Operator method in Quantum Computing

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
Binghamton, NY
(Date: November 27, 2014)

N.D. Mermin, Quantum Computer Science An Introduction (Cambridge, 2007).

1. Definition of the operator $\hat{n}$

We define the operator $\hat{n}$ from the eigenvalue problem

$$\hat{n}|x\rangle = x|x\rangle,$$

with

$$\hat{n}|0\rangle = 0|0\rangle = 0, \qquad \hat{n}|1\rangle = 1|1\rangle = |1\rangle$$

where x = 0 and 1. $\hat{n}$ is the projection operator and is defined by

$$\hat{n} = |1\rangle\langle 1|.$$

The matrix of $\hat{n}$ under the basis of {$|0\rangle$, $|1\rangle$ } is given by

$$\hat{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The operator $\hat{n}$ is the projection operator on the state .$|1\rangle$

2. Definition of the operator $\hat{m} = \hat{1} - \hat{n}$

We define the operator $\hat{m}$ as

$$\hat{m} = \hat{1} - \hat{n}$$

where $\hat{1}$ is the identity matrix of 2x2,

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For simplicity, here, we use $\hat{m}$ instead of $\tilde{n}$. We note that

$$\hat{m}|x\rangle = (\hat{1}-\hat{n})|x\rangle = |x\rangle - x|x\rangle = (1-x)|x\rangle$$

Thus we have

$$\hat{m}|0\rangle = |0\rangle, \qquad \hat{m}|1\rangle = 0|1\rangle = 0$$

The matrix of $\hat{m}$ under the basis of $\{|0\rangle, |1\rangle\}$ is given by

$$\hat{m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The operator $\hat{n}$ is the projection operator on the state $.|0\rangle$

$$\hat{m} = |0\rangle\langle 0|.$$

3. Properties of $\hat{n}$ and $\hat{m}$

$$\hat{n}^2 = \hat{n}$$

$$\hat{m}^2 = \hat{m}$$

$$\hat{n}\hat{m} = \hat{m}\hat{n} = 0$$

$$\hat{m} + \hat{n} = 1$$

4. Pauli matrix

$$\hat{X} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{Y} = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{Z} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{n}\hat{X} = \hat{X}\hat{m}, \qquad \hat{m}\hat{X} = \hat{X}\hat{n}$$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{1}$$

$$\hat{X}\hat{Z} = -\hat{Z}\hat{X}$$

$$\hat{n} = \frac{1}{2}(1 - \hat{Z}), \qquad \hat{m} = \frac{1}{2}(1 + \hat{Z}), \qquad \hat{Z} = \hat{n} - \hat{m}$$

$$\hat{Y} = i\hat{X}\hat{Z} = -i\hat{Z}\hat{X}, \qquad \hat{Z} = i\hat{Y}\hat{X} = -i\hat{X}\hat{Y}, \qquad \hat{X} = i\hat{Z}\hat{Y} = -i\hat{Y}\hat{Z}$$

5. Hadamard operator

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{X} + \hat{Z})$$

$$\hat{H}^2 = \frac{1}{2}(\hat{X} + \hat{Z})(\hat{X} + \hat{Z}) = \frac{1}{2}(\hat{X}^2 + \hat{X}\hat{Z} + \hat{Z}\hat{X} + \hat{Z}^2) = \hat{1}$$

$$\hat{H}\hat{X}\hat{H} = \hat{Z}, \qquad \hat{H}\hat{Z}\hat{H} = \hat{X}.$$

Note that

$$\begin{aligned}
\hat{H}\hat{X}\hat{H} &= \frac{1}{2}(\hat{X} + \hat{Z})\hat{X}(\hat{X} + \hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z})(\ddot{1} + \hat{X}\hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} + \hat{X}^2\hat{Z} + \hat{Z}\hat{X}Z) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} + \hat{X}^2\hat{Z} - \hat{Z}^2\hat{X}) \\
&= \hat{Z}
\end{aligned}$$

$$\begin{aligned}
\hat{H}\hat{Z}\hat{H} &= \frac{1}{2}(\hat{X} + \hat{Z})\hat{Z}(\hat{X} + \hat{Z}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z})(\hat{1} + \hat{Z}\hat{X}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} - \hat{X}^2\hat{Z} + \hat{Z}^2\hat{X}) \\
&= \frac{1}{2}(\hat{X} + \hat{Z} - \hat{Z} + \hat{X}) \\
&= \hat{X}
\end{aligned}$$

$$\hat{H} \otimes \hat{H} = \frac{1}{2}(\hat{X} + \hat{Z}) \otimes (\hat{X} + \hat{Z}) = \frac{1}{2}(\hat{X} \otimes \hat{X} + \hat{X} \otimes \hat{Z} + \hat{Z} \otimes \hat{X} + \hat{Z} \otimes \hat{Z})$$

6. Calculation of matrices by using Mathematica

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
n = (0 0
     0 1); m = I2 - n; X = PauliMatrix[1];
Y = PauliMatrix[2]; Z = PauliMatrix[3]; ϕ0 = (1
                                              0);
ϕ1 = (0
      1);
m // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

```
n // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

```
n.n - n // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
m.m - m // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

n.X − X.m // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

m.X − X.n // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

m − n // MatrixForm

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Z.X + X.Z // MatrixForm

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\mathbf{\frac{1}{2}}$ **(I2 − Z) // MatrixForm**

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathbf{\frac{1}{2}}$ **(X + Z) // MatrixForm**

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

```
H1 = 1/Sqrt[2] (X + Z); H1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
X.X // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Z.Z // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
X.Z + Z.X // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
H1.H1 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
H1.X.H1 - Z // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
H1.Z.H1 - X // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
H1.ϕ0 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
H1.ϕ1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
f11 = KroneckerProduct[m, I2] +
  KroneckerProduct[n, X];
f11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
f12 =
 1/2 (KroneckerProduct[I2, I2 + X] +
    KroneckerProduct[Z, I2 - X]);
f12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
f13 =
 1/2 (KroneckerProduct[I2 + Z, I2] +
    KroneckerProduct[I2 - Z, X]);
f13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
S11 = KroneckerProduct[n, n] +
  KroneckerProduct[m, m] +
  KroneckerProduct[X.n, X.m] +
  KroneckerProduct[X.m, X.n];
S11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
C11 = KroneckerProduct[m, I2] +
  KroneckerProduct[n, X];
C11 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
Z + ⅈ X .Y // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
Y + ⅈ Z.X // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
X + ⅈ Y.Z // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

```
KroneckerProduct[H1, H1] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$U1 = \begin{pmatrix} u11 & u12 \\ u21 & u22 \end{pmatrix};$$

```
KroneckerProduct[I2, U1] // MatrixForm
```

$$\begin{pmatrix} u11 & u12 & 0 & 0 \\ u21 & u22 & 0 & 0 \\ 0 & 0 & u11 & u12 \\ 0 & 0 & u21 & u22 \end{pmatrix}$$

```
h1 =
 KroneckerProduct[H1, H1].KroneckerProduct[X, X] //
  MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

7. The CNOT gate with $\hat{U}_{CNOT}$

The CNOT gate is defined by

$$
\begin{aligned}
\hat{U}_{CNOT} &= \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X} \\
&= \frac{1}{2}(\hat{1}+\hat{Z})\otimes\hat{1}+\frac{1}{2}(\hat{1}-\hat{Z})\otimes\hat{X} \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1}+\hat{Z}\otimes\hat{1}+\hat{1}\otimes\hat{X}-\hat{Z}\otimes\hat{X})
\end{aligned}
$$

with

$$\hat{U}_{CNOT}{}^{2}=\hat{1}$$

$$
\begin{aligned}
\hat{U}_{CNOT}(1-2) &= \hat{C}_{12}\otimes\hat{1} \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1}+\hat{Z}\otimes\hat{1}\otimes\hat{1}+\hat{1}\otimes\hat{X}\otimes\hat{1}-\hat{Z}\otimes\hat{X}\otimes\hat{1})
\end{aligned}
$$

$$\hat{U}_{CNOT}(1-3)=\frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1}+\hat{Z}\otimes\hat{1}\otimes\hat{1}+\hat{1}\otimes\hat{1}\otimes\hat{X}-\hat{Z}\otimes\hat{1}\otimes\hat{X})$$

((**Mathematica**))

```
Clear["Global`*"];
I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3];

