Lecture Note
Rotation matrix
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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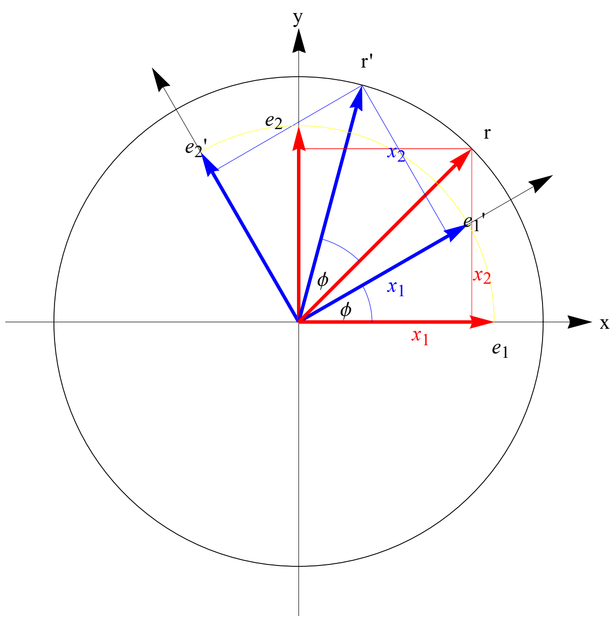
1. 2D rotation matrix

Suppose that the vector $\boldsymbol{r}$ is rotated through θ (counter-clock wise) around the z axis. The position vector $\boldsymbol{r}$ is changed into $\boldsymbol{r}'$ in the same orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.



In this Fig, we have

$$\mathbf{e}_1 \cdot \mathbf{e}_1' = \cos\phi \qquad \mathbf{e}_2 \cdot \mathbf{e}_1' = \sin\phi$$
$$\mathbf{e}_1 \cdot \mathbf{e}_2' = -\sin\phi, \qquad \mathbf{e}_2 \cdot \mathbf{e}_2' = \cos\phi$$

We define $\boldsymbol{r}$ and $\boldsymbol{r}'$ as

$$\boldsymbol{r}' = x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2 = x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2',$$

and

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2.$$

Using the relation

$$\boldsymbol{e}_1 \cdot \boldsymbol{r}' = \boldsymbol{e}_1 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')$$
$$\boldsymbol{e}_2 \cdot \boldsymbol{r}' = \boldsymbol{e}_2 \cdot (x_1'\boldsymbol{e}_1 + x_2'\boldsymbol{e}_2) = \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2')$$

we have

$$x_1' = \boldsymbol{e}_1 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') = x_1 \cos\phi - x_2 \sin\phi$$
$$x_2' = \boldsymbol{e}_2 \cdot (x_1\boldsymbol{e}_1' + x_2\boldsymbol{e}_2') = x_1 \sin\phi + x_2 \cos\phi$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \Re(\phi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

((**Note**))
Rotation around the *z* axis in the complex plane

$$x' + iy' = e^{i\phi}(x+iy) = (\cos\phi + i\sin\phi)(x+iy) = x\cos\phi - y\sin\phi + i(x\sin\phi + y\cos\phi)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

2. 3D rotation matrix

We discuss the three-dimensional (3D) case,

$$\mathbf{r} = \sum_{j=1}^{3} x_j \mathbf{e}_j , \qquad \boldsymbol{r}' = \sum_{j=1}^{3} x_j' \boldsymbol{e}_j = \sum_{j=1}^{3} x_j \boldsymbol{e}_j'$$

$$\boldsymbol{r}' = \Re_z(\phi)\boldsymbol{r} = \Re_z(\phi)(\sum_{j=1}^{3} x_j \boldsymbol{e}_j) = \sum_{j=1}^{3} x_j \Re_z(\phi)\boldsymbol{e}_j = \sum_{j=1}^{3} x_j \boldsymbol{e}_j'$$

where

$$\Re_z(\phi)\mathbf{e}_j = \mathbf{e}_j'$$

Thus we have

$$(\sum_{j=1}^{3} x_j \boldsymbol{e}_j') \cdot \boldsymbol{e}_i = (\sum_{j=1}^{3} x_j' \boldsymbol{e}_j) \cdot \boldsymbol{e}_i$$

or

$$\sum_{j=1}^{3} x_j' \delta_{j,i} = x_i' = \sum_{j=1}^{3} (\boldsymbol{e}_i \cdot \boldsymbol{e}_j') x_j = \sum_{j=1}^{3} \Re_{ij} x_j$$

where

$$\Re_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j'$$

(i) The rotation around the z axis.

