Neutron Interferometry Experiment
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It is known that the operator for rotation through 2π radians for a fermion causes a reversal of the sign of the wave function. Only after two full rotations through 4π does the system come back to its original value. In the early days of quantum mechanics, people reassured themselves that in the experimentally accessible probabilities $\langle\psi|\psi\rangle$ the phase factor cancels, hence there seemed to be no reason to worry about this strange property. Aharonov and Suskind (1967) and independently, Berstein (1967) predicted the possibility of observation of the 4π symmetry for interferometric experiments, and Eder and Zeilinger (1967) gave the theoretical framework for the neutron interferometric realization.

It can be confirmed from the experiment using the precession of 2π radians in a magnetic field using a neutron interferometer (Werner et al. and Rauch et al. (1975)]. Here we discuss the detail of the experiment, a 2π rotation of spin 1/2 system, based on the textbook of Sakurai and Napolitano, Modern Quantum Mechanics (2011, revised version).

1. Spin and magnetic moment of neutron

The neutron spin and the magnetic moment. The neutron is a fermion with spin 1/2. The spin operators are given by

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x, \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y, \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z .$$

The neutron wave function can be expressed by

$$|\psi\rangle = c_+|+z\rangle + c_-|-z\rangle .$$

The neutron has a magnetic moment

$$\hat{\boldsymbol{\mu}}_n = \gamma_n \mu_N \hat{\boldsymbol{\sigma}} = \gamma_n \mu_N \frac{2}{\hbar}\hat{\boldsymbol{S}} ,$$

where

$$\gamma_n = -1.91304272(45).$$

and μ_N is the nuclear magneton.

$$\mu_N = \frac{e\hbar}{2m_p c} = 5.0507832413 \times 10^{-24} \text{ emu},$$

where m_p is the rest mass of proton. So the magnitude of the magnetic moment is

$$|\mu_n| = |\gamma_n|\mu_N = 1.91304272\mu_N = 9.662347055(71)\times 10^{-24} \text{ emu.}$$

or

$$\mu_n = -1.91304272\mu_N$$

where emu = erg/Oe. Here we use the expression for the nuclear magnetic moment as

$$\hat{\boldsymbol{\mu}}_n = -|\mu_n|\frac{2}{\hbar}\hat{\boldsymbol{S}}.$$

((**Note**))

The non-zero magnetic moment of the neutron indicates that it is not an elementary particle. The sign of the neutron's magnetic moment is that of a negatively charged particle. Similarly, the fact that the magnetic moment of the proton, $\mu_p = 2.792847356(23)$ μ_N, is not equal to 1 μ_N indicates that it too is not an elementary particle. Protons and neutrons are composed of quarks, and the magnetic moments of the quarks can be used to compute the magnetic moments of the nucleons.
https://en.wikipedia.org/wiki/Neutron_magnetic_moment

2. Rotation operator

We now consider the rotation operator around the u axis by α for the spin 1/2 system (neutron), which is given by

$$\hat{R}_u(\alpha) = e^{-i\frac{\alpha(\hat{\boldsymbol{\sigma}}\cdot\mathbf{u})}{2}} = \hat{1}\cos\frac{\alpha}{2} - i(\hat{\boldsymbol{\sigma}}\cdot\mathbf{u})\sin\frac{\alpha}{2}.$$

When $\alpha = 2\pi$,

$$\hat{R}_u(2\pi) = e^{-i\pi(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u})} = \hat{1}\cos\pi - i(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{u})\sin\pi = -\hat{1}.$$

For any ket $|\psi\rangle$, we have

$$\hat{R}_u(2\pi)|\psi\rangle = -|\psi\rangle.$$

The ket for the 2π rotated state differs from the original ket by a minus sign. When $|\psi\rangle = |\pm z\rangle$, we have

$$\hat{R}_u(2\pi)\left|+z\right\rangle = -\left|+z\right\rangle, \text{ and } \quad \hat{R}_u(2\pi)\left|-z\right\rangle = -\left|-z\right\rangle.$$

We also note that

$$\hat{R}_u(4\pi)\left|+z\right\rangle = \left|+z\right\rangle, \quad \text{and} \quad \hat{R}_u(4\pi)\left|-z\right\rangle = \left|-z\right\rangle.$$

3 Neutron interferometry experiment

A beam of thermal neutrons is split into two parts, path A-B-D and path A-C-D. That path A-B-D goes through a magnetic-field-free region. In contrast, the path A-C-D enters a small region where a static magnetic field is present. A variable phase shift is obtained by having the neutrons of the path A-C-D pass through a uniform magnetic field over a distance l.