UCNOT =
 1/2 (KroneckerProduct[I2, I2] +
    KroneckerProduct[Z, I2] +
    KroneckerProduct[I2, X] -
    KroneckerProduct[Z, X]);
UCNOT // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
UCNOT12 =
 1/2 (KroneckerProduct[I2, I2, I2] +
    KroneckerProduct[Z, I2, I2] +
    KroneckerProduct[I2, X, I2] -
    KroneckerProduct[Z, X, I2]);
UCNOT12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

```
UCNOT13 =
 1/2 (KroneckerProduct[I2, I2, I2] +
     KroneckerProduct[Z, I2, I2] +
     KroneckerProduct[I2, I2, X] -
     KroneckerProduct[Z, I2, X]) //
  MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

8. Swap (exchange) operator

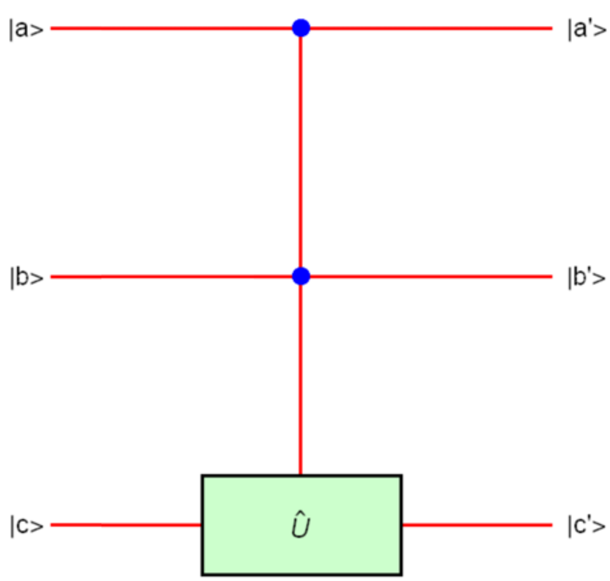
$\hat{G}_{swap}$ is called a swap (exchange) operator, which simply interchanges the states of qubits *1* and 2.

$$\begin{aligned}
\hat{G}_{SWAP} &= \hat{n}\otimes\hat{n} + \hat{m}\otimes\hat{m} + (\hat{X}\hat{n})\otimes(\hat{X}\hat{m}) + (\hat{X}\hat{m})\otimes(\hat{X}\hat{n}) \\
&= \frac{1}{4}(\hat{1}-\hat{Z})\otimes(\hat{1}-\hat{Z}) + \frac{1}{4}(\hat{1}+\hat{Z})\otimes(\hat{1}-\hat{Z}) + \frac{1}{4}[\hat{X}(\hat{1}-\hat{Z})]\otimes[X(\hat{1}+\hat{Z})] \\
&\quad + \frac{1}{4}[\hat{X}(\hat{1}+\hat{Z})]\otimes[X(\hat{1}-\hat{Z})] \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{Z}) + \frac{1}{4}(\hat{X} - \hat{X}\hat{Z})\otimes(X + \hat{X}\hat{Z}) + \frac{1}{4}(\hat{X} + \hat{X}\hat{Z})\otimes(X - \hat{X}\hat{Z}) \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{Z}) + \frac{1}{4}(\hat{X} + i\hat{Y})\otimes(X - i\hat{Y}) + \frac{1}{4}(\hat{X} - i\hat{Y})\otimes(X + i\hat{Y}) \\
&= \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{X}\otimes X + \hat{Y}\otimes\hat{Y} + \hat{Z}\otimes\hat{Z})
\end{aligned}$$

Then the swap operator becomes equivalent to the Dirac exchange spin operator

---

9. Toffoli gate



Only if $|a\cdot b\rangle = |1\rangle$

$$|a'\rangle = |a\rangle, \qquad |b'\rangle = |b\rangle$$

$$|c'\rangle = \hat{U}|c\rangle$$

Other wise

$$|a'\rangle = |a\rangle, \qquad |b'\rangle = |b\rangle$$

$$|c'\rangle = |c\rangle$$

Truth table

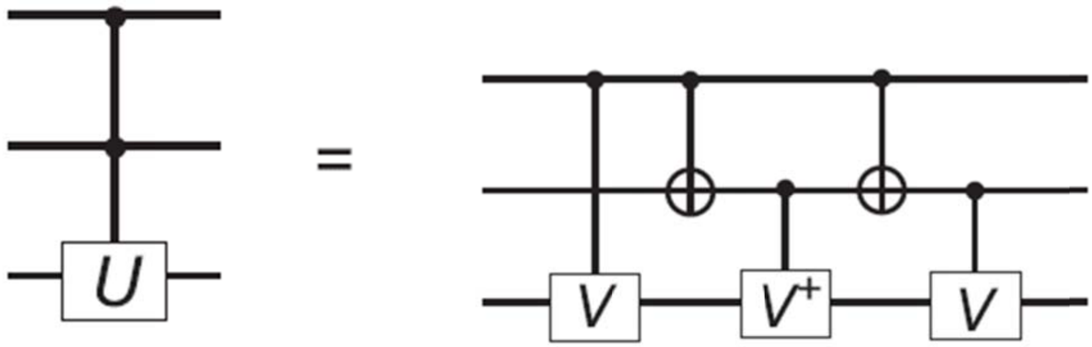
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	$\hat{U}|0\rangle = u_{11}|0\rangle + u_{21}|1\rangle$
1	1	1	1	1	$\hat{U}|1\rangle = u_{12}|0\rangle + u_{22}|1\rangle$

$$\hat{G}_{Toffli}(U)] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{11} & u_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{U} \end{pmatrix}$$

10. Example: equivalent quantum circuits

We show that a two-qubit controlled gate canted using a combination of one qubit controlled gate as

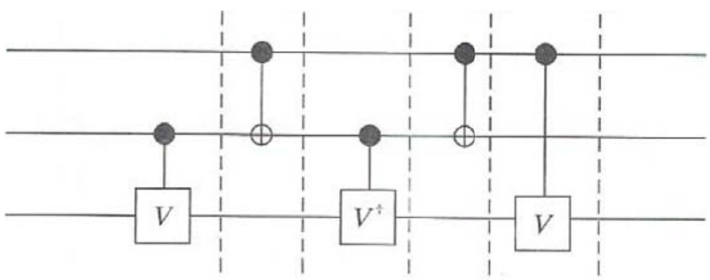
$$\hat{G}_{Toffoli}[U] = \hat{G}_{V23}(\hat{G}_{CNOT} \otimes \hat{1})(\hat{1} \otimes \hat{G}[V^{+}])(\hat{G}_{CNOT} \otimes \hat{1})\hat{G}_{V13}.$$



This can be also described by

$$\hat{G}_{Toffoli}[U] = \hat{G}_{V13}(\hat{G}_{CNOT} \otimes \hat{1})[\hat{1} \otimes \hat{G}(V^{+})](\hat{G}_{CNOT} \otimes \hat{1})\hat{G}_{V23}$$

which means that $\hat{G}_{Toffoli}[U]$ is a universal gate (reversible).



The left hand-side is the Toffoli gate with the matrix $\hat{U}$,

$$\hat{G}_{Toffoli}[U] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$

where

$$\hat{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

Suppose that the matrix $\hat{U}$ is expressed by $\hat{U} = \hat{V}^2$, where $\hat{V}$ is the unitary operator.
The CNOT operator is given by

$$\hat{G}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix},$$

$\hat{G}_{CNOT} \otimes \hat{1}$ is obtained as

$$\hat{G}_{CNOT} \otimes \hat{1} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix} \otimes \hat{1} = \begin{pmatrix} \hat{I}_2 & 0 & 0 & 0 \\ 0 & \hat{I}_2 & 0 & 0 \\ 0 & 0 & 0 & \hat{I}_2 \\ 0 & 0 & \hat{I}_2 & 0 \end{pmatrix}$$

The control *V* gate is given by

$$\hat{G}[V] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{V} \end{pmatrix},. \qquad \hat{G}[V^+] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{V}^+ \end{pmatrix}$$

Then we have

$$\hat{G}_{V23} = \hat{1} \otimes \hat{U}[V] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{11} & V_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & V_{21} & V_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{V} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

$\hat{C}_{V13}$ is the matrix obtained from the matrix $\hat{1} \otimes \hat{G}[V]$ by the appropriate interchange of row and column,

$$\hat{G}_{V13}[V]=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&V_{11}&V_{12}&0&0\\0&0&0&0&V_{21}&V_{22}&0&0\\0&0&0&0&0&0&V_{11}&V_{12}\\0&0&0&0&0&0&V_{21}&V_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}&0&0&0\\0&\boldsymbol{I}&0&0\\0&0&\boldsymbol{V}&0\\0&0&0&\boldsymbol{V}\end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2]; CV = (1 0 0   0
                                                 0 1 0   0
                                                 0 0 v11 v12
                                                 0 0 v21 v22);

CVP = (1 0 0    0
       0 1 0    0
       0 0 v11c v21c
       0 0 v12c v22c);

V1 = (v11 v12
      v21 v22);

CUV13[A_] := Module[{A1, U, U1, U11, U12}, A1 = A;
  U = KroneckerProduct[I2, A1];
  U1 = {U[[All, 1]], U[[All, 2]], U[[All, 5]], U[[All, 6]],
    U[[All, 3]], U[[All, 4]], U[[All, 7]], U[[All, 8]] };
  U11 = Transpose[U1];
  U12 = {U11[[1]], U11[[2]], U11[[5]], U11[[6]], U11[[3]],
    U11[[4]], U11[[7]], U11[[8]]}];

CV13 = CUV13[CV]; CV13 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v11 & v12 & 0 & 0 \\ 0 & 0 & 0 & 0 & v21 & v22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11 & v12 \\ 0 & 0 & 0 & 0 & 0 & 0 & v21 & v22 \end{pmatrix}$$

```
UCNOT = (1 0 0 0
         0 1 0 0
         0 0 0 1
         0 0 1 0); UCNOTI2 = KroneckerProduct[UCNOT, I2];
```

$$\text{UCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
UCNOTI2 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

```
CV23 = KroneckerProduct[I2, CV];
```

```
CV23 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v11 & v12 & 0 & 0 & 0 & 0 \\ 0 & 0 & v21 & v22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11 & v12 \\ 0 & 0 & 0 & 0 & 0 & 0 & v21 & v22 \end{pmatrix}$$

```
CVP23 = KroneckerProduct[I2, CVP]; CVP23 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v11c & v21c & 0 & 0 & 0 & 0 \\ 0 & 0 & v12c & v22c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v11c & v21c \\ 0 & 0 & 0 & 0 & 0 & 0 & v12c & v22c \end{pmatrix}$$

```
K1 = CV23.UCNOTI2.CVP23.UCNOTI2.CV13 // FullSimplify;
```

$$vvc = \begin{pmatrix} v11 & v12 \\ v21 & v22 \end{pmatrix} . \begin{pmatrix} v11c & v21c \\ v12c & v22c \end{pmatrix};$$

$$vcv = \begin{pmatrix} v11c & v21c \\ v12c & v22c \end{pmatrix} . \begin{pmatrix} v11 & v12 \\ v21 & v22 \end{pmatrix};$$

```
U1 = V1.V1 // Simplify;

rule1 = {vvc[[1, 1]] → 1, vvc[[1, 2]] → 0, vvc[[2, 1]] → 0,
   vvc[[2, 2]] → 1, vcv[[1, 1]] → 1, vcv[[1, 2]] → 0, vcv[[2, 1]] → 0,
   vcv[[2, 2]] → 1 };
rule2 = {U1[[1, 1]] → U11, U1[[1, 2]] → U12, U1[[2, 1]] → U21,
   U1[[2, 2]] → U22};

K11 = K1 //. rule1; K12 = K11 //. rule2; K12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U11 & U12 \\ 0 & 0 & 0 & 0 & 0 & 0 & U21 & U22 \end{pmatrix}$$

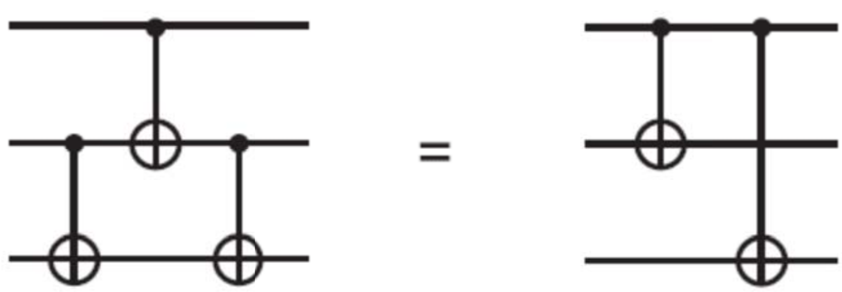
```
L1 = CV13.UCNOTI2.CVP23.UCNOTI2.CV23 // FullSimplify;

L11 = L1 //. rule1; L12 = L11 //. rule2; L12 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U11 & U12 \\ 0 & 0 & 0 & 0 & 0 & 0 & U21 & U22 \end{pmatrix}$$

10. Equivalent circuits

We show the equivalence between two quantum circuits as shown below,



$$(\hat{1}\otimes\hat{G}_{CNOT})(\hat{G}_{CNOT}\otimes\hat{1})(\hat{1}\otimes\hat{G}_{CNOT}) = (\hat{G}_{CNOT}\otimes\hat{1})\hat{C}_{13}(CNOT)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where

$$\hat{G}_{CNOT} \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{1} \otimes \hat{G}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\hat{G}_{CNOT13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \hat{G}[V] &= \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{V} \\ &= \frac{1}{2}(\hat{1} + \hat{Z}) \otimes \hat{1} + \frac{1}{2}(\hat{1} - \hat{Z}) \otimes \hat{V} \\ &= \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} + \hat{1} \otimes \hat{V} - \hat{Z} \otimes \hat{V}) \end{aligned}$$

$$\hat{G}_{12}[V]\otimes\hat{1}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&V_{11}&0&V_{12}&0\\0&0&0&0&0&V_{11}&0&V_{12}\\0&0&0&0&V_{21}&0&V_{22}&0\\0&0&0&0&0&V_{21}&0&V_{22}\end{pmatrix}$$

$$\hat{G}_{23}[V]=\hat{1}\otimes\hat{G}[V]$$

$$=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&V_{11}&V_{12}&0&0&0&0\\0&0&V_{21}&V_{22}&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&V_{11}&V_{12}\\0&0&0&0&0&0&V_{21}&V_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}_2&0&0&0\\0&\boldsymbol{V}&0&0\\0&0&\boldsymbol{I}_2&0\\0&0&0&\boldsymbol{V}\end{pmatrix}$$

$$\hat{G}_{13}[V]=\frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1}+\hat{Z}\otimes\hat{1}\otimes\hat{1}+\hat{1}\otimes\hat{1}\otimes\hat{V}-\hat{Z}\otimes\hat{1}\otimes\hat{V})$$

$$=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&V_{11}&V_{12}&0&0\\0&0&0&0&V_{21}&V_{22}&0&0\\0&0&0&0&0&0&V_{11}&V_{12}\\0&0&0&0&0&0&V_{21}&V_{22}\end{pmatrix}=\begin{pmatrix}\boldsymbol{I}_2&0&0&0\\0&\boldsymbol{I}_2&0&0\\0&0&\boldsymbol{V}&0\\0&0&0&\boldsymbol{V}\end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3];
V = ( V11  V12
      V21  V22 );

UV =
 1/2 (KroneckerProduct[I2, I2] +
    KroneckerProduct[Z, I2] +
    KroneckerProduct[I2, V] -
    KroneckerProduct[Z, V]);
UV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V11 & V12 \\ 0 & 0 & V21 & V22 \end{pmatrix}$$

```
C12V = KroneckerProduct[UV, I2];
C12V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V11 & 0 & V12 & 0 \\ 0 & 0 & 0 & 0 & 0 & V11 & 0 & V12 \\ 0 & 0 & 0 & 0 & V21 & 0 & V22 & 0 \\ 0 & 0 & 0 & 0 & 0 & V21 & 0 & V22 \end{pmatrix}$$

```
C23V = KroneckerProduct[I2, UV];
C23V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & V11 & V12 & 0 & 0 & 0 & 0 \\ 0 & 0 & V21 & V22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V11 & V12 \\ 0 & 0 & 0 & 0 & 0 & 0 & V21 & V22 \end{pmatrix}$$