$$\Re_{11} = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1' = \cos\phi, \quad \Re_{12} = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2' = -\sin\phi, \quad \Re_{13} = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3' = 0$$
$$\Re_{21} = \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' = \sin\phi, \quad \Re_{22} = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2' = \cos\phi, \quad \Re_{23} = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3' = 0$$
$$\Re_{31} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1' = 0, \quad \Re_{32} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_2' = 0, \quad \Re_{33} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3' = 1$$

$$\boldsymbol{r}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \Re_{11} & \Re_{12} & \Re_{13} \\ \Re_{21} & \Re_{22} & \Re_{23} \\ \Re_{31} & \Re_{32} & \Re_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$\Re_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Re_z(\Delta\phi) = \begin{pmatrix} \cos\Delta\phi & -\sin\Delta\phi & 0 \\ \sin\Delta\phi & \cos\Delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{(\Delta\phi)^2}{2} & -\Delta\phi & 0 \\ \Delta\phi & 1 - \frac{(\Delta\phi)^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) Rotation around the x axis

$$\Re_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

$$\Re_x(\Delta\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Delta\phi & -\sin\Delta\phi \\ 0 & \sin\Delta\phi & \cos\Delta\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{(\Delta\phi)^2}{2} & -\Delta\phi \\ 0 & \Delta\phi & 1-\frac{(\Delta\phi)^2}{2} \end{pmatrix}$$

(iii) Rotation around the y axis

$$\Re_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

$$\Re_y(\Delta\phi) = \begin{pmatrix} \cos\Delta\phi & 0 & \sin\Delta\phi \\ 0 & 1 & 0 \\ -\sin\Delta\phi & 0 & \cos\Delta\phi \end{pmatrix} = \begin{pmatrix} 1-\frac{(\Delta\phi)^2}{2} & 0 & \Delta\phi \\ 0 & 1 & 0 \\ -\Delta\phi & 0 & 1-\frac{(\Delta\phi)^2}{2} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"]; Rx = ( 1   0        0      );
                        ( 0   Cos[ϕ]  -Sin[ϕ] )
                        ( 0   Sin[ϕ]   Cos[ϕ] )
Ry = ( Cos[ϕ]   0  Sin[ϕ] );
     ( 0        1  0      )
     ( -Sin[ϕ]  0  Cos[ϕ] )
Rz = ( Cos[ϕ]  -Sin[ϕ]  0 );
     ( Sin[ϕ]   Cos[ϕ]  0 )
     ( 0        0       1 )

Rx.Ry - Ry.Rx // Series[#, {ϕ, 0, 2}] & //
  Normal // MatrixForm
```

$$\begin{pmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
Ry.Rz - Rz.Ry // Series[#, {ϕ, 0, 2}] & //
  Normal // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\phi^2 \\ 0 & \phi^2 & 0 \end{pmatrix}$$

```
Rz.Rx - Rx.Rz // Series[#, {ϕ, 0, 2}] & //
  Normal // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & \phi^2 \\ 0 & 0 & 0 \\ -\phi^2 & 0 & 0 \end{pmatrix}$$

Then we have

$$\begin{aligned} \Re_x(\Delta\phi)\Re_y(\Delta\phi) - \Re_y(\Delta\phi)\Re_x(\Delta\phi) &= \begin{pmatrix} 0 & -(\Delta\phi)^2 & 0 \\ (\Delta\phi)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \Re_z((\Delta\phi)^2) - 1 \end{aligned}$$

in the limit of $\phi \to 0$. since

$$\Re_z(\phi^2) = \begin{pmatrix} \cos\phi^2 & -\sin\phi^2 & 0 \\ \sin\phi^2 & \cos\phi^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\phi^2 & 0 \\ \phi^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\phi^2 & 0 \\ \phi^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Example of the sequential geometrical rotation

Here we show an example of the geometrical rotations. Suppose that the initial vector is given by

$$\boldsymbol{r}_0 = \frac{1}{\sqrt{2}}(0,1,1)$$

which lies in the *y*-*z* plane. We apply the two kinds of rotation to the vector $\boldsymbol{r}_0$.

