Rutherford scattering and hyperbola orbit

Masatsugu Sei Suzuki and Itsuko S. Suzuki
Department of Physics, SUNY at Binghamton

(Date: May 27, 2016)

Ernest Rutherford, 1st Baron Rutherford of Nelson OM, FRS (30 August 1871 – 19 October 1937) was a New Zealand-born British chemist and physicist who became known as the father of nuclear physics. In early work he discovered the concept of radioactive half life, proved that radioactivity involved the transmutation of one chemical element to another, and also differentiated and named alpha and beta radiation. This work was done at McGill University in Canada. It is the basis for the Nobel Prize in Chemistry he was awarded in 1908 "for his investigations into the disintegration of the elements, and the chemistry of radioactive substances".

http://en.wikipedia.org/wiki/Ernest_Rutherford

CONTENT

1. Introduction

One of us (M.S.) had an opportunity to teach Phys.323 (Modern Physics) in Fall 2011 and 2012 at the Binghamton University. Rutherford scattering is one of the most experiments in the quantum mechanics. During this class, I prepared the lecture note on the Rutherford scattering. We read a lot of textbooks on this matter, including the textbooks of modern physics and quantum mechanics. We read a book written by Segre (x-ray to quark). There is one figure (as shown below) of Rutherford scattering which was published by Rutherford (1911). We realize that the definition of the scattering angle (θ) is different from that used for the conventional x-ray and neutron scattering, except for θ in the scattering (the quantum mechanics) and 2θ in the x-ray and neutron scattering (condensed matter physics). Using this angle, Segre shows that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\sin^4(\frac{\theta}{2})}, \tag{1}$$

in spite of the difference of the definition of the scattering angle. We realize that even for a great scientist such as Segre, they had such an careless mistake for such as Rutherford scattering which is so well-known and so well-discussed in the modern physics textbooks.

Here we start with the nomenclature of hyperbola orbit. Using Mathematica, we examine the features of the hyperbola, which are closely related to the essential points of the Rutherford scattering. There have been so many books on the Rutherford scattering. In particular, We the quantum mechanics textbook by Tomonaga (geometrical discussion) and the book by Longair (historical background) are very useful for our understanding the physics. We also make use of the Mathematica to discuss the geoemetry of hyperbola orbit.