```
C13V =
  1/2 (KroneckerProduct[I2, I2, I2] +
    KroneckerProduct[Z, I2, I2] +
    KroneckerProduct[I2, I2, V] -
    KroneckerProduct[Z, I2, V]);
C13V // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V11 & V12 & 0 & 0 \\ 0 & 0 & 0 & 0 & V21 & V22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V11 & V12 \\ 0 & 0 & 0 & 0 & 0 & 0 & V21 & V22 \end{pmatrix}$$

11 Toffoli gate:

The Toffoli gate is given by

$$
\begin{aligned}
\hat{G}_{toffoli} &= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{G}_{CNOT} \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}) \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{X} \end{pmatrix}
\end{aligned}
$$

where

$$
\hat{G}_{CNOT} = \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}
$$

((**Mathematica**))

```
Clear["Global`*"]; I2 = IdentityMatrix[2]; X = PauliMatrix[1];
Y = PauliMatrix[2]; Z = PauliMatrix[3]; n = (0 0
                                             0 1);
m = (1 0
     0 0);
         (1 0 0 0
UCNOT =   0 1 0 0
          0 0 0 1
          0 0 1 0);

TOF1 = KroneckerProduct[m, I2, I2] +
  KroneckerProduct[n, UCNOT];
TOF1 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

12. Fredkin gate

The Fredkin gate is given by

$$\hat{G}_{Fredkin} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{U}_{SWAP} \otimes \hat{n}$$

$$= \hat{1} \otimes \hat{1} \otimes \hat{m} + \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{X} \otimes X + \hat{Y} \otimes \hat{Y} + \hat{Z} \otimes \hat{Z}) \otimes \hat{n}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\hat{G}_{SWAP}$ is the SWAP operator,

$$\hat{G}_{SWAP} = \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{X} \otimes X + \hat{Y} \otimes \hat{Y} + \hat{Z} \otimes \hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; X = PauliMatrix[1]; Y = PauliMatrix[2];
Z = PauliMatrix[3]; n = {{0, 0}, {0, 1}}; m = {{1, 0}, {0, 0}};
I2 = IdentityMatrix[2];
SWAP = {{1, 0, 0, 0}, {0, 0, 1, 0}, {0, 1, 0, 0}, {0, 0, 0, 1}};

Fredkin =
 KroneckerProduct[I2, I2, m] +
   KroneckerProduct[SWAP, n] // Simplify;
Fredkin // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
SWAP =
 1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[X, X] +
    KroneckerProduct[Y, Y] + KroneckerProduct[Z, Z]);
SWAP // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

13. Controlled V gate

$$\hat{C}[V] = \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{Z} \otimes \hat{1} + \hat{1} \otimes \hat{V} - \hat{Z} \otimes \hat{V}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & V_{21} & V_{22} \end{pmatrix}$$

$$\hat{R}[V] = \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{1} \otimes \hat{Z} + \hat{V} \otimes \hat{1} - \hat{V} \otimes \hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 1 & 0 \\ 0 & V_{21} & 0 & V_{22} \end{pmatrix}$$

```
Clear["Global`*"]; I2 = IdentityMatrix[2];
X = PauliMatrix[1]; Y = PauliMatrix[2]; Z = PauliMatrix[3];
V = (V11 V12
     V21 V22);

UV =
  1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[Z, I2] +
     KroneckerProduct[I2, V] - KroneckerProduct[Z, V]);

RUV =
  1/2 (KroneckerProduct[I2, I2] + KroneckerProduct[I2, Z] +
     KroneckerProduct[V, I2] - KroneckerProduct[V, Z]);