(a) First we apply the rotation $\Re_y(\theta = 0 \rightarrow \frac{\pi}{3})$ to $\boldsymbol{r}_0$.

$$\boldsymbol{r}_1 = \Re_y(\theta = 0 \rightarrow \frac{\pi}{3})\boldsymbol{r}_0$$

After that we apply the rotation $\Re_z(\phi = 0 \rightarrow \frac{\pi}{3})$ to $\boldsymbol{r}_1$.

$$\boldsymbol{r}_2 = \Re_z(\phi = 0 \rightarrow \frac{\pi}{3})\boldsymbol{r}_1$$

This procedure corresponds to the change of position vectors, $\boldsymbol{r}_0 \rightarrow \boldsymbol{r}_1 \rightarrow \boldsymbol{r}_2$

(b) First we apply the rotation $\Re_z(\phi = 0 \rightarrow \frac{\pi}{3})$ to $\boldsymbol{r}_0$.

$$\boldsymbol{r}_1' = \Re_z(\phi = 0 \rightarrow \frac{\pi}{3})\boldsymbol{r}_0$$

After that we apply the rotation $\Re_y(\theta = 0 \to \frac{\pi}{3})$ to $\boldsymbol{r}_1$'.

$$\boldsymbol{r}_2' = \Re_z(\theta = 0 \to \frac{\pi}{3})\boldsymbol{r}_1'$$

This procedure corresponds to the change of position vectors, $\boldsymbol{r}_0 \to \boldsymbol{r}_1' \to \boldsymbol{r}_2'$

((**Mathematica**))

Using the Mathematica, we can draw the process of the rotation as follows. It is clear that the position vector $\boldsymbol{r}_2$' is different from the position vector $\boldsymbol{r}_2$. This implies that

$$\boldsymbol{r}_2 = \Re_z(\frac{\pi}{3})\Re_y(\frac{\pi}{3})\boldsymbol{r}_0 \neq \boldsymbol{r}_2' = \Re_z(\frac{\pi}{3})\Re_y(\frac{\pi}{3})\boldsymbol{r}_0$$

or

$$\Re_z(\frac{\pi}{3})\Re_y(\frac{\pi}{3})\boldsymbol{r}_0 \neq \boldsymbol{r}_2' = \Re_y(\frac{\pi}{3})\Re_z(\frac{\pi}{3})\boldsymbol{r}$$

