Projection operator for the photon polarization

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For the Stern-Gerlach experiment with spin 1/2, after the measurement of the spin direction Sz, the state of the system collapses into one of two states, $\left|+z\right\rangle$ and $\left|-z\right\rangle$. The situation is rather different for the light polarization, the state of the system collapses into one state depending on the direction of polarizer such as *x*-axis polarizer; the eigenket $\left|x\right\rangle$ for the *x*-polarizer,and the eigenket $\left|y\right\rangle$ for the *y*-polarizer. The final state is given by the projection operator which is applied to the initial state $\left|\psi\right\rangle$. Here we discuss the role of the projection operator for the photon polarization.

1. Stern-Gerlach experiment for the spin 1/2 (as an example)

First we consider the Stern-Gerlach experiment (SG_z). We have two eigenkets of the spin operator $\hat{S}_z$, $\left|+z\right\rangle$ and $\left|-z\right\rangle$.

$$\hat{S}_z\left|+z\right\rangle = \frac{\hbar}{2}\left|+z\right\rangle, \qquad \hat{S}_z\left|-z\right\rangle = -\frac{\hbar}{2}\left|-z\right\rangle$$

The projection operators are defined by

$$\hat{P}_{+z} = \left|+z\right\rangle\left\langle +z\right|, \qquad \hat{P}_{-z} = \left|-z\right\rangle\left\langle -z\right|$$

The spin operator $\hat{S}_z$ can be expressed by

$$\hat{S}_z = \frac{\hbar}{2}(\left|+z\right\rangle\left\langle +z\right| - \left|-z\right\rangle\left\langle -z\right|) = \frac{\hbar}{2}(\hat{P}_{+z} - \hat{P}_{-z})$$

(a) **The measurements: eigenvalue problem**

The eigenkets $\left|+z\right\rangle$ and $\left|-z\right\rangle$ are determined from the eigenvalue problem. After the measurements the system collapses

(b) **Projection operator**

When the initial state of the system is is given by $\left|\psi\right\rangle$, the final states after the SG_z are

$$\hat{P}_{+z}|\psi\rangle, \qquad \hat{P}_{-z}|\psi\rangle$$

using the projection operators.

(c) **The probability**

The probability of finding the system in the state $|+z\rangle$ is

$$\langle +z|\hat{P}_{+z}|\psi\rangle = \langle +z|\psi\rangle$$

The probability of finding the system in the state $|-z\rangle$ is

$$\langle -z|\hat{P}_{-z}|\psi\rangle = \langle -z|\psi\rangle$$

2. Photon polarization

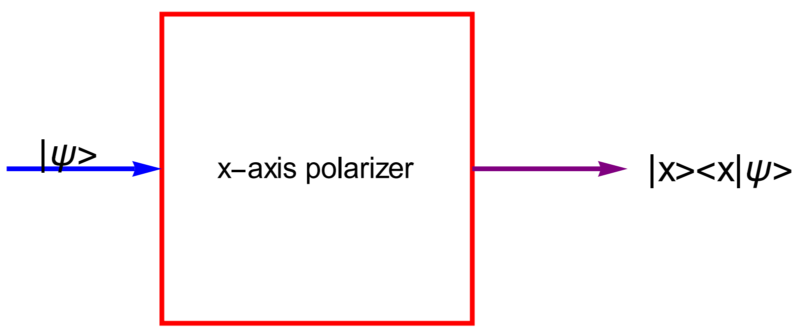
(a) Projection operator

The final state of the light after passing the *x*-axis polarizer is given by

$$\hat{P}_x|\psi\rangle = |x\rangle\langle x|\psi\rangle .$$

The probability of finding the system in the state $|x\rangle$ is

$$\left|\langle x|\hat{P}_x|\psi\rangle\right|^2 = \left|\langle x|\psi\rangle\right|^2 .$$



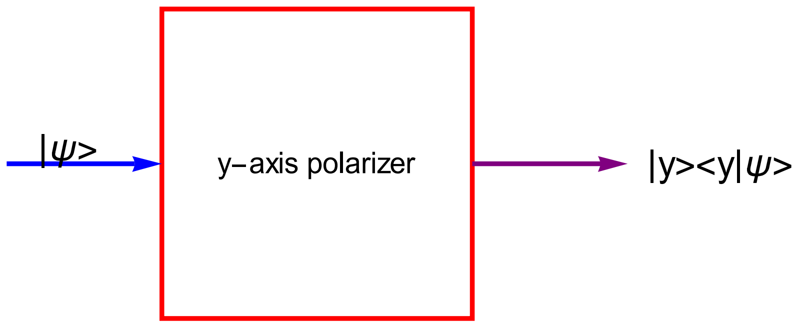
(b) The use of *y*-axis polarizer

The final state of the light after passing the *y*-axis polarizer is given by

$$\hat{P}_y|\psi\rangle = |y\rangle\langle y|\psi\rangle .$$

The probability of finding the system in the state $|y\rangle$ is

$$\left|\langle y|\hat{P}_y|\psi\rangle\right|^2 = \left|\langle y|\psi\rangle\right|^2 .$$