UV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & V11 & V12 \\ 0 & 0 & V21 & V22 \end{pmatrix}$$

```
RUV // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & V11 & 0 & V12 \\ 0 & 0 & 1 & 0 \\ 0 & V21 & 0 & V22 \end{pmatrix}$$

---

14. Controlled-*U* gate

$$\hat{C}[U] = \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{1} + \hat{1}\otimes\hat{U} - \hat{Z}\otimes\hat{U})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{U} \end{pmatrix}$$

$$\hat{R}[U] = \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{1}\otimes\hat{Z} + \hat{U}\otimes\hat{1} - \hat{U}\otimes\hat{Z}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & 1 & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix}$$

$\hat{R}(H)$

$$\hat{R}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

15. Controlled-CNOT

$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix}$$

16. Fredkin gate

$$\begin{aligned} \hat{G}_{Fredkin} &= \hat{1}\otimes\hat{1}\otimes\hat{m} + \hat{G}_{SWAP}\otimes\hat{n} \\ &= \hat{1}\otimes\hat{1}\otimes\hat{m} + \frac{1}{2}(\hat{1}\otimes\hat{1} + \hat{X}\otimes X + \hat{Y}\otimes\hat{Y} + \hat{Z}\otimes\hat{Z})\otimes\hat{n} \\ &== \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

17. Toffoli gate

$$\hat{G}_{toffoli} = \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{G}_{CNOT}$$
$$= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X})$$
$$= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X}$$
$$== \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{X} \end{pmatrix}$$

18. R-CNOT

$$\hat{R}_{CNOT} = \hat{1}\otimes\hat{m}+\hat{X}\otimes\hat{n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

19. Swap gate

$$\hat{G}_{SWAP} = \hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{n}+\hat{X}m\otimes\hat{X}\hat{n}+\hat{X}\hat{n}\otimes\hat{X}\hat{m}$$
$$= \frac{1}{2}(\hat{1}\otimes\hat{1}+\hat{X}\otimes X+\hat{Y}\otimes\hat{Y}+\hat{Z}\otimes\hat{Z})$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

20. Hadamard gate:

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\hat{P}_\alpha = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$\hat{T}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$\hat{S} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$\hat{G}_{V12}[V] = \hat{G}[V] \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} \end{pmatrix}$$

$$\hat{G}_{V23} = \hat{1} \otimes \hat{G}[V]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{11} & 0 & V_{12} \\ 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{21} & 0 & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{V} & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

$$\hat{G}_{V13} = \frac{1}{2}(\hat{1}\otimes\hat{1}\otimes\hat{1} + \hat{Z}\otimes\hat{1}\otimes\hat{1} + \hat{1}\otimes\hat{1}\otimes\hat{V} - \hat{Z}\otimes\hat{1}\otimes\hat{V})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{11} & V_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{11} & V_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{I}_2 & 0 & 0 \\ 0 & 0 & \boldsymbol{V} & 0 \\ 0 & 0 & 0 & \boldsymbol{V} \end{pmatrix}$$

---

10. Expression for $\hat{G}_{CNOT}$ in terms of Pauli operators $\hat{X}$ and $\hat{Z}$

We know that the controlled-CNOT gate can be expressed by

$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}.$$

Here we consider another expressions for the controlled-CNOT gate. We start with

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The projection operators are defined by

$$\hat{P}_+ = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\hat{P}_- = |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

We note that

$$\hat{P}_+ + \hat{P}_- = \hat{1}.$$

The operator $\hat{X}$ can be described as

$$\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{P}_+ - \hat{P}_- .$$

The operator $\hat{Z}$ can be expressed by

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{m} - \hat{n}$$

Note that

$$\hat{m} = \frac{1}{2}(\hat{1} + \hat{Z}), \qquad \hat{n} = \frac{1}{2}(\hat{1} - \hat{Z})$$

since $\hat{m} + \hat{n} = \hat{1}$.

We now introduce the operator (controlled-CNOT gate), which can be generated from

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\hat{G}_{CNOT}$ is equivalent to the controlled-X gate

$$\hat{G}_{CNOT} = \hat{C}[X] = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & \boldsymbol{X} \end{pmatrix}.$$

This can be derived using the Kronecker product. Since

$$\hat{P}_+ + \hat{P}_- = \hat{1}, \qquad \hat{P}_+ - \hat{P}_- = \hat{X}$$

we get

$$\hat{P}_+ = \frac{1}{2}(\hat{1} + \hat{X}), \qquad \hat{P}_- = \frac{1}{2}(\hat{1} - \hat{X})$$

Then the controlled-CNOT operator can be rewritten as

$$
\begin{aligned}
\hat{G}_{CNOT} &= \hat{m} \otimes (\hat{P}_+ + \hat{P}_-) + \hat{n} \otimes (\hat{P}_+ - \hat{P}_-) \\
&= \hat{m} \otimes \hat{P}_+ + \hat{m} \otimes \hat{P}_- + \hat{n} \otimes \hat{P}_+ - \hat{n} \otimes \hat{P}_- \\
&= (\hat{m} + \hat{n}) \otimes \hat{P}_+ + (\hat{m} - \hat{n}) \otimes \hat{P}_- \\
&= \hat{1} \otimes \hat{P}_+ + \hat{Z} \otimes \hat{P}_-
\end{aligned}
$$

or

$$\hat{G}_{CNOT} = \frac{1}{2}[\hat{1} \otimes (\hat{1} + \hat{X}) + \hat{Z} \otimes (\hat{1} - \hat{X})]$$

11. The expression of $\hat{G}_{CNOT}$ in terms of Hadamard gate $\hat{H}$

The Hadamard gate is defined by

$$\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{Z} + \hat{X}).$$

Then we get

$$\hat{m}\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\hat{H}\hat{m}\hat{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \hat{P}_+ = \frac{1}{2}(\hat{1} + \hat{X})$$

Noting that $\hat{1}^2 = \hat{1}\hat{1}$ and using the property of the Kronecker product, we get

$$\hat{1} \otimes \hat{P}_+ = \hat{1} \otimes (\hat{H}\hat{m}\hat{H}) = (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m}\hat{H}) = (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m})(1 \otimes \hat{H}) \qquad (1)$$

Similarly, we have

$$\hat{n}\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

$$\hat{H}\hat{n}\hat{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \hat{P}_- = \frac{1}{2}(\hat{1} - \hat{X}).$$

Noting that $\hat{Z} = \hat{1}\hat{Z}$ and using the property of the Kronecker product, we get

$$\hat{Z} \otimes \hat{P}_{-} = \hat{Z} \otimes (\hat{H}\hat{n}\hat{H}) = (\hat{Z} \otimes \hat{H})(\hat{1} \otimes \hat{n}\hat{H}) = (\hat{1} \otimes \hat{H})(\hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H}) \quad (2)$$

From Eqs.(1) and (2), thus we have

$$\begin{aligned}\hat{G}_{CNOT} &= \hat{1} \otimes \hat{P}_{+} + \hat{Z} \otimes \hat{P}_{-} \\ &= (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m})(1 \otimes \hat{H}) + (\hat{1} \otimes \hat{H})(\hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H}) \\ &= (\hat{1} \otimes \hat{H})(\hat{1} \otimes \hat{m} + \hat{Z} \otimes \hat{n})(\hat{1} \otimes \hat{H})\end{aligned}$$

which can be rewritten as

$$\hat{G}_{CNOT} = \hat{G}_{X} = (\hat{1} \otimes \hat{H})\hat{R}_{Z}(\hat{1} \otimes \hat{H}) = (\hat{1} \otimes \hat{H})\hat{G}_{Z}(\hat{1} \otimes \hat{H}).$$

since

$$\hat{R}_{Z} = \hat{G}_{Z}.$$