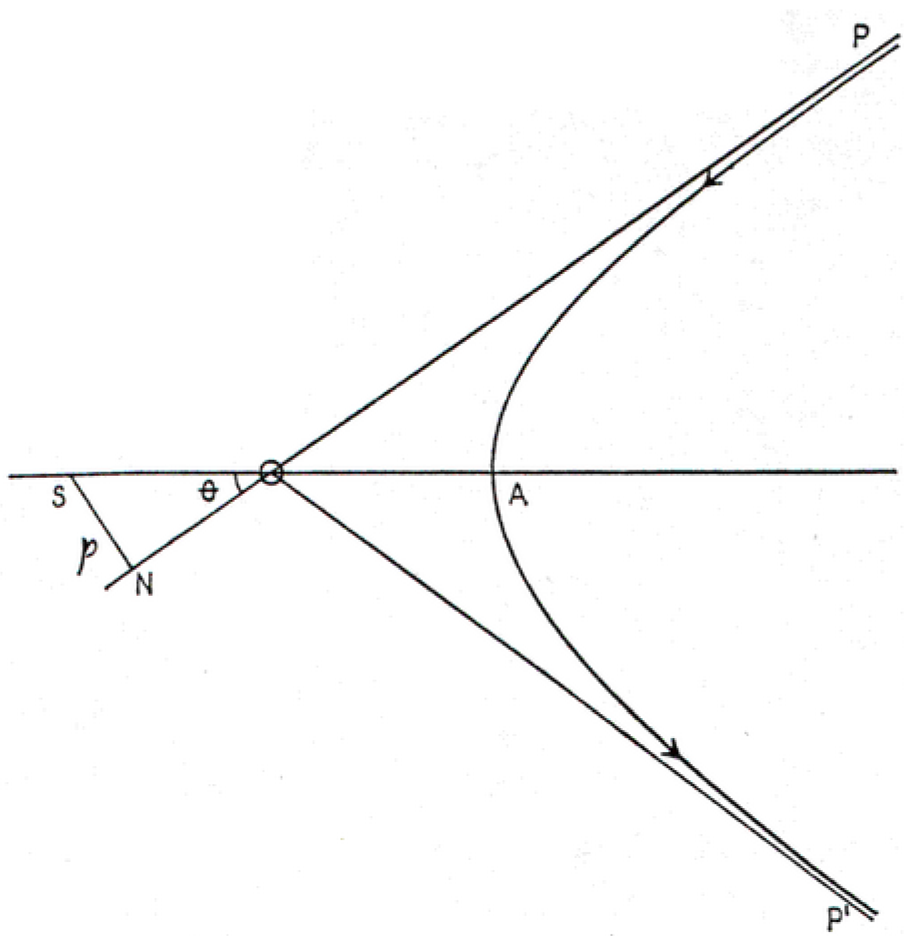

Fig.1 Original figure for the Rutherford scattering (Rutherford, 1911). Consider the passage of a positive electrified particle close to the center of an atom. Supposing that the velocity of the particle is not appreciably changed by its passage through the atom, the path of the particle under the influence of a repulsive force varying inversely as the square of the distance will be a hyperbola with the center of the atom S as the external focus. The particle to enter the atom in the direction PO, and that the direction of motion on escaping the atom is OP'. OP and OP' make equal angles with the line SA, where A is the apse of the hyperbola. p = SN = perpendicular distance from center on direction of initial motion of particle. The scattering angle θ is related to the angle $\angle SON = \varphi$ as $2\varphi + \theta = \pi$. So that φ is not the scattering angle in the conventional Rutherford scattering.

2. Historical Background (Longair)
M. Longair, Quantum Concepts in Physics (Cambridge, 2013).

The discovery of the nuclear structure of atoms resulted from a series of experiments carried out by Rutherford and his colleagues, Hans Geiger and Ernest Marsden, in the period 1909–1912.

Rutherford had been impressed by the fact that α-particles could pass through thin films rather easily, suggesting that much of the volume of atoms is empty space, although there was clear evidence for small-angle scattering. Rutherford persuaded Marsden, who was still an undergraduate, to investigate whether or not α-particles were deflected through large angles on being fired at a thin gold foil target. To Rutherford's astonishment, a few particles were deflected by more than 90◦, and a very small number almost returned along the direction of incidence.

Rutherford realized that it required a very considerable force to send the α-particle back along its track. In 1911 he hit upon the idea that, if all the positive charge were concentrated in a compact nucleus, the scattering could be attributed to the repulsive electrostatic force between the incoming α-particle and the positive nucleus. Rutherford was no theorist, but he used his knowledge of central orbits in inverse-square law fields of force to work out the properties of what became known as *Rutherford scattering* (Rutherford, 1911). The orbit of the α-particle is a *hyperbola*, the angle of deflection θ being

$$b = \frac{\kappa}{2K_0}\cot\frac{\theta}{2}, \tag{2}$$

where b is the impact parameter, K_0 is the kinetic energy of the α-particle, $\kappa = 2Zq_e{}^2$, and Z the nuclear charge. The eccentricity of the hyperbola is given by

$$e = \frac{1}{\sin\dfrac{\theta}{2}}, \tag{3}$$

where $a = \dfrac{\kappa}{2K_0}$ and $b = ae\cos\dfrac{\theta}{2}$. The hyperbola orbit can be expressed as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{4}$$

where

$$ae = \sqrt{a^2 + b^2}. \tag{5}$$

It is straightforward to work out the probability that the α-particle is scattered through an angle θ. The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^2}{16K_0^{\ 2}} \frac{1}{\sin^4 \frac{\theta}{2}}. \tag{6}$$

This famous $\csc^4 \frac{\theta}{2}$ law derived by Rutherford, was found to explain precisely the observed distribution of scattering angles of the α-particles (Geiger and Marsden, 1913). Rutherford had, however, achieved much more. The fact that the scattering law was obeyed so precisely, even for large angles of scattering, meant that the inverse-square law of electrostatic repulsion held good to very small distances indeed. They found that the nucleus had to have size less than about 10^{-14} m, very much less than the sizes of atoms, which are typically about 10^{-10} m.

3. Nomenclature and feature of the hyperbola orbit

An alpha particle considered as a massive point charge, incident on the nucleus, is repelled according to a Coulomb's law, and, as Newton had already calculated, it follows a hyperbolic orbit, with the nucleus, with the nucleus as one of the focal points of the hyperbola. It seems that Rutherford had learned this as a student in New Zealand. Before we discuss the physics of Rutherford scattering, we discuss the properties of the hyperbola orbit.