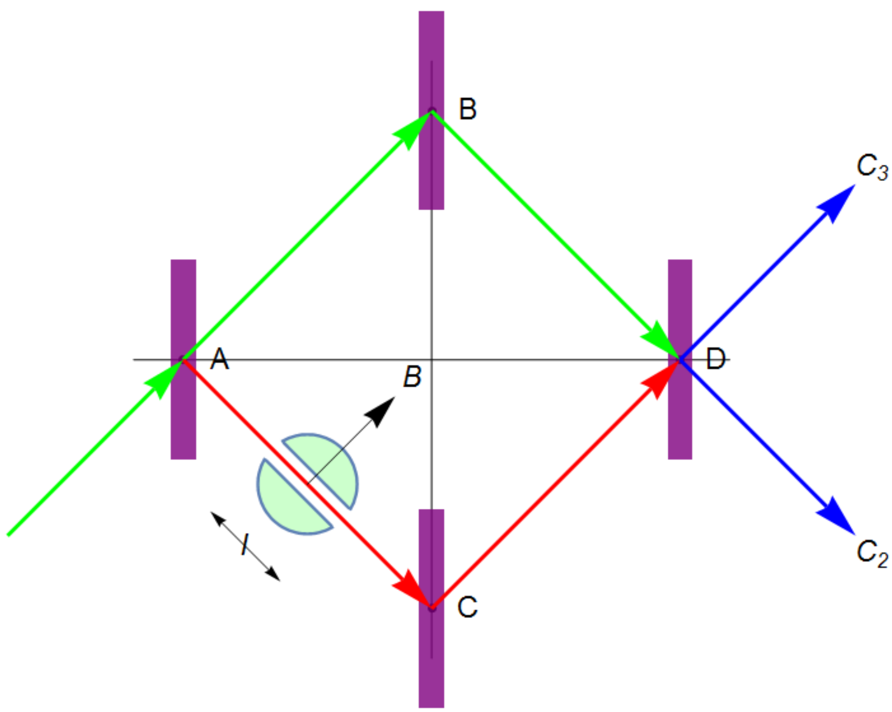

Fig. A schematic diagram of the neutron interferometer. On the path AC the neutrons are in a magnetic field B (0 to 500 Oe) for a distance l (= 2 cm).

The spin Hamiltonian $\hat{H}$ of the neutron in the presence of a magnetic field $\boldsymbol{B}$ (//z)

$$\hat{H} = -\hat{\boldsymbol{\mu}}_n \cdot \boldsymbol{B} = -(-\frac{2\left|\mu_n\right|}{\hbar}\hat{\boldsymbol{S}}) \cdot \boldsymbol{B} = \frac{2\left|\mu_n\right|B}{\hbar}\hat{S}_z = \frac{2\left|\mu_n\right|B}{\hbar}\frac{\hbar}{2}\hat{\sigma}_z = \frac{\hbar\omega}{2}\hat{\sigma}_z.$$

The Larmor angular frequency ω is defined as

$$\omega = \frac{2|\mu_n|B}{\hbar} = \frac{2|\gamma_n|\mu_N B}{\hbar} = \frac{e|\gamma_n|B}{m_p c}.$$

or

ω = 1.83247 x 10^4 B(Oe), [rad/s]

or

$$f = \frac{\omega}{2\pi} = 2.914647 \text{ x } 10^3\ B(\text{Oe}). \qquad [\text{Hz}]$$

We note that

$\hbar\omega$ = 1.20616 x 10^{-3} B(Oe) [neV].