12. Matrix representation of Kronecker product for two qubits

$$\hat{m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{m} + \hat{n} = \hat{1}.$$

$$\hat{P}_{+} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \hat{P}_{-} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\hat{m} \otimes \hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{n} \otimes \hat{1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{1} \otimes \hat{m} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \hat{1} \otimes \hat{n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{m}\otimes\hat{m}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{m}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\hat{n}\otimes\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$$\hat{m}\otimes\hat{X}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{X}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},$$

$$\hat{X}\otimes\hat{m}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{X}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{pmatrix}$$

$$\hat{m}\otimes\hat{U}=\begin{pmatrix}u_{11}&u_{12}&0&0\\u_{21}&u_{22}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{U}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&u_{11}&u_{12}\\0&0&u_{21}&u_{22}\end{pmatrix}$$

$$\hat{U}\otimes\hat{m}=\begin{pmatrix}u_{11}&0&u_{12}&0\\0&0&0&0\\u_{21}&0&u_{22}&0\\0&0&0&0\end{pmatrix},\qquad \hat{U}\otimes\hat{n}=\begin{pmatrix}0&0&0&0\\0&u_{11}&0&u_{12}\\0&0&0&0\\0&u_{21}&0&u_{22}\end{pmatrix}$$

$$\hat{X}\hat{m}\otimes\hat{X}\hat{n}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{X}\hat{n}\otimes\hat{X}\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{X}\hat{m}\otimes\hat{X}\hat{n}+\hat{X}\hat{n}\otimes\hat{X}\hat{m}=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$\hat{m}\otimes\hat{P}_{+}=\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{P}_{+}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&1\\0&0&1&1\end{pmatrix}$$

$$\hat{m}\otimes\hat{P}_{-}=\frac{1}{2}\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad \hat{n}\otimes\hat{P}_{-}=\frac{1}{2}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&-1\\0&0&-1&1\end{pmatrix}$$

13. Equivalence of quantum circuits between $\hat{G}_Z$ and $\hat{R}_Z$

We show that the controlled-*Z* gate $\hat{G}_Z$ is equivalent to the quantum circuit with $\hat{R}_Z$.
((Method-1)) The use of matrices

$$\begin{aligned}\hat{G}_Z&=\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z}\\&=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\\&=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
\hat{R}_Z &= \hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

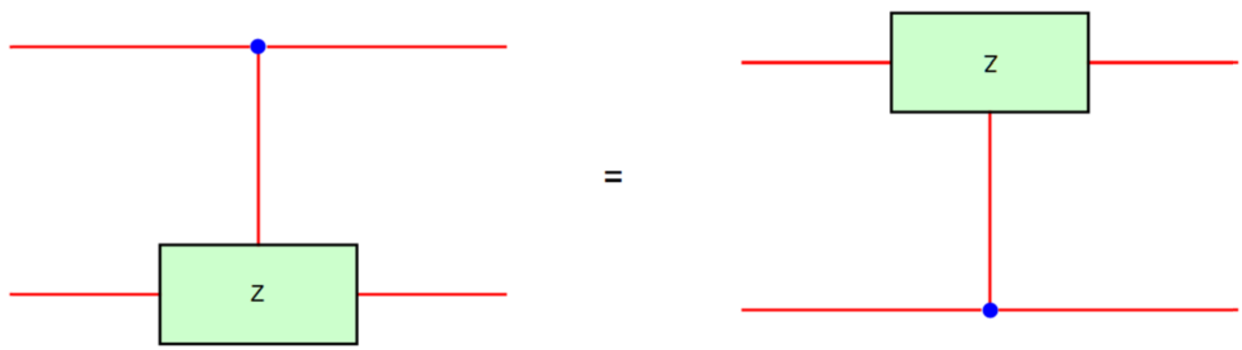


Fig. Quantum gates with $\hat{G}_Z$ and $\hat{R}_Z$.

((**Method-II**) Operation method

$$
\begin{aligned}
\hat{G}_Z{}^2 &= (\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z})(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z}) \\
&= \hat{m}^2\otimes\hat{1}+\hat{m}\hat{n}\otimes\hat{Z}+\hat{n}\hat{m}\otimes\hat{Z}+\hat{n}^2\otimes\hat{Z}^2 \\
&= \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{1} \\
&= (\hat{m}+\hat{n})\otimes\hat{1} \\
&= \hat{1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_Z{}^2 &= (\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n})(\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n}) \\
&= \hat{1}\otimes\hat{m}^2+\hat{Z}\otimes\hat{m}\hat{n}+\hat{Z}\otimes\hat{n}\hat{m}+\hat{Z}^2\otimes\hat{n}^2 \\
&= \hat{1}\otimes\hat{m}+\hat{1}\otimes\hat{n} \\
&= \hat{1}\otimes(\hat{m}+\hat{n}) \\
&= \hat{1}
\end{aligned}
$$

$$\begin{aligned}
\hat{G}_Z\hat{R}_Z &= (\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z})(\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n}) \\
&= \hat{m}\otimes\hat{m}+\hat{m}\hat{Z}\otimes\hat{n}+\hat{n}\otimes\hat{Z}\hat{m}+\hat{n}\hat{Z}\otimes\hat{Z}\hat{n} \\
&= \hat{m}\otimes\hat{m}+\hat{m}\otimes\hat{n}+\hat{n}\otimes\hat{m}+\hat{n}\otimes\hat{n} \\
&= (\hat{m}+\hat{n})\otimes(\hat{m}+\hat{n}) \\
&= \hat{1}
\end{aligned}$$

$$\begin{aligned}
\hat{R}_Z\hat{G}_Z &= (\hat{1}\otimes\hat{m}+\hat{Z}\otimes\hat{n})(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{Z}) \\
&= \hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{m}\hat{Z}+\hat{Z}\hat{m}\otimes\hat{n}+\hat{Z}\hat{n}\otimes\hat{n}\hat{Z} \\
&= \hat{m}\otimes\hat{m}+\hat{m}\otimes\hat{n}+\hat{n}\otimes\hat{m}+\hat{n}\otimes\hat{n} \\
&= (\hat{m}+\hat{n})\otimes(\hat{m}+\hat{n}) \\
&= \hat{1}
\end{aligned}$$

where

$$\hat{m}\hat{n}=0, \qquad \hat{m}+\hat{n}=\hat{1}$$

$$\hat{m}\hat{Z}=\hat{Z}\hat{m}=\hat{m}, \qquad \hat{n}\hat{Z}=\hat{Z}n=-\hat{n}.$$

Since

$$\hat{G}_Z(\hat{G}_Z\hat{R}_Z)=\hat{G}_Z, \qquad \hat{R}(\hat{R}_Z\hat{G}_Z)=\hat{R},$$

we get the relation

$$\hat{G}_Z=\hat{R}_Z.$$

The two quantum circuits are equivalent.

---

14. Quantum circuits related to the controlled *U*-gate

(a) Quantum circuit with $\hat{G}_U=\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{U}$.

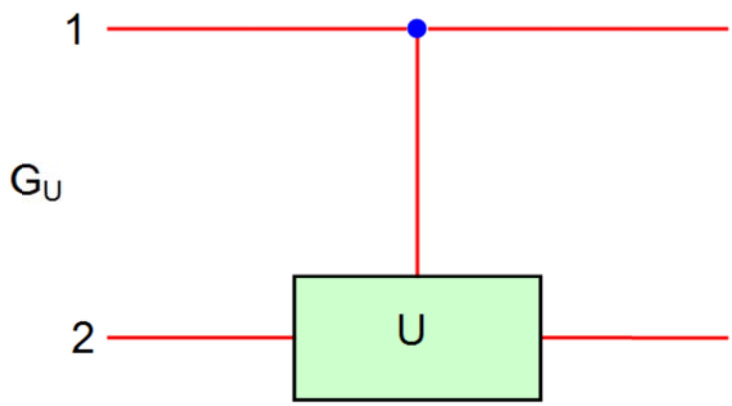


Fig. Controlled-U gate with $\hat{G}_U = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{U}$. $\hat{U}$ is the 2x2 matrix.

$$\begin{aligned}
\hat{G}_U &= \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{U} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u_{11} & u_{12} \\ 0 & 0 & u_{21} & u_{22} \end{pmatrix}
\end{aligned}$$

(b) Quantum circuit with $\hat{R}_U = \hat{1} \otimes \hat{m} + \hat{U} \otimes \hat{n}$

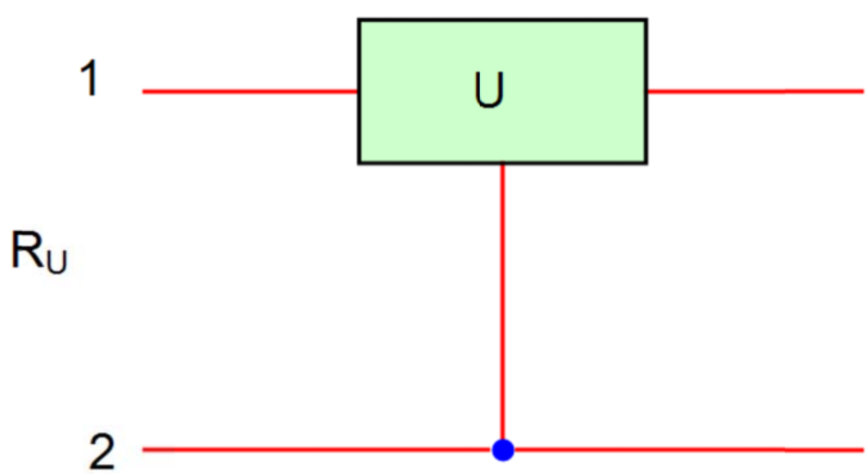


Fig. Quantum circuit with $\hat{R}_U = \hat{1} \otimes \hat{m} + \hat{U} \otimes \hat{n}$. $\hat{U}$ is the 2x2 matrix.

$$\hat{R}_U = \hat{1} \otimes \hat{m} + \hat{U} \otimes \hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & 0 & u_{12} \\ 0 & 0 & 1 & 0 \\ 0 & u_{21} & 0 & u_{22} \end{pmatrix}$$

15. Qauntum circuits related to the controlled-CNOT

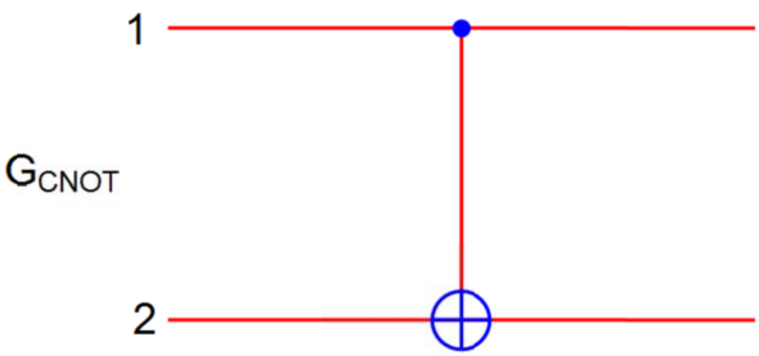


Fig. Controlled-CNOT with $\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X}$

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

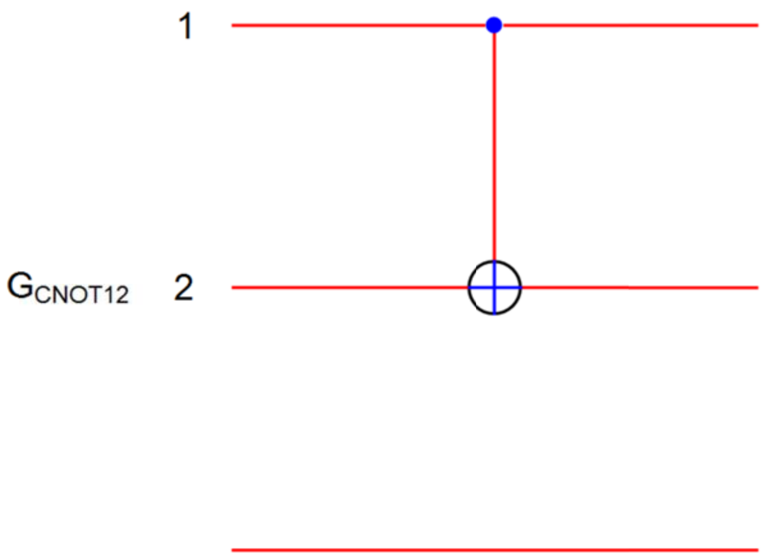


Fig. Quantum gate with Controlled-CNOT between 1 and 2. with $\hat{G}_{CNOT12} = \hat{G}_{CNOT} \otimes \hat{1} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{X} \otimes \hat{1}$.

$$
\begin{aligned}
\hat{G}_{CNOT12} &= \hat{G}_{CNOT} \otimes \hat{1} \\
&= \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{X} \otimes \hat{1} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

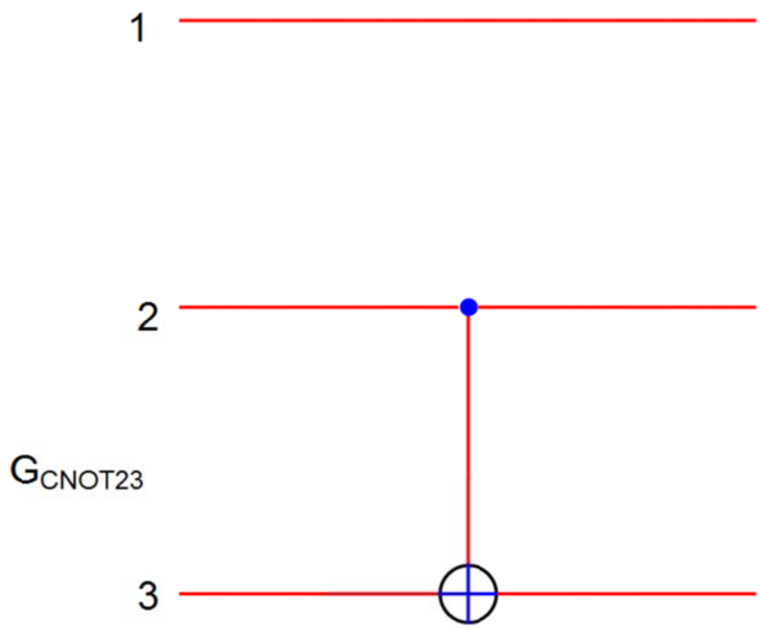


Fig. Quantum gate with Controlled-CNOT between 2 and 3. with $\hat{G}_{CNOT\,23} = \hat{1} \otimes \hat{G}_{CNOT} = \hat{1} \otimes \hat{m} \otimes \hat{1} + \hat{1} \otimes \hat{n} \otimes \hat{X}$.

$$
\begin{aligned}
\hat{G}_{CNOT\,23} &= \hat{1} \otimes \hat{G}_{CNOT} \\
&= \hat{1} \otimes \hat{m} \otimes \hat{1} + \hat{1} \otimes \hat{n} \otimes \hat{X} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{aligned}
$$

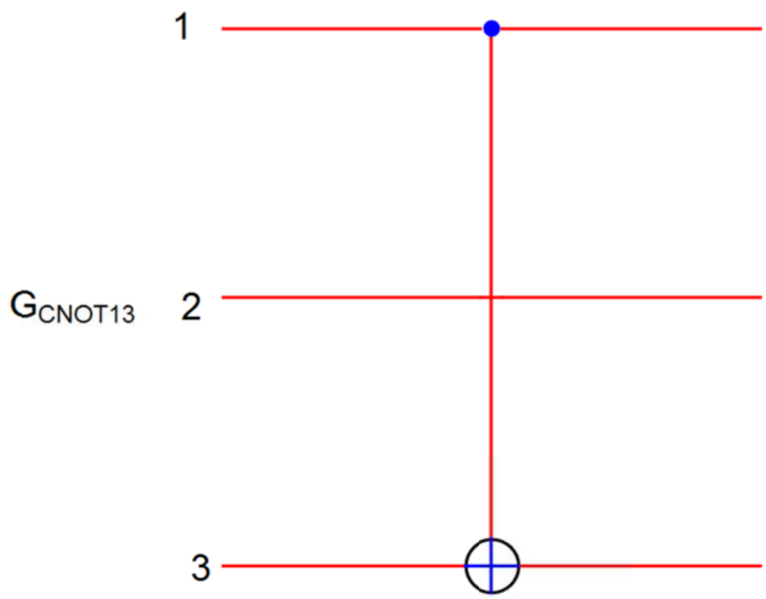