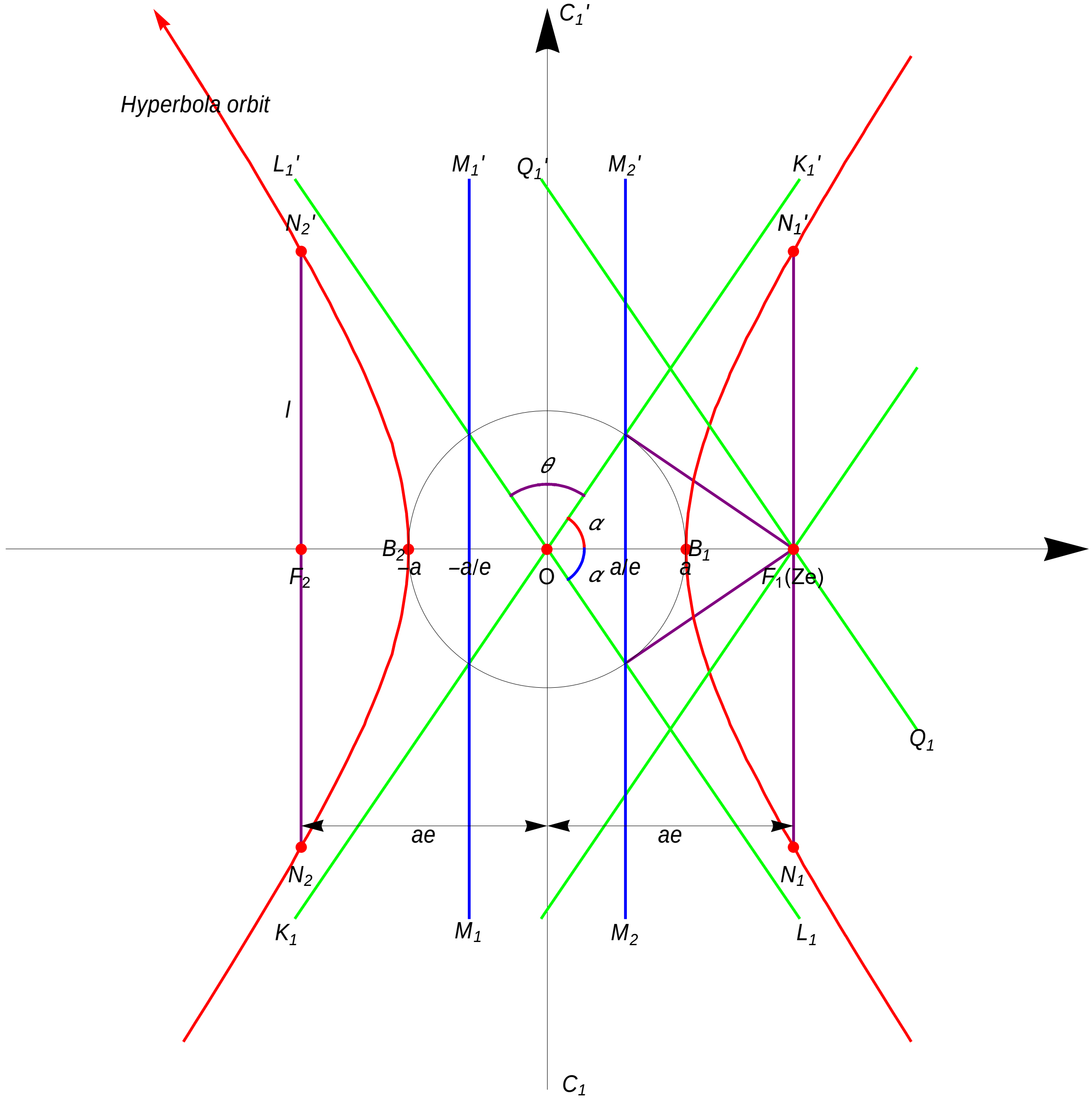

Fig.2(a) Nomenclature of the hyperbola.