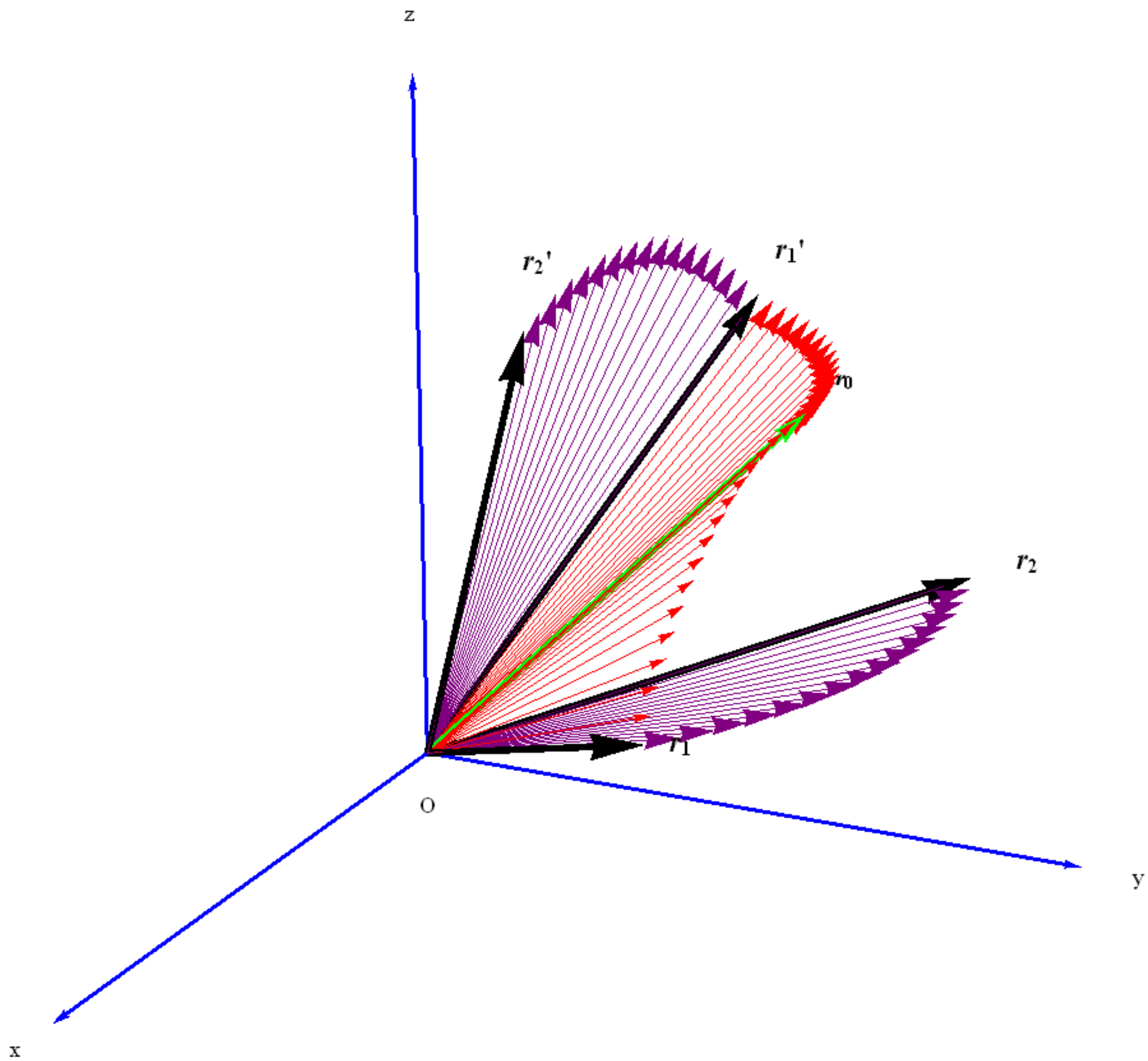
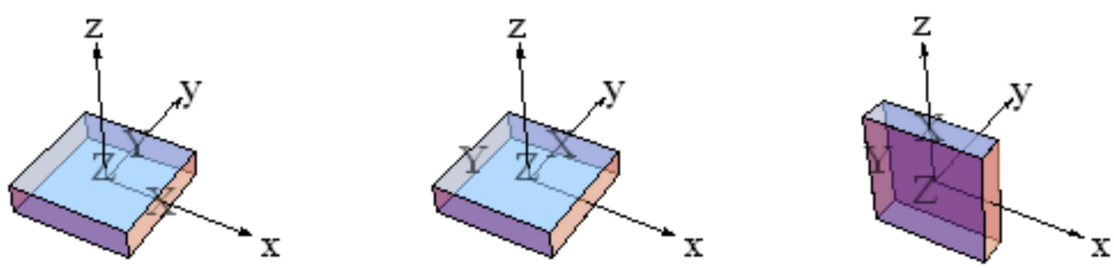


Fig. Example of the sequential geometrical rotations to explain the noncommunitivity. This procedure corresponds to the change of position vectors, $\boldsymbol{r}_0 \rightarrow \boldsymbol{r}_1 \rightarrow \boldsymbol{r}_2$ and , $\boldsymbol{r}_0 \rightarrow \boldsymbol{r}_1' \rightarrow \boldsymbol{r}_2'$.