((**Example**))

When B = 165 Oe, we have

ω = 3.02358 x 10^6 rad/s, f = 0.48122 MHz,

and

$$\varepsilon = \frac{1}{2}\hbar\omega = 0.995 \text{ neV}.$$

We now consider the time dependence of the ket vector $|\psi(t)\rangle$,

$$|\psi(t)\rangle = \hat{U}(t)|\psi(t=0)\rangle,$$

where the time evolution operator $\hat{U}(t)$ is given by

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{2}\hat{\sigma}_z \omega t}.$$

Let the angle ϕ be defined as

$$\phi = \omega T = \frac{e|\gamma_n|B}{m_p c}T = \frac{e|\gamma_n|B}{m_p c}\frac{\ell m_n \lambda\!\!\!^{-}}{\hbar} = \frac{e|\gamma_n|Bl\lambda}{2\pi c\hbar},$$

where the time T during which the neutron passes the magnetic-field region, is given by

$$T = \frac{l}{v} = \frac{l}{\dfrac{\hbar}{m_n \lambda\!\!\!^{-}}} = \frac{l m_n \lambda\!\!\!^{-}}{\hbar}.$$

Note that the momentum of the neutron is given from the de Broglie relation,

$$p = m_n v = \frac{h}{\lambda} = \frac{\hbar}{\lambda\!\!\!^{-}}, \qquad \text{and} \qquad \lambda\!\!\!^{-} = \frac{\lambda}{2\pi}.$$

When $t = T$, the unitary operator $\hat{U}(t)$ is equivalent to the rotation operator $\hat{R}_z(\phi)$,

$$\hat{U}(T) = \hat{R}_z(\phi) = e^{-i\frac{\phi\hat{\sigma}_z}{2}} = \hat{1}\cos\frac{\phi}{2} - i\hat{\sigma}_z \sin\frac{\phi}{2}.$$

since

$$\begin{aligned}
e^{-i\frac{\phi\hat{\sigma}_z}{2}} &= e^{-i\frac{\phi\hat{\sigma}_z}{2}}(\left|+z\right\rangle\left\langle +z\right| + \left|-z\right\rangle\left\langle -z\right|) \\
&= e^{-i\frac{\phi}{2}}\left|+z\right\rangle\left\langle +z\right| + e^{i\frac{\phi}{2}}\left|-z\right\rangle\left\langle -z\right| \\
&= \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \\
&= \cos\frac{\phi}{2}\hat{1} - i\sin\frac{\phi}{2}\hat{\sigma}_z
\end{aligned}$$

Then we have

$$\hat{R}_z(\phi)\left|+z\right\rangle = e^{-i\frac{\phi\hat{\sigma}_z}{2}}\left|+z\right\rangle = \cos\frac{\phi}{2}\left|+z\right\rangle - i\sin\frac{\phi}{2}\hat{\sigma}_z\left|+z\right\rangle = e^{-i\frac{\phi}{2}}\left|+z\right\rangle,$$

and

$$\hat{R}_z(\phi)\left|-z\right\rangle = e^{-i\frac{\phi\hat{\sigma}_z}{2}}\left|-z\right\rangle = \cos\frac{\phi}{2}\left|-z\right\rangle - i\sin\frac{\phi}{2}\hat{\sigma}_z\left|-z\right\rangle = e^{i\frac{\phi}{2}}\left|+\right\rangle.$$

For any ket given by

$$\left|\psi\right\rangle = C_1\left|+z\right\rangle + C_2\left|-z\right\rangle,$$

we have

$$. \hat{R}_z(\phi)|\psi\rangle = e^{-i\frac{\phi\sigma_z}{2}}|\psi\rangle = C_1 e^{-i\frac{\phi}{2}}|+z\rangle + C_2 e^{i\frac{\phi}{2}}|-z\rangle .$$

The appearance of the half-angle $\phi/2$ has an extremely interesting consequence.