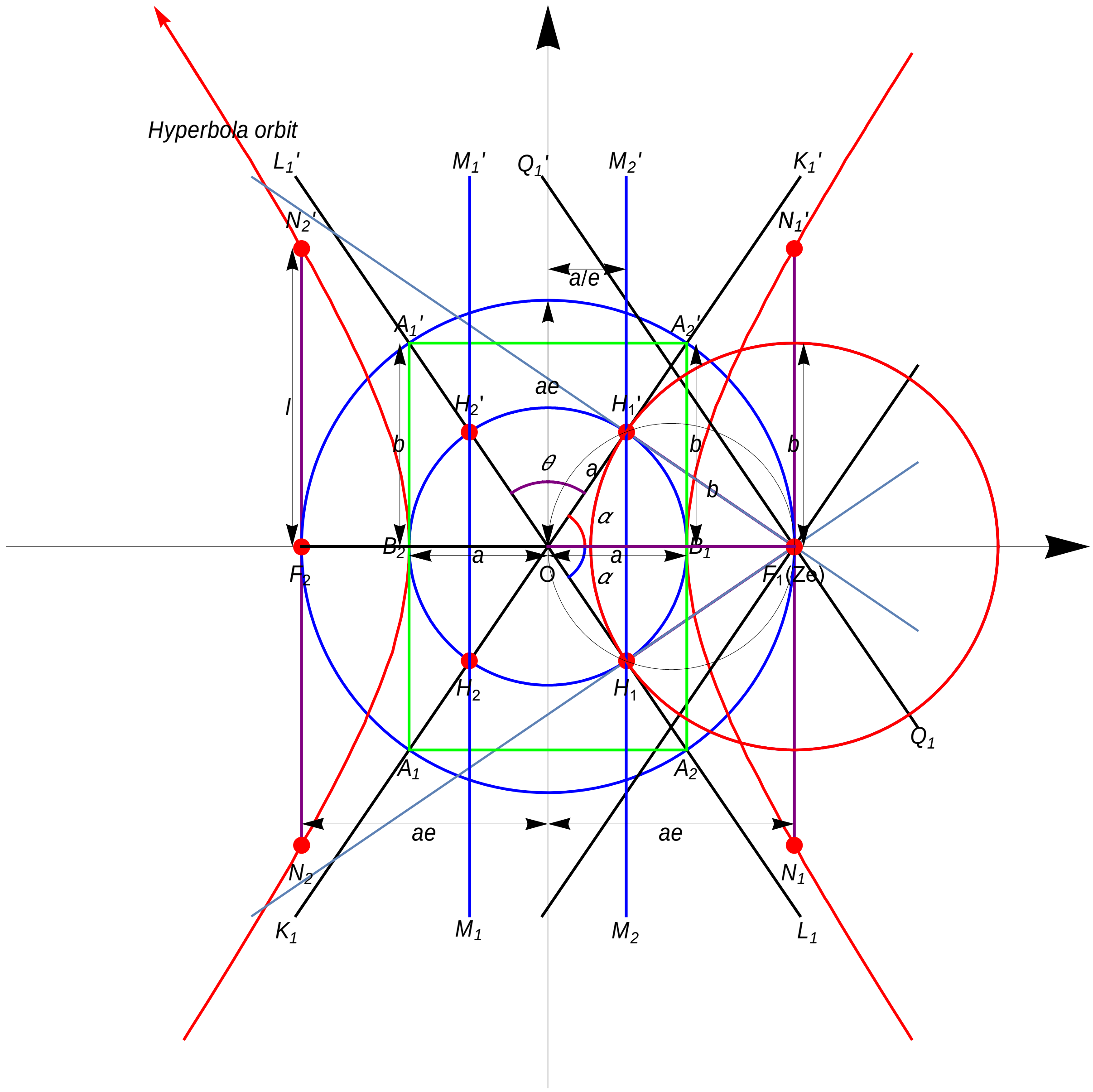

Fig.2(b) Detail in the geometry of hyperbola

e:	eccentricity
Line $\overline{F_1F_2}$	transverse axis
Line $\overline{C_1C_1'}$	conjugate
O:	center of the hyperbola
F_1, F_2	focal point
B_1, B_2	vertrices
$\overline{M_1M_1'}$, $\overline{M_2M_2'}$	directrices

l semi-latus rectum

The hyperbola consists of the two red curves. The asymptotes of the hyperbola are denoted by the lines K_1K_1' and L_1L_1'. They intersect at the center of the hyperbola, O. The two focal points are labeled F_1 (atom with Zq_e) and F_2, and the line joining them is the transverse axis. The line through the center, perpendicular to the transverse axis is the conjugate axis. The two lines parallel to the conjugate axis (thus, perpendicular to the transverse axis) are the two directrices, M_1M_1'and M_2M_2'. The eccentricity e equals the ratio of the distances from a point **P** on the hyperbola to one focus and its corresponding directrix line. The two vertices (B_1 and B_2) are located on the transverse axis at $\pm a$ relative to the center. θ is the angle formed by each asymptote with the transverse axis. The length F_2N_2' (l) is called the semi-latus rectum,