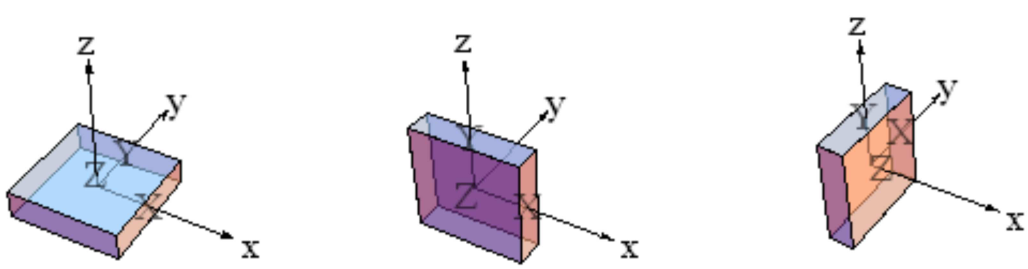
4. Non-communitivity of finite rotation

We show another example for the non-commutivity of sequential rotations.

Let us consider a 90° rotation around the z axis, denoted by $R(\frac{\pi}{2},\boldsymbol{k})$, followed by a 90° rotation around the x axis, denoted by $R(\frac{\pi}{2},\boldsymbol{i})$; compare this with a 90° rotation around the x axis, denoted by $R(\frac{\pi}{2},\boldsymbol{i})$, followed by a 90° rotation around the z axis, denoted by $R(\frac{\pi}{2},\boldsymbol{k})$. The net results are different.



(a) $R(\frac{\pi}{2},\boldsymbol{k}) \rightarrow$ $R(\frac{\pi}{2},\boldsymbol{i})$



(b) $R(\frac{\pi}{2},\boldsymbol{i}) \rightarrow$ $R(\frac{\pi}{2},\boldsymbol{k})$

((Note))

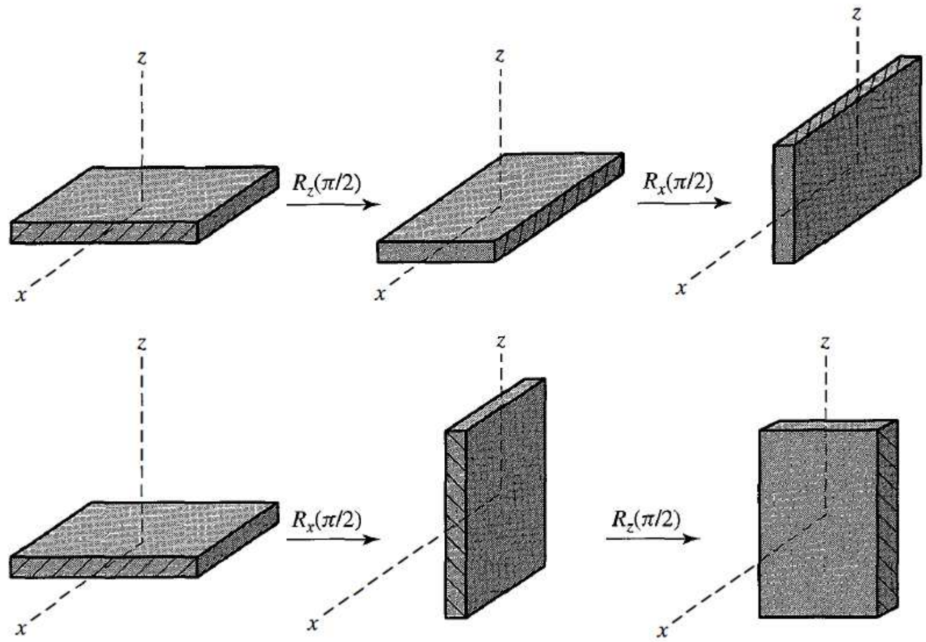


Fig. Example to illustrate the non-communitivity of finite rotations (Sakurai, 2011).