The state $|\chi\rangle$ in the region D is given by

$$|\chi\rangle = |\psi\rangle + \hat{R}_z(\phi)|\psi\rangle .$$

The intensity is proportional to $\langle\chi|\chi\rangle$,

$$\langle\chi|\chi\rangle = (\langle\psi| + \langle\psi|\hat{R}_z^{+}(\phi))(|\psi\rangle + \hat{R}_z(\phi)|\psi\rangle) = 2 + \langle\psi|\hat{R}_z^{+}(\phi) + \hat{R}_z(\phi)|\psi\rangle ,$$

where $\langle\psi|\psi\rangle = 1$.

$$\hat{R}_z(\phi) = \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}, \qquad \hat{R}_z^{+}(\phi) = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix},$$

$$\hat{R}_z(\phi) + \hat{R}_z^{+}(\phi) = \begin{pmatrix} 2\cos\frac{\phi}{2} & 0 \\ 0 & 2\cos\frac{\phi}{2} \end{pmatrix} = 2\cos\frac{\phi}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$\langle\chi|\chi\rangle = 2 + 2\cos\frac{\phi}{2} = 4\cos^2\frac{\phi}{4} .$$

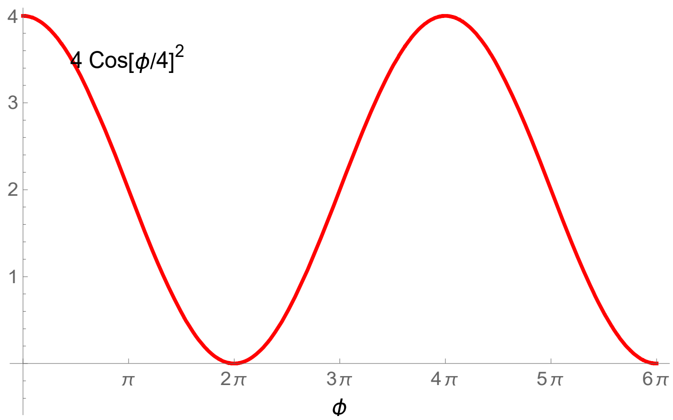

Fig. Plot of $\langle \chi | \chi \rangle = 2 + 2\cos\frac{\phi}{2} = 4\cos^2\frac{\phi}{4}$ as a function of the phase difference ϕ.

The period of ϕ is

$$\phi = 4\pi ,$$

or

$$\frac{\omega T}{4} = \frac{e|\gamma_n|TB}{4m_n c} = \pi ,$$

or

$$B = \frac{4\pi m_n c}{e|\gamma_n|T} = \frac{4\pi m_n c}{e|\gamma_n|(\frac{l m_n \bar{\lambda}}{\hbar})} = \frac{4\pi c\hbar}{e|\gamma_n| l\bar{\lambda}} ,$$

or

$$\boxed{l\Delta B = \frac{4\pi c\hbar}{e|\gamma_n|\bar{\lambda}}}.$$

((**Experimental result**)) Werner et al. (1975)