$$l = \frac{b^2}{a} = a(e^2 - 1) .$$

4. Property of hyperbola

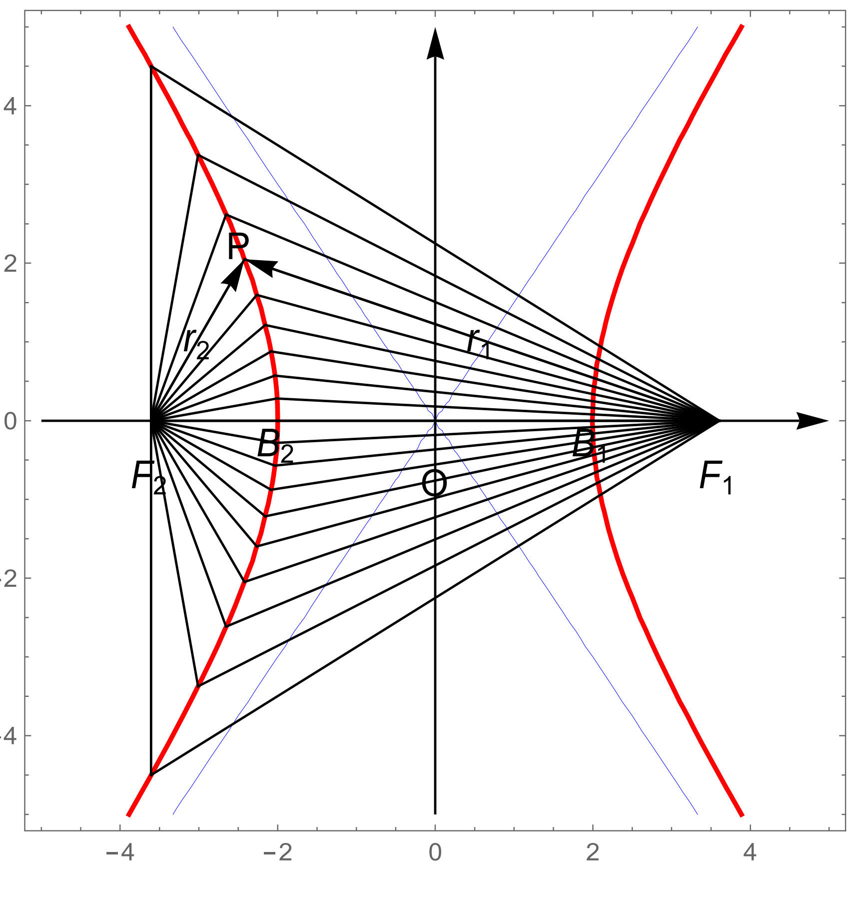

Fig.3 (a) Definition of hyperbpora. $r_1 - r_2 = 2a$.