Fig. Quantum gate with Controlled-CNOT between 1 and 3. with $\hat{G}_{CNOT13} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{1} \otimes \hat{X}$.

$$\hat{G}_{CNOT13} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{1} \otimes \hat{X}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

16 R-CNOT gate with $\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$

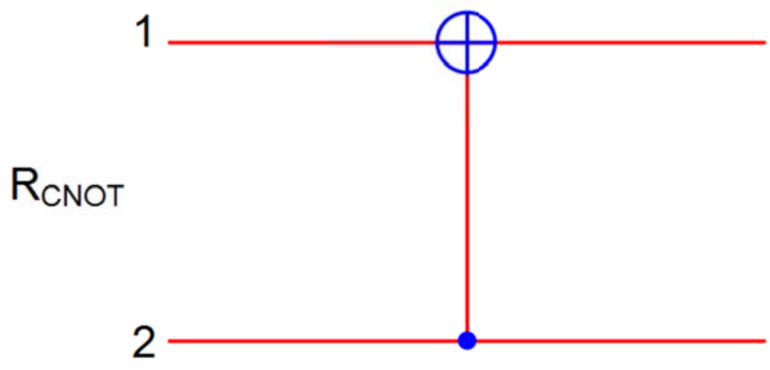


Fig. Quantum gate with $\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

17. Quantrum circuits related to the SWAP gate

(a) Swap gate with $\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$

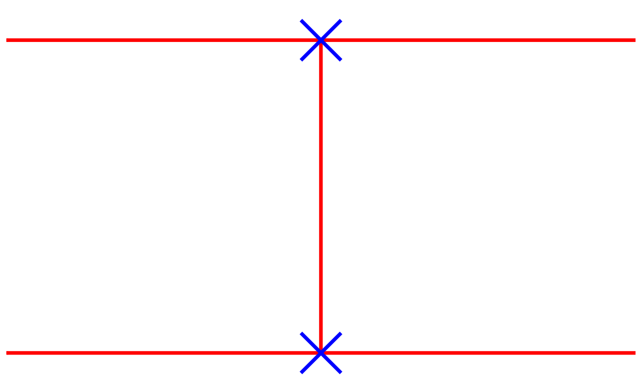


Fig. Swap gate with $\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$

$$\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Quantum circuit $\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$

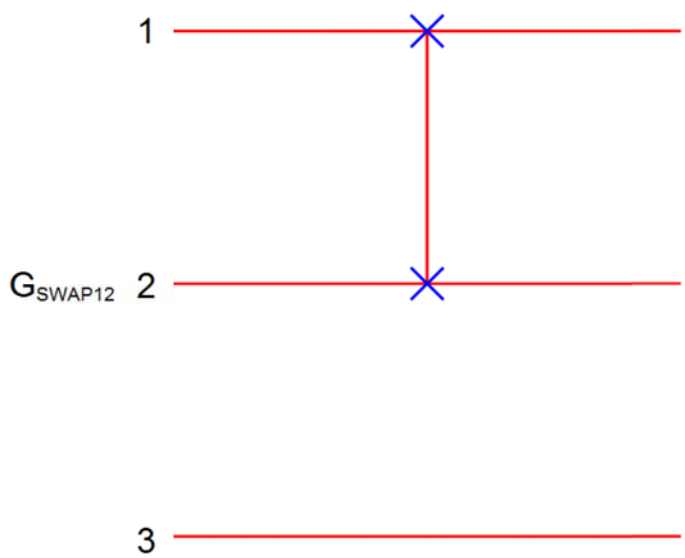


Fig. Quantum circuit including Swap gate between 1 and 2. $\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$

$$\hat{G}_{SWAP12} = \hat{G}_{SWAP} \otimes \hat{1}$$
$$= \hat{m} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{1} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} \otimes \hat{1} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} \otimes \hat{1}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) Quantum circuit with $\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$

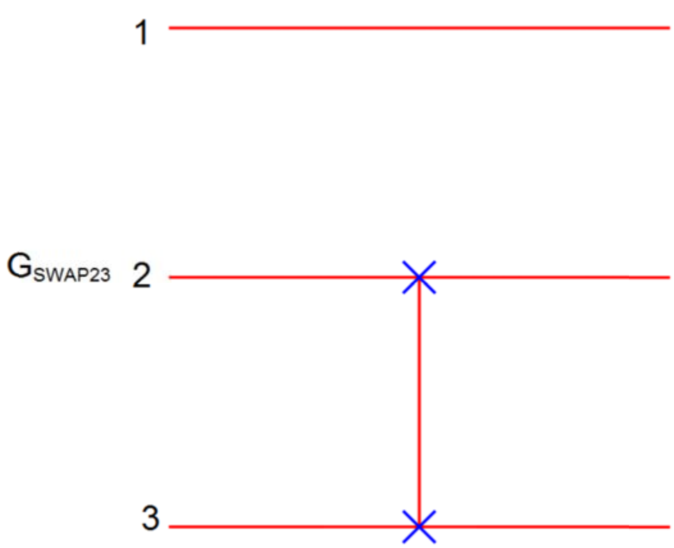


Fig. Quantum circuit including Swap gate between 2 and 3. $\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$

$$\hat{G}_{SWAP23} = \hat{1} \otimes \hat{G}_{SWAP}$$
$$= \hat{1} \otimes \hat{m} \otimes \hat{m} + \hat{1} \otimes \hat{n} \otimes \hat{n} + \hat{1} \otimes \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{1} \otimes \hat{X}\hat{n} \otimes \hat{X}\hat{m}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(d) Quantum gate with $\hat{G}_{SWAP13}$

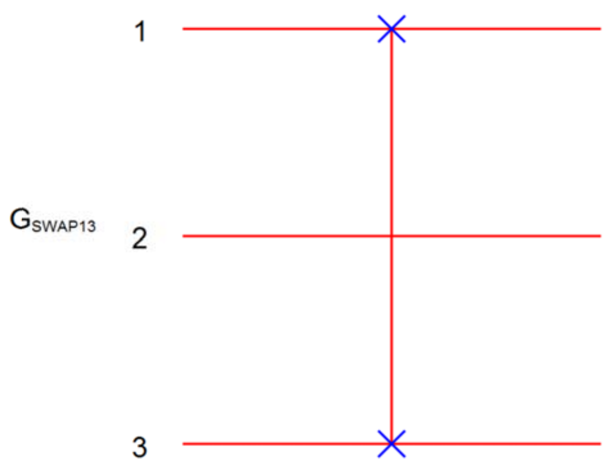


Fig. Quantum circuit including Swap gate between 1 and 3. $\hat{G}_{SWAP13}$

$$\hat{G}_{SWAP13} =$$

$$= \hat{m}\otimes\hat{1}\otimes\hat{m} + \hat{n}\otimes\hat{1}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{1}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{1}\otimes\hat{X}\hat{m}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

18. Matrix representation of typical tri-qubits

$$\hat{1}\otimes\hat{1}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{m}\otimes\hat{1}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{m}\otimes\hat{m}\otimes\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{m}\otimes\hat{m}\otimes\hat{m} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\hat{1}\otimes\hat{1}\otimes\hat{m}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0\end{pmatrix},\quad \hat{1}\otimes\hat{m}\otimes\hat{m}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\hat{1}\otimes\hat{m}\otimes\hat{n}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

$$\hat{n}\otimes\hat{1}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},\quad \hat{n}\otimes\hat{n}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix},$$

$$\hat{n}\otimes\hat{n}\otimes\hat{n}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}$$

$$\hat{m}\otimes\hat{n}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix},\quad \hat{n}\otimes\hat{m}\otimes\hat{1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}$$

19 Quantum gates related to the Toffoli gate

(a) Quantum gate $\hat{G}_{toffoli}$

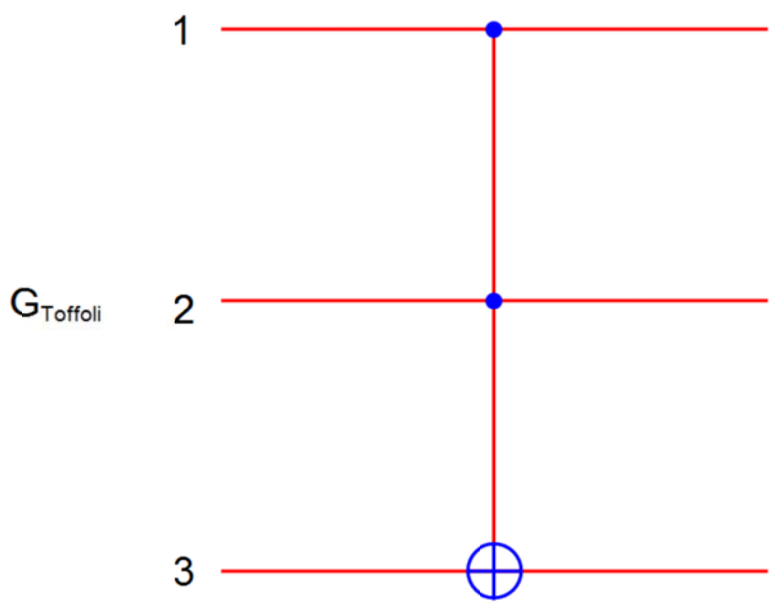


Fig. Toffoli gate with $\hat{G}_{toffoli} == \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X}$

$$\hat{G}_{toffoli} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{X}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{X} \end{pmatrix}$$

(b) Quantum gate with $\hat{R}_{toffoli}$

In the expression of

$$\hat{G}_{toffoli} = \hat{m}_1 \otimes \hat{1}_2 \otimes \hat{1}_3 + \hat{n}_1 \otimes \hat{m}_2 \otimes \hat{1}_3 + \hat{n}_1 \otimes \hat{n}_2 \otimes \hat{X}_3$$

we change the number of subscript as 1→3, 2→2, 3→1,

$$\hat{m}_3 \otimes \hat{1}_2 \otimes \hat{1}_1 + \hat{n}_3 \otimes \hat{m}_2 \otimes \hat{1}_1 + \hat{n}_3 \otimes \hat{n}_2 \otimes \hat{X}_1$$

This can be rewrirren as

$$\begin{aligned} \hat{R}_{toffoli} &= \hat{1}_1 \otimes \hat{1}_2 \otimes \hat{m}_3 + \hat{1}_1 \otimes \hat{m}_2 \otimes \hat{n}_3 + \hat{X}_1 \otimes \hat{n}_2 \otimes \hat{n}_3 \\ &= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{m} \otimes \hat{n} + \hat{X} \otimes \hat{n} \otimes \hat{n} \end{aligned}$$

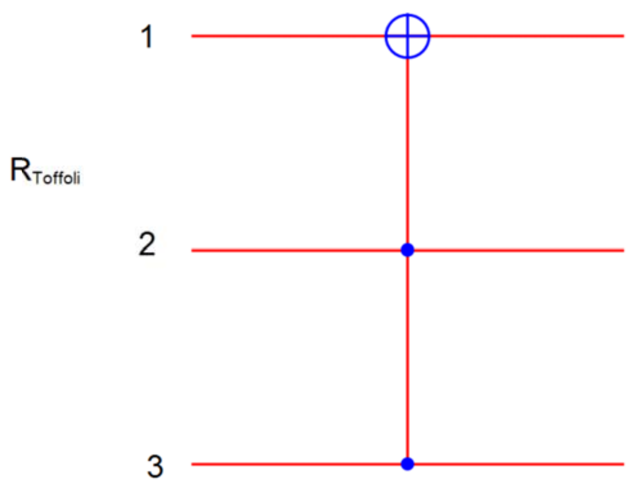


Fig. Quantum gate $\hat{R}_{toffoli} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{m} \otimes \hat{n} + \hat{X} \otimes \hat{n} \otimes \hat{n}$

We note that

$$\hat{R}_{toffoli} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{m} \otimes \hat{n} + \hat{X} \otimes \hat{n} \otimes \hat{n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

20. Quantum gate related to the Fredkin gate

(a) Quantum gate with $\hat{G}_{Fredkin} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{G}_{SWAP}$

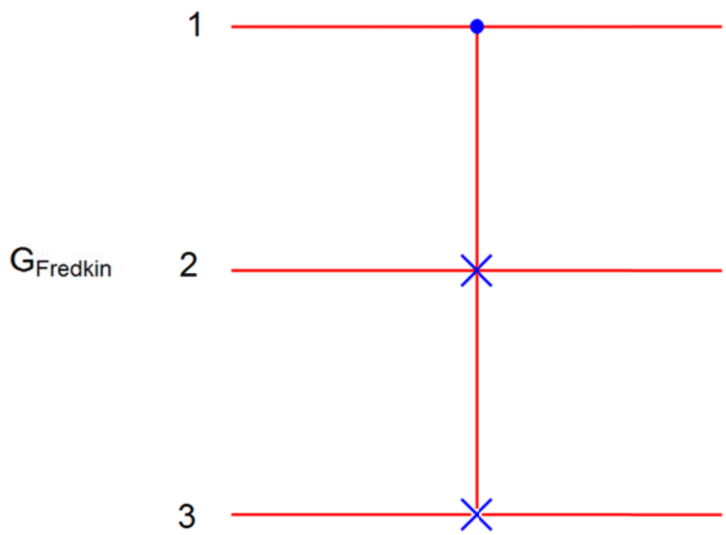


Fig. Fredkin gate with $\hat{G}_{Fredkin} = \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{G}_{SWAP}$,

$$
\begin{aligned}
\hat{G}_{Fredkin} &= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{G}_{SWAP} \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes(\hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m}) \\
&= \hat{m}\otimes\hat{1}\otimes\hat{1} + \hat{n}\otimes\hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n}\otimes\hat{n} + \hat{n}\otimes\hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{n}\otimes\hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{G}_{SWAP} &= \hat{m}\otimes\hat{m} + \hat{n}\otimes\hat{n} + \hat{X}\hat{m}\otimes\hat{X}\hat{n} + \hat{X}\hat{n}\otimes\hat{X}\hat{m} \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

(b) Quantum gate with $\hat{R}_{Fredkin}$

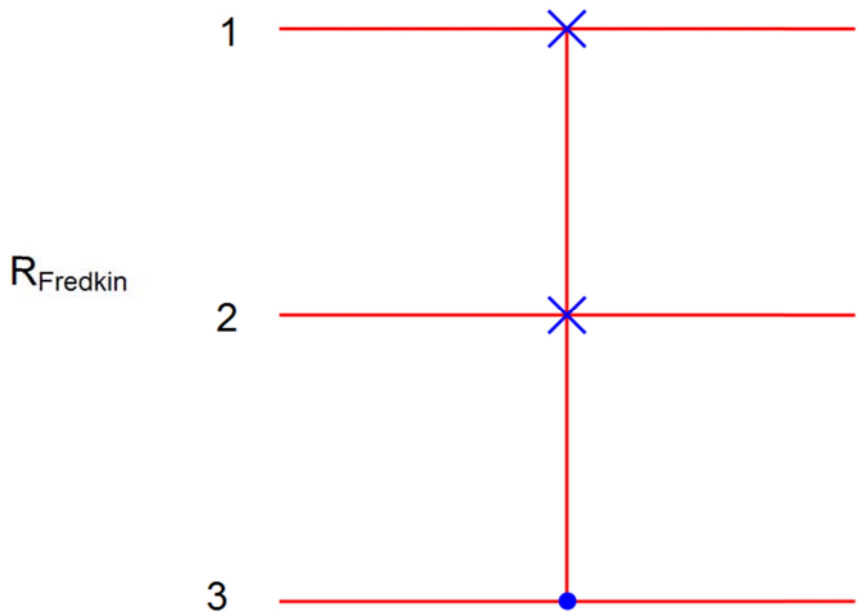
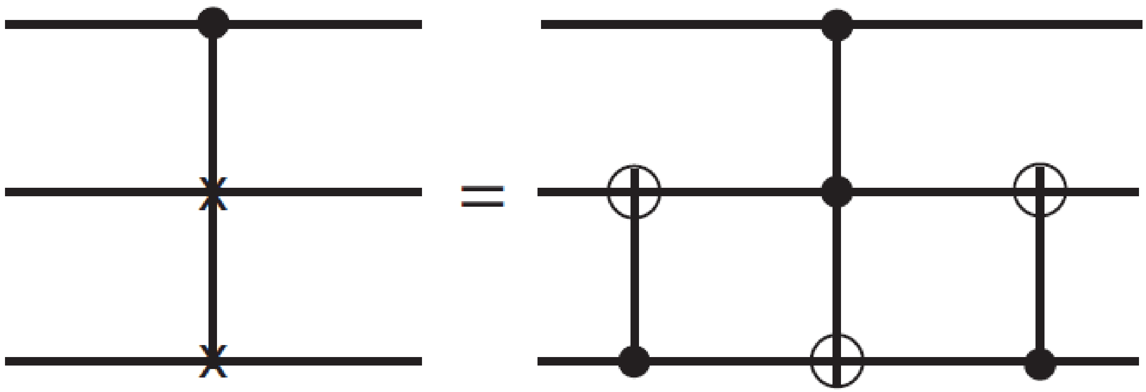


Fig. Modified Fredkin gate with $\hat{R}_{Fredkin} = \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{G}_{SWAP} \otimes \hat{n}$

$$
\begin{aligned}
\hat{R}_{Fredkin} &= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{G}_{SWAP} \otimes \hat{n} \\
&= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{m} \otimes \hat{m} \otimes \hat{n} + \hat{n} \otimes \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} \otimes \hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} \otimes \hat{n} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

21. Quantum circuit equivalence to the Fredkin gate



$$
\begin{aligned}
\hat{R}_{CNOT23}\cdot\hat{G}_{Toffoli}\cdot\hat{R}_{CNOT23} &= (\hat{1}\otimes\hat{1}\otimes\hat{m}+\hat{1}\otimes\hat{X}\otimes\hat{n})\cdot(\hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{n}\otimes\hat{X})\\
&\quad\cdot(\hat{1}\otimes\hat{1}\otimes\hat{m}+\hat{1}\otimes\hat{X}\otimes\hat{n})\\