5. Rotation operator

The rotation operators is defined by

$$\hat{R}_x(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_x\phi), \qquad \text{[related to } \Re_x(\Delta\phi)\text{]}$$

$$\hat{R}_y(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_y\phi), \qquad \text{[related to } \Re_y(\Delta\phi)\text{]}$$

$$\hat{R}_z(\phi) = \exp(-\frac{i}{\hbar}\hat{J}_z\phi). \qquad \text{[related to } \Re_z(\Delta\phi)\text{]}$$

where $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ are angular momentum.

Using the relation

$$\Re_x(\Delta\phi)\Re_y(\Delta\phi) - \Re_y(\Delta\phi)\Re_x(\Delta\phi) = \begin{pmatrix} 0 & -(\Delta\phi)^2 & 0 \\ (\Delta\phi)^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \Re_z((\Delta\phi)^2) - 1$$

we get the relation for the rotation operator for the infinitesimal angle ϕ,

$$\hat{R}_x(\Delta\phi)\hat{R}_y(\Delta\phi) - \hat{R}_y(\Delta\phi)\hat{R}_x(\Delta\phi) = \hat{R}_z((\Delta\phi)^2) - 1.$$

Noting that

$$\hat{R}_x(\Delta\phi) = \hat{1} - \frac{i}{\hbar}\hat{J}_x\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_x\Delta\phi}{\hbar}\right)^2,$$

$$\hat{R}_y(\Delta\phi) = \hat{1} - \frac{i}{\hbar}\hat{J}_y\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_y\Delta\phi}{\hbar}\right)^2$$

$$\hat{R}_z(\Delta\phi) = \hat{1} - \frac{i}{\hbar}\hat{J}_z\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_z\Delta\phi}{\hbar}\right)^2$$

we have

$$\begin{aligned}
&[\hat{1} - \frac{i}{\hbar}\hat{J}_x\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_x\Delta\phi}{\hbar}\right)^2][\hat{1} - \frac{i}{\hbar}\hat{J}_y\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_y\Delta\phi}{\hbar}\right)^2] \\
&-[\hat{1} - \frac{i}{\hbar}\hat{J}_y\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_y\Delta\phi}{\hbar}\right)^2][\hat{1} - \frac{i}{\hbar}\hat{J}_x\Delta\phi + \frac{1}{2}\left(\frac{\hat{J}_x\Delta\phi}{\hbar}\right)^2] \\
&= \hat{1} - \frac{i}{\hbar}\hat{J}_z(\Delta\phi)^2 + \frac{1}{2}\left(\frac{\hat{J}_z(\Delta\phi)^2}{\hbar}\right)^2 - \hat{1} \\
&= -\frac{i}{\hbar}\hat{J}_z(\Delta\phi)^2
\end{aligned}$$

The lowest-order non-vanishing terms involve $(\Delta\phi)^2$. Equating these terms , we get

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$$

Similarly, we have the commutation relations,

$$[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x, \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

APPENDIX

2D rotation matrix (type-I rotation)

First we consider the type-I rotation for the two-dimensional (2D) system. Suppose that the rotation of the orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by angle θ around the z axis (counter clock wise) yields to the new orthogonal basis $\{\boldsymbol{e}_1', \boldsymbol{e}_2'\}$ as shown in Fig. We note that the position vector $\boldsymbol{r}$ is fixed under the rotation. This implies that $\boldsymbol{r}$ in the old basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is equal to $\mathbf{r}'$ in the new basis $\{\boldsymbol{e}_1', e_2'\}$; $\boldsymbol{r} = \boldsymbol{r}'$.