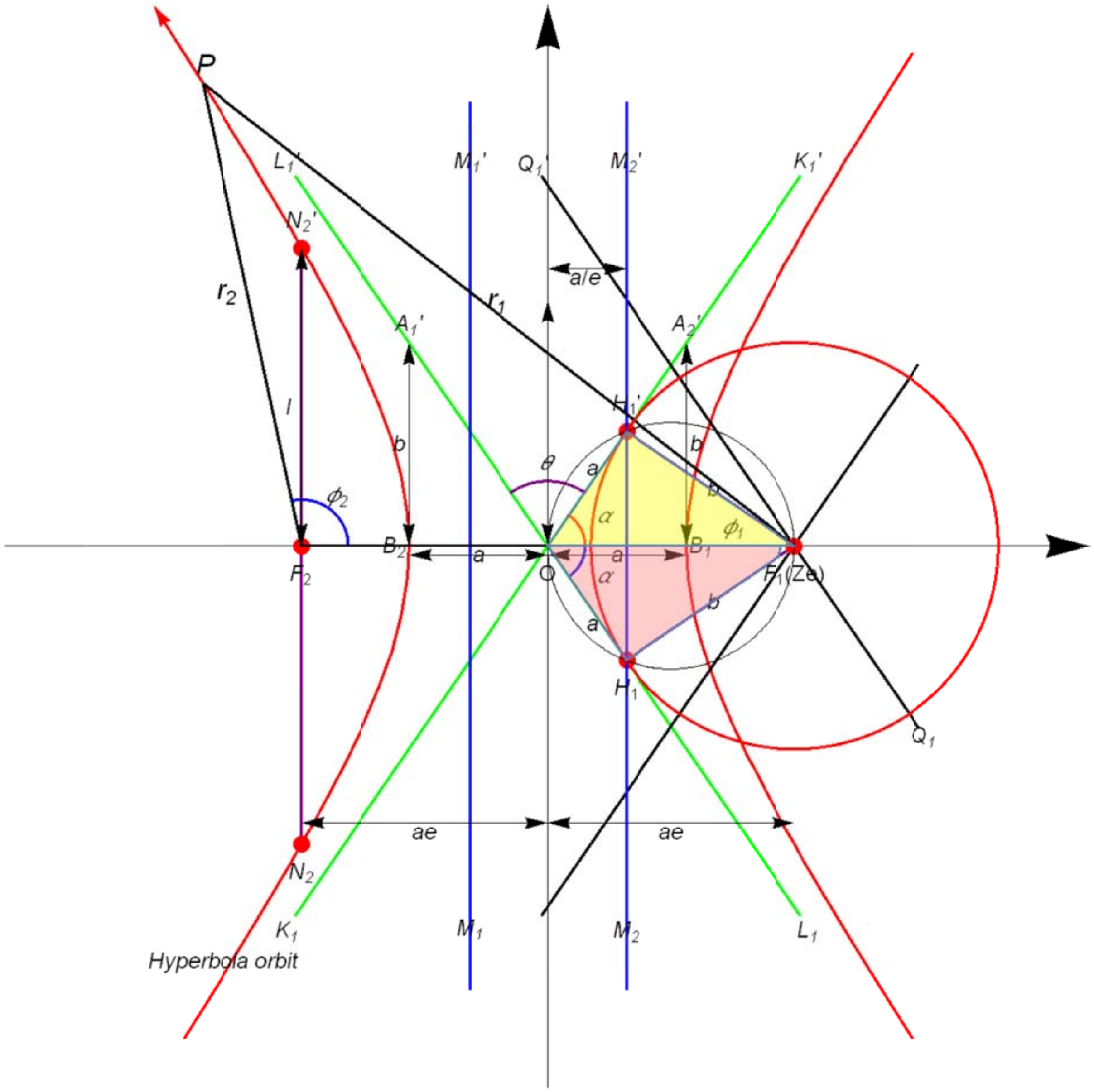

Fig.3(b) $\overline{OA_1} = a = k_0$, $\overline{A_1B_1} = b$. $\overline{OF_1} = \overline{OF_2} = \sqrt{a^2+b^2} = a\varepsilon$, $\overline{OB_1} = a\varepsilon$. $\overline{OS_1} = a$, $\overline{OS_2} = a$, $\overline{F_2S_1} = \overline{F_2S_2} = b$. O: target nucleus. F_1 and F_2: focal points.