&= (\hat{1}\otimes\hat{1}\otimes\hat{m}+\hat{1}\otimes\hat{X}\otimes\hat{n})\cdot(\hat{m}\otimes\hat{1}\otimes\hat{m}+\hat{m}\otimes\hat{X}\otimes\hat{n}\\
&\quad+\hat{n}\otimes\hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{m}\hat{X}\otimes\hat{n}+\hat{n}\otimes\hat{n}\otimes\hat{X}\hat{m}+\hat{n}\otimes\hat{n}\hat{X}\otimes\hat{X}\hat{n})\\
&= (\hat{m}\otimes\hat{1}\otimes\hat{m}^2+\hat{m}\otimes\hat{X}\otimes\hat{m}\hat{n}+\hat{n}\otimes\hat{m}\otimes\hat{m}^2+\hat{n}\otimes\hat{n}\otimes\hat{m}\hat{X}\hat{m}\\
&\quad+\hat{n}\otimes\hat{n}\hat{X}\otimes\hat{m}\hat{X}\hat{n}+\hat{m}\otimes\hat{X}\otimes\hat{n}\hat{m}+\hat{m}\otimes\hat{X}^2\otimes\hat{n}^2+n\otimes Xm\otimes\hat{n}\hat{m}\\
&\quad+\hat{n}\otimes\hat{X}\hat{n}\otimes\hat{n}\hat{X}\hat{m}+\hat{n}\otimes\hat{X}\hat{n}\hat{X}\otimes\hat{n}\hat{X}\hat{n})\\
&= \hat{m}\otimes\hat{1}\otimes\hat{m}++\hat{n}\otimes\hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{n}\otimes\hat{m}\hat{n}\hat{X}\\
&\quad+\hat{n}\otimes\hat{n}\hat{X}\otimes\hat{m}^2\hat{X}+\hat{m}\otimes\hat{1}\otimes\hat{n}\\
&\quad+\hat{n}\otimes\hat{X}\hat{n}\otimes\hat{n}^2\hat{X}+\hat{n}\otimes\hat{X}\hat{n}\hat{X}\otimes\hat{n}\hat{m}\hat{X}\\
&= \hat{m}\otimes\hat{1}\otimes\hat{1}+\hat{n}\otimes\hat{m}\otimes\hat{m}+\hat{n}\otimes\hat{n}\otimes\hat{n}+\hat{n}\otimes\hat{X}\hat{n}\otimes\hat{X}\hat{m}++\hat{n}\otimes\hat{X}\hat{m}\otimes\hat{X}\hat{n}
\end{aligned}
$$

where

$$\hat{m}\hat{n}=0\,,\qquad \hat{n}\hat{m}=0\,,\qquad \hat{m}^2=\hat{m}\,,\qquad \hat{n}^2=\hat{n}\,,\qquad \hat{X}^2=\hat{1}\,,$$

$$\hat{m}+\hat{n}=\hat{1}\,,$$

$$\hat{n}\hat{X}=\hat{X}\hat{m}\,,\qquad \hat{m}\hat{X}=\hat{X}\hat{n}\,,$$

and

$$\hat{R}_{CNOT\,23} = \hat{1} \otimes \hat{R}_{CNOT}$$
$$= \hat{1} \otimes \hat{1} \otimes \hat{m} + \hat{1} \otimes \hat{X} \otimes \hat{n}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\hat{G}_{toffoli} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{n} \otimes \hat{X}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus we have

$$\hat{R}_{CNOT\,23} \cdot \hat{G}_{Toffoli} \cdot \hat{R}_{CNOT\,23} = \hat{G}_{Fredkin}$$

where

$$\hat{G}_{Fredkin} = \hat{m} \otimes \hat{1} \otimes \hat{1} + \hat{n} \otimes \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} \otimes \hat{n} + \hat{n} \otimes \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{n} \otimes \hat{X}\hat{n} \otimes \hat{X}\hat{m}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

22. Quantum circuit $\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT}$ equivalent to $\hat{G}_{SWAP}$

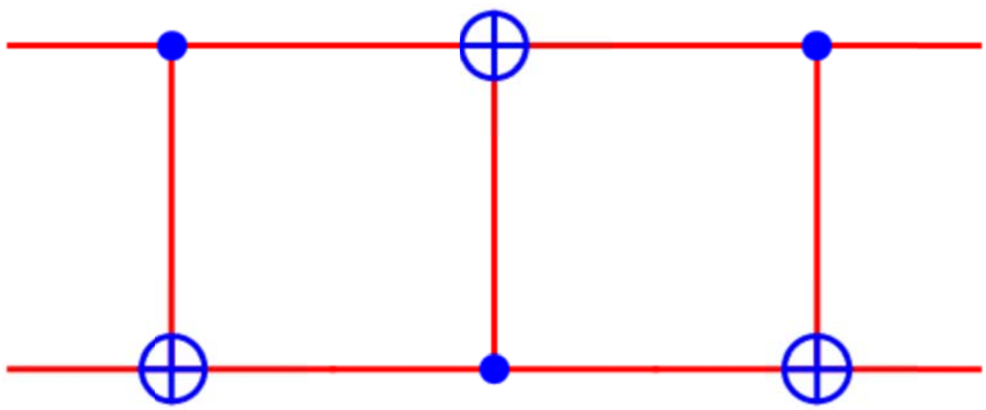


Fig.a Quantum circuit with $\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT}$.

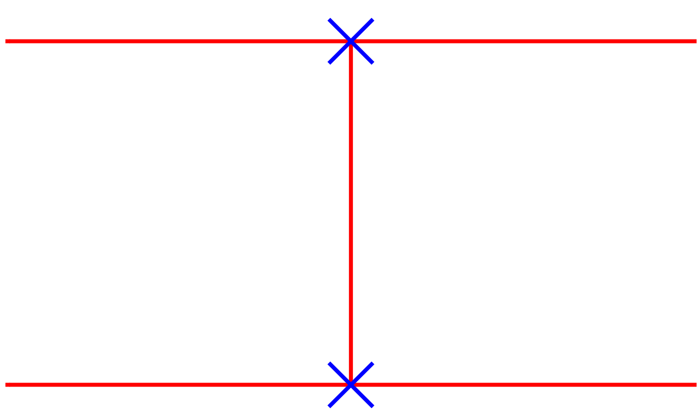


Fig.b Swap gate with $\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}$.

We show that this quantum circuit is equivalent to the SWAP gate. We note that

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n},$$

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X}.$$

Then we get

$$\begin{aligned}\hat{G}_{CNOT}\hat{R}_{CNOT}\hat{G}_{CNOT} &= (\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X})(\hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n})(\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X}) \\ &= \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m} + \hat{X}\hat{m} \otimes \hat{X}\hat{n}\end{aligned}$$

which is the same as

$$\hat{G}_{SWAP} = \hat{m} \otimes \hat{m} + \hat{n} \otimes \hat{n} + \hat{X}\hat{m} \otimes \hat{X}\hat{n} + \hat{X}\hat{n} \otimes \hat{X}\hat{m}.$$

23. Equivalent quantum circuits: $(\hat{X} \otimes \hat{1})\hat{G}_{CNOT}(\hat{X} \otimes \hat{1}) = \hat{n} \otimes \hat{1} + \hat{m} \otimes \hat{X}$

We show that these two quantum circuits are equivalent to each other.

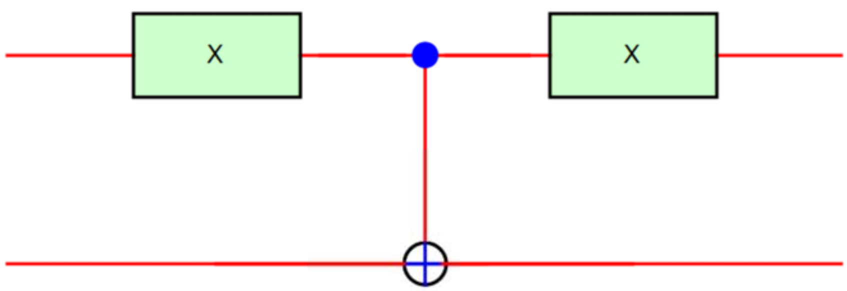


Fig.a Quantum circuit with $(\hat{X} \otimes \hat{1})\hat{G}_{CNOT}(\hat{X} \otimes \hat{1})$

which is equivalent to a new type of quantum gate

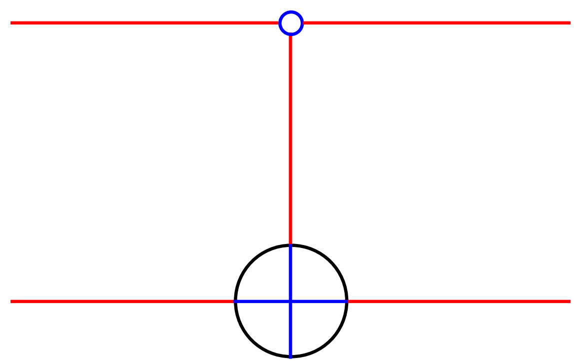


Fig.b Quantum circuit with $\hat{n}\otimes\hat{1}+\hat{m}\otimes\hat{X}$. Controlled operation with a NOT gate being performed on the second qubit, conditional on the first qubit being set to zero.

We note that

$$\hat{G}_{CNOT}=\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X}\ ,$$

Then we have

$$\begin{aligned}(\hat{X}\otimes\hat{1})\hat{G}_{CNOT}(\hat{X}\otimes\hat{1}) &= (\hat{X}\otimes\hat{1})(\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X})(\hat{X}\otimes\hat{1})\\ &= (\hat{X}\otimes\hat{1})(\hat{m}\hat{X}\otimes\hat{1}+\hat{n}\hat{X}\otimes\hat{X})\\ &= \hat{X}\hat{m}\hat{X}\otimes\hat{1}+\hat{X}\hat{n}\hat{X}\otimes\hat{X}\\ &= \hat{n}\hat{X}^2\otimes\hat{1}+\hat{m}\hat{X}^2\otimes\hat{X}\\ &= \hat{n}\otimes\hat{1}+\hat{m}\otimes\hat{X}\end{aligned}$$

In fact we obtain

$$(\hat{X}\otimes\hat{1})\hat{G}_{CNOT}(\hat{X}\otimes\hat{1})=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

24. Equivalence of two quantum circuits: $(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H})=\hat{G}_X$

We show that these two quantum circuits are equivalent to each other.

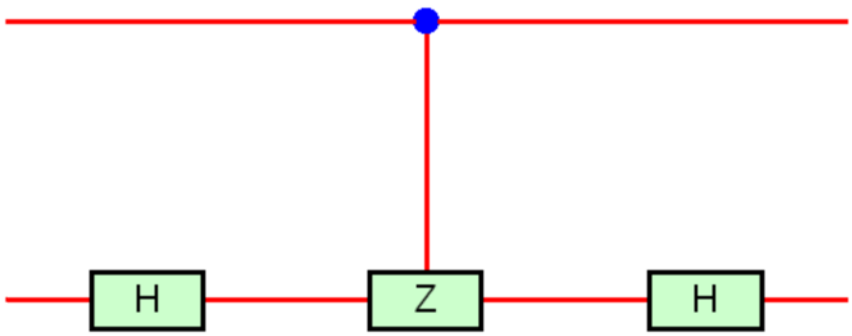


Fig.a Quantum circuit with $(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H})$.

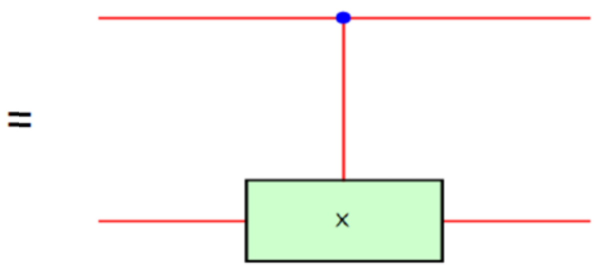


Fig.b Equivalent quantum circuit with $\hat{G}_X$

$$\hat{G}_Z = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{Z}\,, \qquad \hat{G}_X = \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X}\,.$$

The quantum circuit can be represented by

$$\begin{aligned}(\hat{1}\otimes\hat{H})\hat{G}_Z(\hat{1}\otimes\hat{H}) &= (\hat{1}\otimes\hat{H})(\hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{Z})(\hat{1}\otimes\hat{H}) \\ &= (\hat{1}\otimes\hat{H})(\hat{m}\otimes\hat{H} + \hat{n}\otimes\hat{Z}\hat{H}) \\ &= \hat{m}\otimes\hat{H}^2 + \hat{n}\otimes\hat{H}\hat{Z}\hat{H} \\ &= \hat{m}\otimes\hat{1} + \hat{n}\otimes\hat{X} \\ &= \hat{G}_X\end{aligned}$$

Note that

$$\hat{H}^2 = \frac{1}{2}(\hat{X}+\hat{Z})(\hat{X}+\hat{Z}) = \frac{1}{2}(\hat{X}^2 + \hat{Z}^2 + \hat{X}\hat{Z} + \hat{Z}\hat{X}) = \hat{1}$$

$$
\begin{aligned}
\hat{H}\hat{Z}\hat{H} &= \frac{1}{2}(\hat{X}+\hat{Z})\hat{Z}(\hat{X}+\hat{Z}) \\
&= \frac{1}{2}(\hat{X}+\hat{Z})(\hat{Z}\hat{X}+\hat{Z}^2) \\
&= \frac{1}{2}(\hat{X}+\hat{Z})(i\hat{Y}+\hat{1}) \\
&= \frac{1}{2}(i\hat{X}\hat{Y}+\hat{X}+i\hat{Z}\hat{Y}+\hat{Z}) \\
&= \hat{X}
\end{aligned}
$$

where

$$
\hat{X}\hat{Y} = -\hat{Y}\hat{X} = i\hat{Z}\,, \qquad \hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = i\hat{X}\,, \qquad \hat{Z}\hat{X} = -\hat{X}\hat{Z} = i\hat{Y}
$$

25. Equivalence of two quantum circuits; $(\hat{H}\otimes\hat{H})\hat{G}_{CNOT}(\hat{H}\otimes\hat{H}) = \hat{R}_{CNOT}$

We show that these two quantum circuits are equivalent to each other.

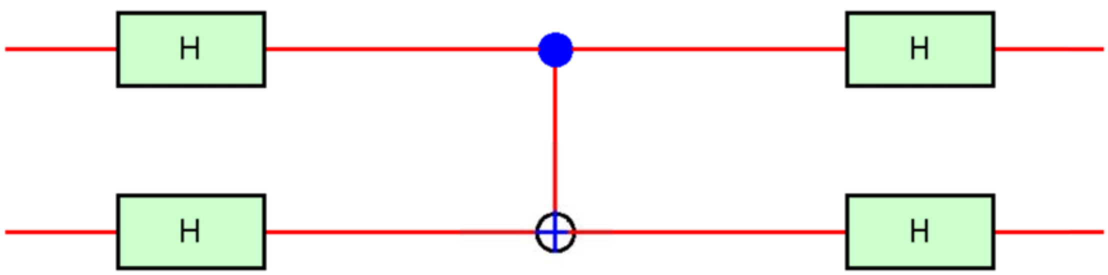


Fig.a Quantum circuit with $(\hat{H}\otimes\hat{H})\hat{G}_{CNOT}(\hat{H}\otimes\hat{H})$.

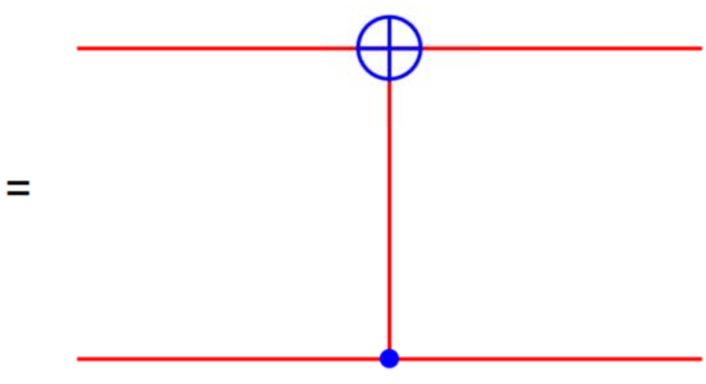


Fig.b Equivalent quantum circuit with $\hat{R}_{CNOT} = \hat{1}\otimes\hat{m} + \hat{X}\otimes\hat{n}$.

