

$$\psi(x, y, z) = e^{ik_y y + ik_z z}\phi(x)$$

$$x = \frac{\xi}{\beta},$$

with $\beta = \sqrt{\frac{m\omega_c}{\hbar}} = \sqrt{\frac{eB}{\hbar c}} = \frac{1}{\ell}$ and $\omega_c = \frac{eB}{mc}$

$$\xi_0 = \beta \frac{c\hbar k_y}{eB} = \sqrt{\frac{c\hbar}{eB}} k_y = \ell k_y$$

We assume the periodic boundary condition along the *y* axis.

$$\psi(x, y + L_y, z) = \psi(x, y, z)$$

or

$$e^{ik_y L_y} = 1$$

or

$$k_y = \frac{2\pi}{L_y} n_y \qquad (n_y\text{: intergers})$$

Then we have

$$\phi''(\xi) = [(\xi - \xi_0)^2 + \frac{c}{e\hbar B}(-2mE_1 + \hbar^2 k_z^{\ 2})]\phi(\xi)$$

We put

$$E_1 = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hbar^2 k_z^{\ 2}}{2m}$$

(Landau level)

or

$$2mE_1 = \hbar^2 k_z^{\ 2} + 2m\hbar\omega_c(n + \frac{1}{2}) = \hbar^2 k_z^{\ 2} + \frac{2eB\hbar}{c}(n + \frac{1}{2})$$

$$\phi''(\xi) = [(\xi - \xi_0)^2 - (2n + 1)]\phi(\xi)$$

Finally we get a differential equation for $\phi(\xi)$.

$$\phi''(\xi) + [2n + 1 - (\xi - \xi_0)^2]\phi(\xi)$$

The solution of this differential equation is

$$\phi_n(\xi) = (\sqrt{\pi}\, 2^n n!)^{-1/2} e^{-\frac{(\xi-\xi_0)^2}{2}} H_n(\xi - \xi_0)$$

with

$$\xi_0 = \sqrt{\frac{c\hbar}{eB}} k_y = \ell k_y$$

$$\ell = \sqrt{\frac{c\hbar}{eB}}$$

$$x_0 = \frac{\xi_0}{\beta} = \ell \xi_0 = \ell^2 k_y$$

The coordinate x_0 is the center of orbits. Suppose that the size of the sysyem along the x axis is L_x. The coordinate x_0 should satisfy the condition, $0<x_0<L_x$. Since the eneygy of the system is independent of x_0, this state is degenerate.

$$0 < x_0 = \frac{\xi_0}{\beta} = \ell \xi_0 = \ell^2 k_y < L_x$$

or

$$\ell^2 k_y = \frac{2\pi}{L_y} \ell^2 n_y < L_x$$

or

$$n_y < \frac{L_x L_y}{2\pi \ell^2}$$

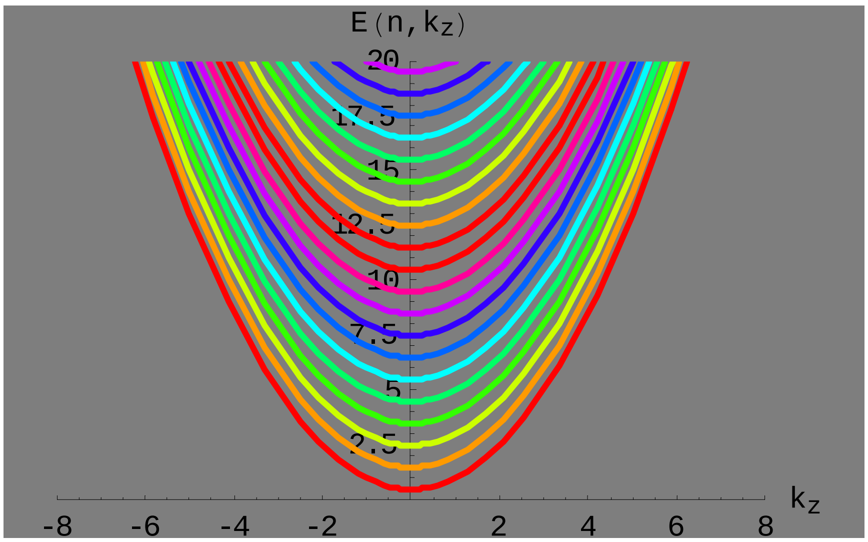
Thus the degeneracy is given by the number of allowed ky values for the system.

$$g = \frac{L_x L_y}{2\pi \ell^2} = \frac{A}{2\pi \ell^2} = \frac{A}{2\pi \frac{c\hbar}{eB}} = \frac{BA}{\Phi_0} = \frac{\Phi}{\Phi_0}$$

where

$$\Phi_0 = \frac{2\pi\hbar c}{e} = 4.13563\times 10^{-7}\ \text{Gauss cm}^2$$

The value of g is the total magnetic flux. There is one state per a quantum magnetic flux Φ_0.



((**Another method**))

$$\hat{H} = \frac{1}{2m}(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A})^2 = \frac{1}{2m}[\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e}{c}(\hat{\boldsymbol{p}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\hat{\boldsymbol{p}})]$$

$$\begin{aligned}\hat{\boldsymbol{p}}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\hat{\boldsymbol{p}} &= \hat{p}_x A_x + \hat{p}_y A_y + \hat{p}_z A_z + A_x\hat{p}_x + A_y\hat{p}_y + A_z\hat{p}_z \\ &= [\hat{p}_x, A_x] + [\hat{p}_y, A_y] + [\hat{p}_z, A_z] + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}} \\ &= \frac{\hbar}{i}\nabla\cdot\boldsymbol{A} + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}}\end{aligned}$$

Then we have

$$\begin{aligned}\hat{H} &= \frac{1}{2m}[\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e}{c}(\frac{\hbar}{i}\nabla\cdot\boldsymbol{A} + 2\boldsymbol{A}\cdot\hat{\boldsymbol{p}})] \\ &= \frac{1}{2m}(\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{e\hbar}{ic}\nabla\cdot\boldsymbol{A} + \frac{2e}{c}\boldsymbol{A}\cdot\hat{\boldsymbol{p}})\end{aligned}$$

Since $\nabla\cdot\boldsymbol{A} = 0$,

$$\hat{H} = \frac{1}{2m}(\hat{\boldsymbol{p}}^2 + \frac{e^2}{c^2}\boldsymbol{A}^2 + \frac{2e}{c}\boldsymbol{A}\cdot\hat{\boldsymbol{p}}) = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{e^2B^2}{2mc^2}\hat{x}^2 + \frac{eB}{mc}\hat{x}\hat{p}_y$$

where

$$\ell^2 = \frac{c\hbar}{eB}, \qquad \hbar\omega_c = \frac{\hbar^2}{m\ell^2} = \frac{\hbar eB}{mc}, \qquad m\omega_c{}^2 = \frac{e^2B^2}{mc^2}$$

$$\hat{H} = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{e^2B^2}{2mc^2}\hat{x}^2 + \omega_c\hat{x}\hat{p}_y = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + \frac{m\omega_c{}^2}{2}\hat{x}^2 + \omega_c\hat{x}\hat{p}_y$$

The first and second terms of this Hamiltonian are that of the simple harmonics along the *x* axis. Thus the wave function is described by the form,

$$\psi(x,y,z) = \phi_n(x)e^{ik_y y + ik_z z}$$