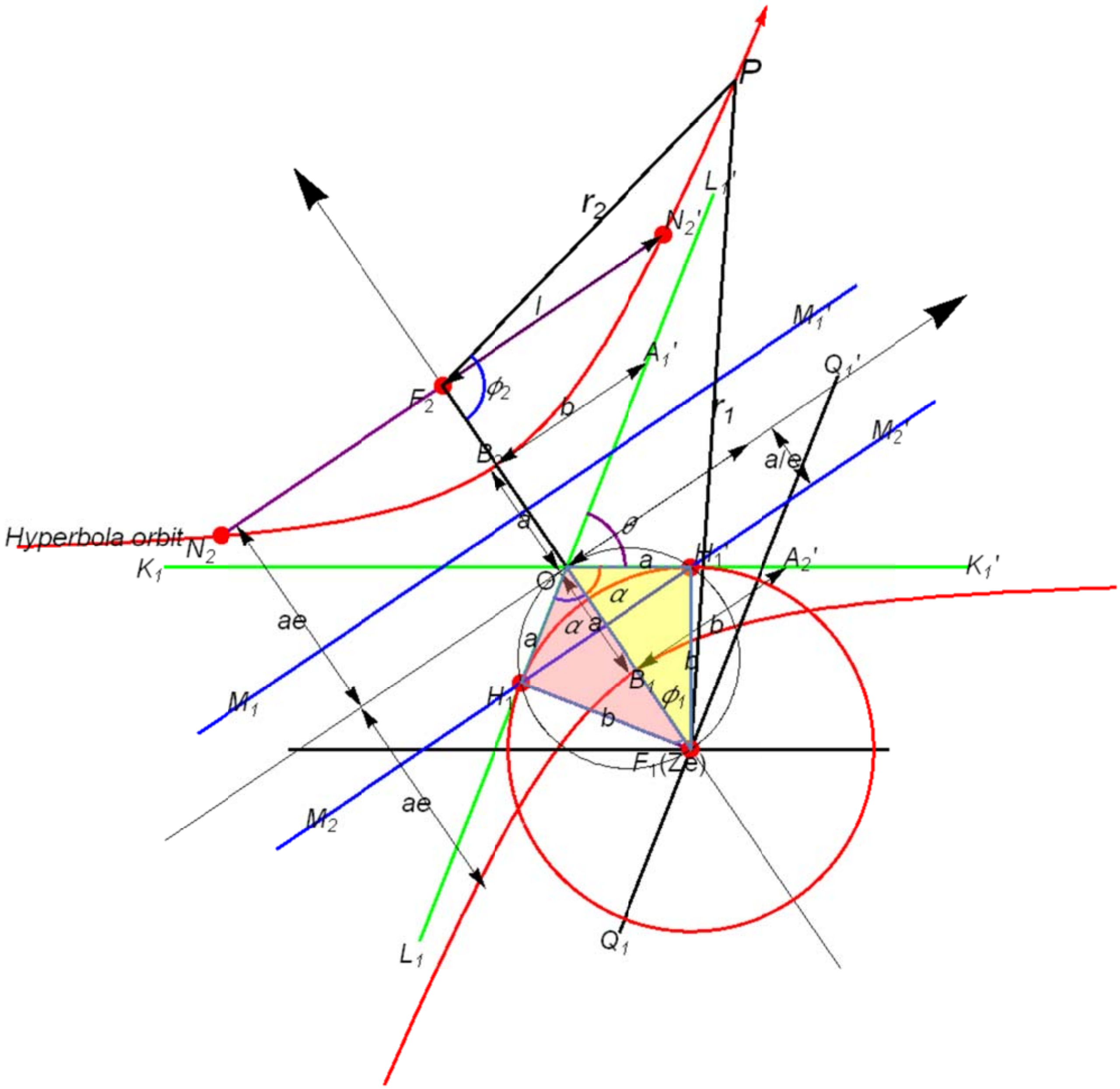

Fig.3(c) Hyperbola orbit. $r_1 - r_2 = 2a$. The target nucleus with Zq_e is located at the focal point F_1. The a particle is at the point P on the hyperbola. $ae = \sqrt{a^2 + b^2}$.

(a) Evaluation of the distance *l*

Suppose that the α particle is at the point N_2' of the hyperbola.

$$r_1 - r_2 = 2a,$$

from the definition of hyperbola, and

$$r_2 = l \,.$$

Then we get

$$r_1^{\ 2} = (r_2 + 2a)^2 = (l + 2a)^2 = l^2 + 4al + 4a^2 \,.$$

Applying the Pythagorean theorem to the triangle $\Delta F_1F_2N_2'$ with the right angle $\angle N_2'F_2F_1$, we have

$$r_1^{\ 2} = l^2 + 4a^2e^2 \,.$$

From these two equations, we get

$$l^2 + 4al + 4a^2 = l^2 + 4a^2e^2 \,,$$

or

$$l = a(e^2 - 1) \,.$$

l is called the semi-latus rectum,

(b) Equation of hyperbola with r_1

We apply the cosine law for the triangle ΔF_2PF_1 such that

$$r_2^{\ 2} = (r_1 - 2a)^2 = 4a^2e^2 - 4aer_1\cos\phi_1 + r_1^{\ 2} = r_1^{\ 2} - 4ar_1 + 4a^2 \,,$$

or

$$ae^2 - er_1\cos\phi_1 = -r_1 + a \,,$$

leading to the result

$$r_1 = \frac{a(e^2 - 1)}{e\cos\phi_1 - 1} = \frac{l}{e\cos\phi_1 - 1} \,.$$

when

where r_1 is the distance between the point P on the hyperbola and the focal point F_1 (the target with the charge Zq_e. When $\phi_1 = 0$, we get the minimum distance $(r_2)_{\min} = a(e+1)$.

(c) Equation of hyperbola with r_2

For the same hyperbola, we apply the cosine law for the triangle $\Delta F_2 P F_1$ such that

$$r_1^{\ 2} = (r_2 + 2a)^2 = 4a^2 e^2 - 4aer_2 \cos\phi_2 + r_2^{\ 2} = r_2^{\ 2} + 4ar_2 + 4a^2 ,$$

or

$$a(e^2 - 1) = er_2 \cos\phi_2 + r_2 .$$

Then we have

$$r_2 = \frac{a(e^2-1)}{1+e\cos\phi_2} = \frac{l}{1+e\cos\phi_2} ,$$

where r_2 is the distance between the point P and the focal point F_2. When $\phi_2 = 0$, we get the minimum distance $(r_2)_{\min} = a(e-1)$.

5. Rutherford scattering experiment

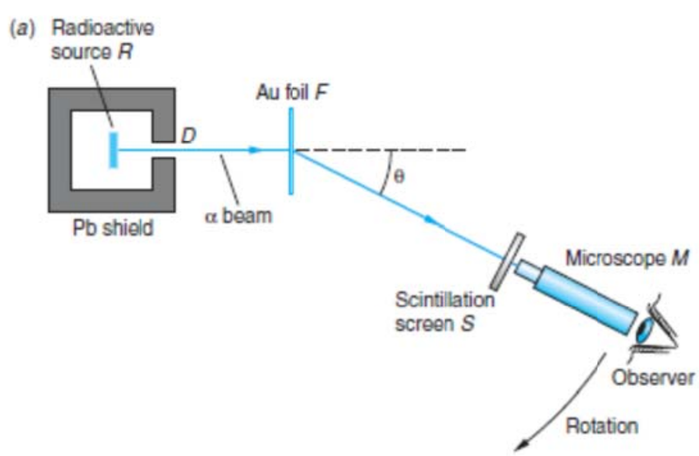

Fig.4 Experimental configuration of the Rutherford scattering [P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.4)]

Rutherford scattering is the scattering of α-particle (light-particle with charge $2q_e$>0) by a nucleus (heavy particle with charge Zq_e). The mass of nucleus is much larger than that of the α-particle. Thus the nucleus remains unmoved before and after collision. There is a repulsive Coulomb interaction between the nucleus and the α particle, leading to the hyperbolic orbit of the α-particle. The potential energy of the interaction (repulsive) is given by

$$U = \frac{2Zq_e^{\,2}}{r} = \frac{\kappa}{r}, \qquad \text{(in cgs units)}$$

where $\kappa = 2Zq_e^{\,2}$. Here we use the charge q_e (>0) instead of e since we use e as the eccentricity of hyperbola. The boundary conditions can be specified by the kinetic energy K_0 and the angular momentum L of the α-particles, or by the initial velocity v_0 and impact parameter b,

$$K_0 = \frac{1}{2} m v_0^{\,2}, \quad \text{and} \qquad L = m v_0 b,$$

where m is the mass of the α-particle.