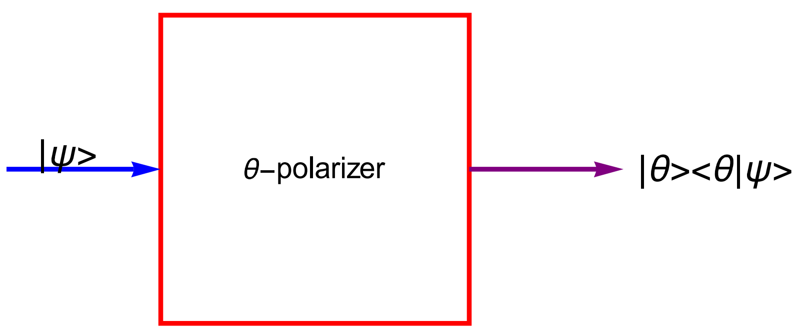
(c) Th polarizer with the angle θ

The final state of the light after passing the angle θ polarizer is given by

$$\hat{P}_\theta|\psi\rangle = |\theta\rangle\langle \theta|\psi\rangle .$$

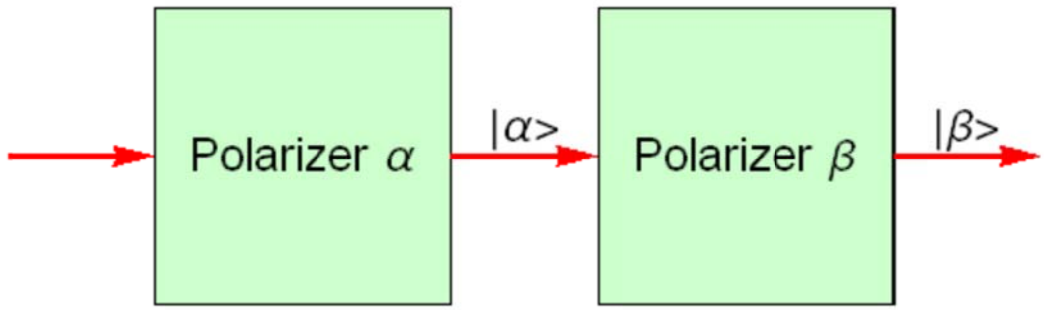
The probability of finding the system in the state $|\theta\rangle$ is

$$\left|\langle \theta|\hat{P}_\theta|\psi\rangle\right|^2 = \left|\langle \theta|\psi\rangle\right|^2 .$$



3. Examples-1

Suppose we use two polarizers (angles α and β) in series; $\alpha - \beta$



The projection operators for the polarizers α and β are

$$\hat{P}_\alpha = |\alpha\rangle\langle\alpha| = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\end{pmatrix} = \begin{pmatrix}\cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha\end{pmatrix}$$

and

$$\hat{P}_\beta = |\beta\rangle\langle\beta| = \begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix}\begin{pmatrix}\cos\beta & \sin\beta\end{pmatrix} = \begin{pmatrix}\cos^2\beta & \sin\beta\cos\beta \\ \sin\beta\cos\beta & \sin^2\beta\end{pmatrix}$$

where

$$|\alpha\rangle = \begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}, \qquad |\beta\rangle = \begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix}$$

The initial state is given by $|\psi\rangle$. After passing through two polarizers α and β, the final state is obtained as

$$\hat{P}_\beta\hat{P}_\alpha|\psi\rangle$$

The probability amplitude for the system in the final state $|\beta\rangle$, is given by

$$\langle\beta|\hat{P}_\beta\hat{P}_\alpha|\psi\rangle = \langle\beta|\alpha\rangle\langle\alpha|\psi\rangle$$

The corresponding probability is

$$P_{\alpha\beta} = |\langle\beta|\alpha\rangle|^2 |\langle\alpha|\psi\rangle|^2 = |\langle\alpha|\psi\rangle|^2 \cos^2(\beta-\alpha) . \qquad \text{(Malus' law)}$$

where

$$\langle\beta|\alpha\rangle = \begin{pmatrix}\cos\beta & \sin\beta\end{pmatrix}\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix} = \cos(\beta-\alpha)$$

4. Example-II

Next we consider n polarizers (angles α_1, α_2, α_3, ..., α_{n-1}, α_n) in series. The initial state is given by $|\psi\rangle$. After passing through *n* polarizers (angles α_1, α_2, α_3, ..., α_{n-1}, α_n) in series, the final state can be obtained as

$$\hat{P}_{\alpha_n}\hat{P}_{\alpha_{n-1}}\cdots\hat{P}_{\alpha_4}\hat{P}_{\alpha_3}\hat{P}_{\alpha_2}\hat{P}_{\alpha_1}|\psi\rangle$$

The probability amplitude for the system in the final state $|\alpha_n\rangle$, is given by

$$\langle\alpha_n|\hat{P}_{\alpha_n}\hat{P}_{\alpha_{n-1}}\cdots\hat{P}_{\alpha_4}\hat{P}_{\alpha_3}\hat{P}_{\alpha_2}\hat{P}_{\alpha_1}|\psi\rangle = \langle\alpha_n|\alpha_{n-1}\rangle\cdots\langle\alpha_3|\alpha_2\rangle\langle\alpha_2|\alpha_1\rangle\langle\alpha_1|\psi\rangle$$

where

$$\hat{P}_{\alpha_k} = |\alpha_k\rangle\langle\alpha_k| = \begin{pmatrix}\cos^2\alpha_k & \sin\alpha_k\cos\alpha_k \\ \sin\alpha_k\cos\alpha_k & \sin^2\alpha_k\end{pmatrix}$$

The corresponding probability is

$$P = |\langle\alpha_1|\psi\rangle|^2 \cos^2(\alpha_n-\alpha_{n-1})\cos^2(\alpha_{n-1}-\alpha_{n-2})\cdots\cos^2(\alpha_3-\alpha_2)\cos^2(\alpha_2-\alpha_1) .$$

REFERENCES

Richard P. Feynman and Albert R. Hibbs, *Quantum Mechanics and Path Integrals*, emended by Daniel F. Styer, Emended edition (Dover Publications, Inc. New York, 2010).