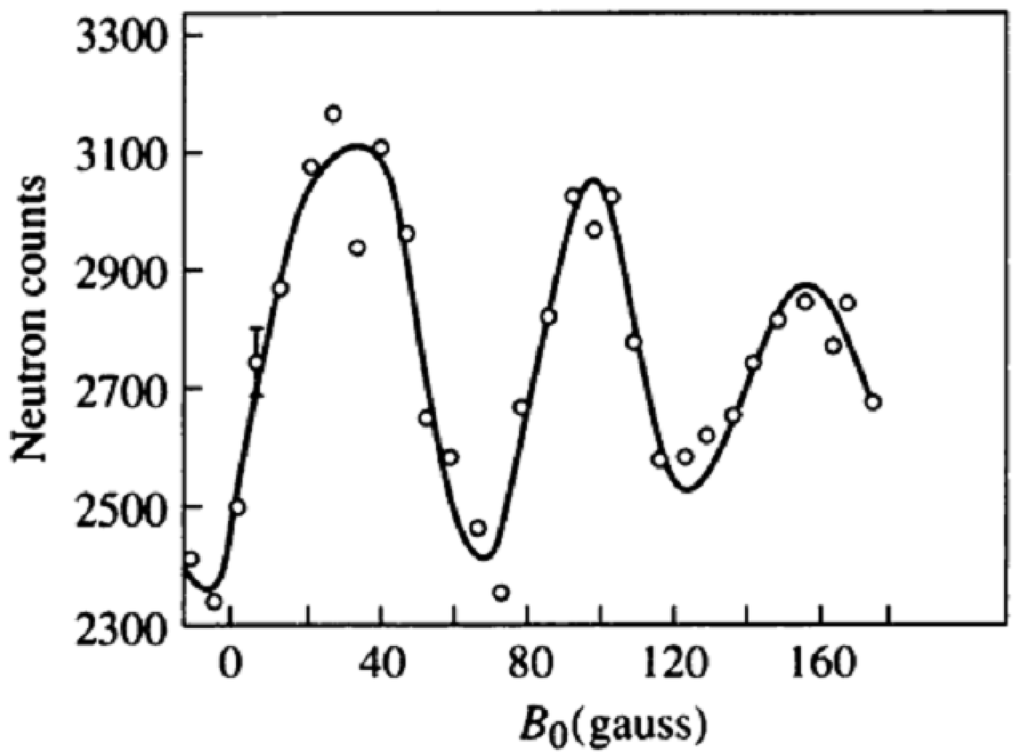

Fig. The difference in counts between the counters C_3 and C_2 as a function of B.

4. Numerical calculation

Using the Mathematica we get the relation

$$(\Delta B)l = \frac{4\pi c\hbar}{e|\gamma_n|\frac{\lambda}{2\pi}} = \frac{27.525}{\lambda(A)}, \tag{1}$$

where ΔB (in units of Oe) and l (in units of cm), λ (in units of Å). Note that in the paper of Werner et al., on the other hand, they obtained

$$(\Delta B)l = \frac{272}{\lambda(A)}, \qquad \text{(Werner et al)} \tag{2}$$

which is different from our calculation. We are not sure how Werner et al. derived such a relation.

(a) If $\lambda = 1.445$ Å and $l = 2.0$ cm, the value of ΔB can be evaluated as

$$\Delta B = 94.1 \text{ Oe},$$

from Eq.(2). (Werner et al. 1975). This value of ΔB is rather different form the experimental value

$\Delta B = 62 \pm 2$ Oe. (experimental result).

(b) Using the same values of λ and l, the value of ΔB can be evaluated as

$$\Delta B = 9.52 \text{ Oe.}$$

from Eq.(1). This value of ΔB is rather different form the experimental value.

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APPENDIX

((**Mathematica**)) The calculation of $(\Delta B)l = \dfrac{4\pi c\hbar}{eg_n \dfrac{\lambda}{2\pi}} = \dfrac{27.525}{\lambda(A)}$, and $\dfrac{\hbar\omega}{2}$ (neV)

```
Clear["Global`*"];
rule1 = {c → 2.99792 × 10^10, ħ → 1.054571628 10^-27,
   gn → 1.9130427, mn → 1.674927211 × 10^-24,
   qe → 4.8032068 × 10^-10, Å → 10^-8, μn → 9.66237055 × 10^-24,
   neV → 1.602176487 × 10^-12 10^-9};
```

$$\omega 1 = \frac{2\,\mu n}{\hbar}\ B\ /.\ \mathrm{rule1}$$

```
18324.7 B
```

```
f1 = ω1 / (2 π) /. rule1
```

```
2916.47 B
```

$$\epsilon 1 = \frac{\hbar\,\omega 1 / 2}{\mathrm{neV}}\ /.\ \mathrm{rule1}$$

```
0.00603078 B
```

$$a = \frac{4\,\pi\, c\ \hbar}{qe\ gn\ (Å / 2\,\pi)}\ /.\ \mathrm{rule1}$$

```
27.5252
```

```
ω1 /. B → 165
```

3.02358×10^{6}

```
f1 /. B → 165 // ScientificForm
```

4.81218×10^{5}

```
ϵ1 /. B → 165
```

```
0.995078
```