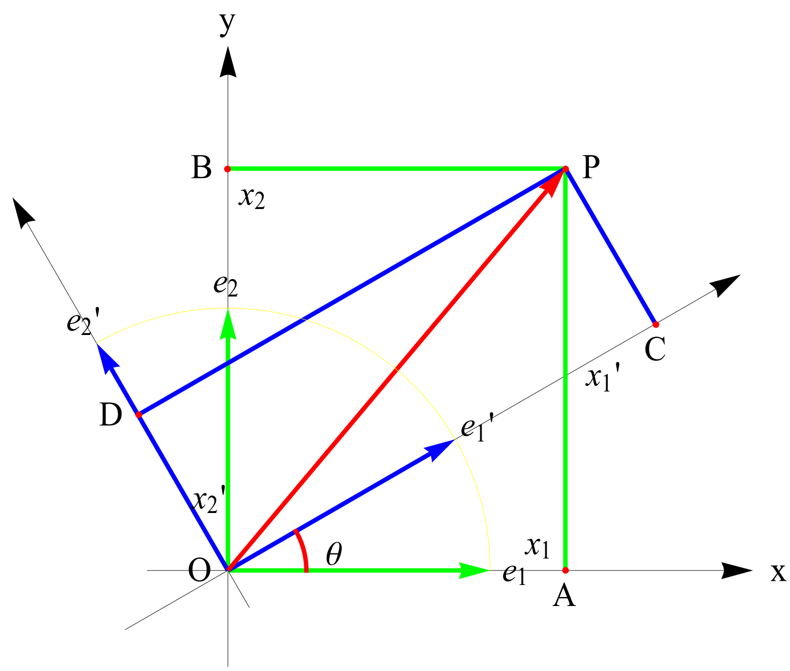


Fig. Rotation of the coordinate axes. $\overrightarrow{OP} = \boldsymbol{r} = \boldsymbol{r}'$. $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$; the old orthogonal basis. $\{\boldsymbol{e}_1', \boldsymbol{e}_2',\}$;. and the new orthogonal basis.

We assume that

$$\mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2$$
$$\mathbf{e}_2' = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2$$

with

$$a_{11} = (\mathbf{e}_1.\mathbf{e}_1') = \cos\theta$$
$$a_{12} = (\mathbf{e}_2.\mathbf{e}_1') = \sin\theta$$
$$a_{21} = (\mathbf{e}_1.\mathbf{e}_2') = -\sin\theta$$
$$a_{22} = (\mathbf{e}_1.\mathbf{e}_2') = \cos\theta$$

or

$$\Re(-\theta) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where the matrix elements $\{a_{ij}\}$ are real and $\Re(-\theta)$ is the rotation matrix. We use $(-\theta)$ for convenience. The transpose of the matrix $\Re(-\theta)$ is givrn by

$$\Re^T(-\theta) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then we have

$$\boldsymbol{e}_1' = a_{11}\boldsymbol{e}_1 + a_{12}\boldsymbol{e}_2 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \Re^T(-\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\boldsymbol{e}_2' = a_{21}\boldsymbol{e}_1 + a_{22}\boldsymbol{e}_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \Re^T(-\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Suppose that the vector $\boldsymbol{r}$ can be expressed by

$$\boldsymbol{r} = \sum_i x_i \boldsymbol{e}_i = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$$
$$\boldsymbol{r}' = \sum_i x_i' \boldsymbol{e}_i' = x_1'\boldsymbol{e}_1' + x_2'\boldsymbol{e}_2'$$
$$\boldsymbol{r} = \boldsymbol{r}'$$

in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and the basis $\{\boldsymbol{e}_1', \boldsymbol{e}_2'\}$, respectively. Then we have

$$x_1' = \boldsymbol{e}_1'\cdot\boldsymbol{r}' = \boldsymbol{e}_1'\cdot\boldsymbol{r} = \boldsymbol{e}_1'\cdot(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) = a_{11}x_1 + a_{12}x_2$$
$$x_2' = \boldsymbol{e}_2'\cdot\boldsymbol{r}' = \boldsymbol{e}_2'\cdot\boldsymbol{r} = \boldsymbol{e}_2'\cdot(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2) = a_{21}x_1 + a_{22}x_2$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \Re(-\theta) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad \text{(A)}$$

((**Interpretation**))

This is interpreted as an orthogonal transformation as a rotation of the vector, leaving the coordinate system unchanged. We can rotate ***r***' clockwise by an angle θ to a new vector *r*'. The component of new vector r' will then be related to the component of old by the same equations (A).