We note that

$$\hat{G}_{CNOT} = \hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X} , \qquad \hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n} .$$

The quantum circuit (Fig.a) is expressed by

$$\begin{aligned}(\hat{H} \otimes \hat{H})\hat{G}_{CNOT}(\hat{H} \otimes \hat{H}) &= (\hat{H} \otimes \hat{H})(\hat{m} \otimes \hat{1} + \hat{n} \otimes \hat{X})(\hat{H} \otimes \hat{H}) \\ &= (\hat{H} \otimes \hat{H})(\hat{m}\hat{H} \otimes \hat{H} + \hat{n}\hat{H} \otimes \hat{X}\hat{H}) \\ &= \hat{H}\hat{m}\hat{H} \otimes \hat{H}^2 + \hat{H}\hat{n}\hat{H} \otimes \hat{H}\hat{X}\hat{H} \\ &= \hat{H}\hat{m}\hat{H} \otimes \hat{1} + \hat{H}\hat{n}\hat{H} \otimes \hat{H}\hat{X}\hat{H} \\ &= \hat{H}\hat{m}\hat{H} \otimes \hat{1} + \hat{H}\hat{n}\hat{H} \otimes \hat{Z}\end{aligned}$$

or

$$\begin{aligned}(\hat{H} \otimes \hat{H})\hat{G}_{CNOT}(\hat{H} \otimes \hat{H}) &= \frac{1}{2}(\hat{1} + \hat{X}) \otimes \hat{1} + \frac{1}{2}(\hat{1} - \hat{X}) \otimes \hat{Z} \\ &= \frac{1}{2}(\hat{1} \otimes \hat{1} + \hat{X} \otimes \hat{1} + \hat{1} \otimes \hat{Z} - \hat{X} \otimes \hat{Z}) \\ &= \hat{1} \otimes \frac{1}{2}(\hat{1} + \hat{Z}) + \hat{X} \otimes \frac{1}{2}(\hat{1} - \hat{Z}) \\ &= \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n}\end{aligned}$$

This agrees with

$$\hat{R}_{CNOT} = \hat{1} \otimes \hat{m} + \hat{X} \otimes \hat{n} .$$

((**Note**))

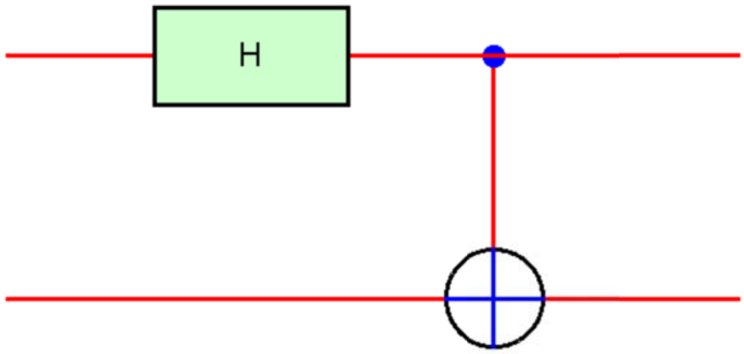
$$\hat{H}\hat{X}\hat{H} = \hat{Z} , \qquad \hat{H}\hat{Z}\hat{H} = \hat{X} ,$$

$$\hat{m} = \frac{1}{2}(\hat{1} + \hat{Z}) , \qquad \hat{n} = \frac{1}{2}(\hat{1} - \hat{Z}) .$$

$$\hat{H}\hat{m}\hat{H} = \frac{1}{2}\hat{H}(\hat{1} + \hat{Z})\hat{H} = \frac{1}{2}(\hat{H}^2 + \hat{H}\hat{Z}\hat{H}) = \frac{1}{2}(\hat{1} + \hat{X}) ,$$

$$\hat{H}\hat{n}\hat{H} = \frac{1}{2}\hat{H}(\hat{1} - \hat{Z})\hat{H} = \frac{1}{2}(\hat{H}^2 - \hat{H}\hat{Z}\hat{H}) = \frac{1}{2}(\hat{1} - \hat{X}) .$$

26. Construction of the Bell's states



Entangled qubits

$$\hat{G}_{CNOT}(\hat{H}\otimes\hat{1}) = (\hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X})(\hat{H}\otimes\hat{1})$$
$$= \hat{m}\hat{H}\otimes\hat{1}+\hat{n}\hat{H}\otimes\hat{X}$$
$$= \frac{1}{\sqrt{2}}[(\hat{1}+\hat{Z})\hat{H}\otimes\hat{1}+(\hat{1}-\hat{Z})\hat{H}\otimes\hat{X}]$$

where

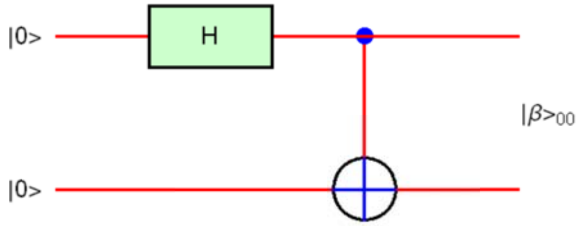
$$\hat{G}_{CNOT} = \hat{m}\otimes\hat{1}+\hat{n}\otimes\hat{X},$$

$$\hat{H} = \frac{1}{\sqrt{2}}(\hat{X}+\hat{Z}), \qquad \hat{Z}\hat{X} = i\hat{Y},$$

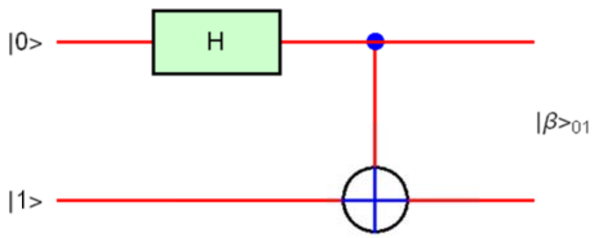
$$\hat{m}\hat{H} = \frac{1}{\sqrt{2}}(\hat{1}+\hat{Z})\hat{H}, \qquad \hat{H}\hat{n} = \frac{1}{\sqrt{2}}\hat{H}(\hat{1}-\hat{Z})$$

The matrix of $\hat{G}_{CNOT}(\hat{H}\otimes\hat{1})$ is given by

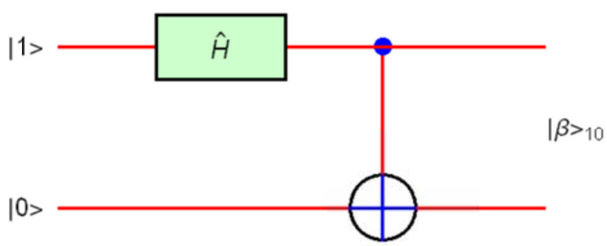
$$\hat{G}_{CNOT}(\hat{H}\otimes\hat{1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$



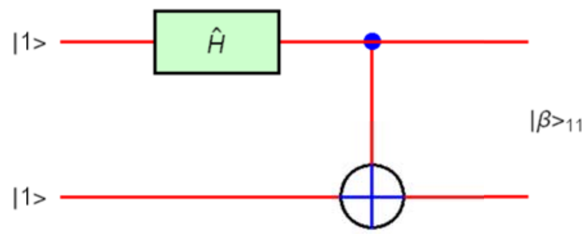
$$|\beta\rangle_{00} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$



$$|\beta\rangle_{01} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$$



$$|\beta\rangle_{10} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$$



$$|\beta\rangle_{11} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

REFERENCES

R.P. Feynman, Feynman Lectures on Computation (Addison-Wesley, 1996).

G.J. Milburn, The Feynman Processor (Perseus Books, 1998).

G.P. Berman, G.D. Doolen, R. Mainieri, and V.I. Tsifrinovich, Introduction to Quantum Computers (World Scientific, 1998).

J. Stolze and D. Suter, Quantum Computing A Short Course from Theory to Experiment (Wiley-VCH, 2004).

G. Bennet, G. Casati, and G. Strini, Principles of Quantum Computation and Information vol. I: Basic Concepts (World Scientific, 2005).

D.C. Marinescu and G.M. Marinescu, Approaching Quantum Computing (Peason Education, Inc., 2005).

M. Hayashi, Quantum Information An Introduction (Springer, 2006).

M. Pavičić, Quantum Computation and Quantum Communication: Theory and Experiments (Springer 2006).

P. Kaye, R. Laflamme, and M. Mosca, An Introduction to Quantum Computing (Oxford, 2007).

M.L. Bellac, A Short Introduction to Quantum Information (Oxford University Press, 2006).

N.D. Mermin, Quantum Computer Science An Introduction (Cambridge, 2007)

L. Diosi, A Short Course in Quantum Information Theory (Springer, 2007).

G. Bennet, G. Casati, and G. Strini, Principles of Quantum Computation and Information vol. II: Basic Tools and Special Topics (World Scientific, 2007).

G. Jaeger, Quantum Information An Overview (Springer, 2007).

D. McMahon, Quantum Computing Explained (Wiley-Interscience, 2008)

Z. Meglicki, Quantum Computing without Magic Devices (MIT Press, 2008).

N.S. Yanofsky and M.A. Mannucci, Quantum Computing for Computer Scientists (Cambridge, 2008).

S.M. Barnett, Quantum Information (Oxford University Press, 2009).

A.D. Vos Reversible Computing Fundamentals, Quantum Computing, and Applications (Wiley-VCH, 2010).
M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge, 2010).
D.C. Marinescu and G.M. Marinescu, Classical and Quantum Information (Elsevier, 2010).
E. Rieffel and W. Polak, Quantum Computing A General Introduction (The MIT Press, 2011).
M. Fayngold and V. Fayngold, Quantum Mechanics and Quantum Information (Wiley-VCH, 2013).
S. Aaronson, Quantum Computing since Democritus (Cambridge University Press, 2013).

Jacques Salomon Hadamard;(8 December 1865 – 17 October 1963) was a French mathematician who made major contributions in number theory, complex function theory, differential geometry and partial differential equations.

Tommaso Toffoli is a professor of electrical and computer engineering at Boston University where he joined the faculty in 1995. He has worked on cellular automata and the theory of artificial life (with Edward Fredkin and others), and is known for the invention of the Toffoli gate.

Edward Fredkin (born 1934) is a distinguished career professor at Carnegie Mellon University (CMU) and an early pioneer of digital physics. His primary contributions include his work on reversible computing and cellular automata. While Konrad Zuse's book, *Calculating Space* (1969), mentioned the importance of reversible computation, the Fredkin gate represented the essential breakthrough. In recent work, he uses the term digital philosophy (DP). During his career Fredkin also served on the faculties of MIT in Computer Science, was a Fairchild Distinguished Scholar at Caltech, and Research Professor of Physics at Boston University.

David Elieser Deutsch, FRS (born 18 May 1953) is a British physicist at the University of Oxford. He is a non-stipendiary Visiting Professor in the Department of Atomic and Laser Physics at the Centre for Quantum Computation (CQC) in the Clarendon Laboratory of the University of Oxford. He pioneered the field of quantum computation by formulating a description for a quantum Turing machine, as well as specifying an algorithm designed to run on a quantum computer. He is a proponent of the many-worlds interpretation of quantum mechanics.