((**Note**)) α particle is He nucleus consisting of two protons and two neutrons (He^{2+})

6. Illustration of the Rutherford scattering experiment (H.E. White)

H.E. White, Introduction to Atomic and Nuclear Physics (D. Van Nostrand Company, Inc., Princeton, NJ, 1964).

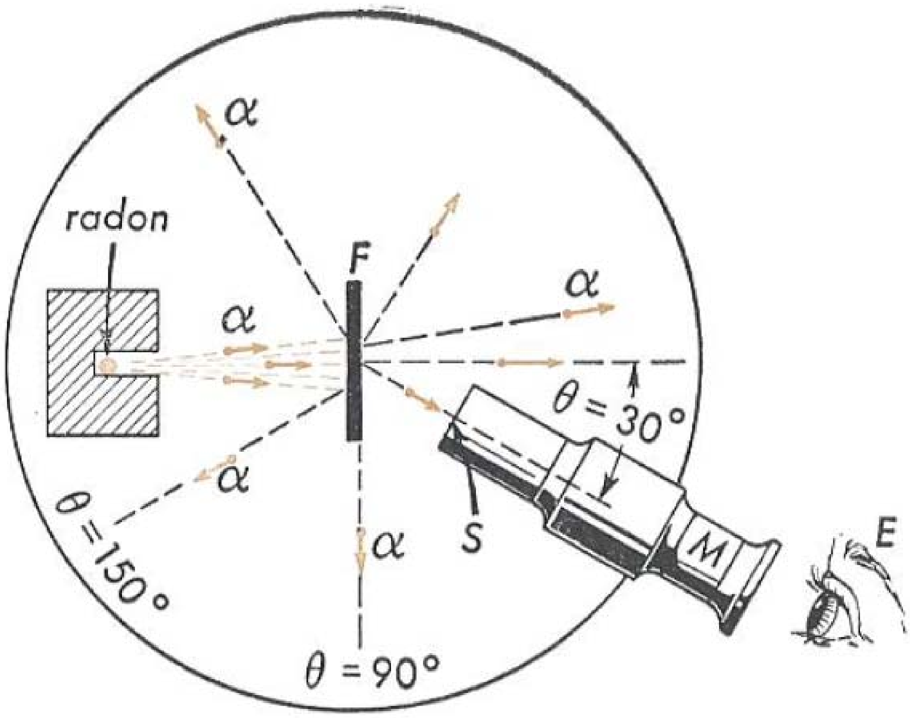

Fig.5 Schematic diagram of a particles being scattered by the atomic nuclei in a thin metallic film (White, 1962)

A schematic diagram of the scattering experiments is given in Fig.5. High-speed a particles from the radioactive element radon, confined to a narrow beam by a hole in a lead block were made to strike a very thin gold foil F, while most of the *a* particles go straight through the foil as if there were nothing, some of them collide with atoms of the foil and bounce off at some angle. The latter phenomenon is known as Rutherford scattering. The observations and measurements made in the experiment consisted of counting the number of particles scattered off at different angles. of particles scattered off at different angle θ. This was done by the scintillation method of observation. Each a particle striking the fluorescent screen S produces a tiny flash of light, called a scintillation, and is observed as such by the microscope M. With the microscope fixed in one position the number of scintillation observed with a period of several minutes was counted; then the microscope was turned to another angle, and the number was again counted for an equal period of time.

In the schematic diagram of Fig.6, a particles as shown passing through a foil three atomic layers thick. Although the nuclear atom was not known at the time the experiments were performed, each atom is drawn in Fig.6 with the positively charged nucleus at the center and surrounded by a number of electrons. Since most of the film is free space, the majority of the α particles go through with little or no deflection as indicated by ray-1. Other α's like ray-2 passing relatively close to an atom nucleus are deflected at an angle of a few degrees. Occasionally, however, an almost head-on collision occurs as shown by ray-4 and the incoming α particle is turned back toward the source. As an a particle approaches an atom, as represented by ray-6, it is repelled by the heavy positively charged nucleus and deflected in such a way as to make it follow a curved path.

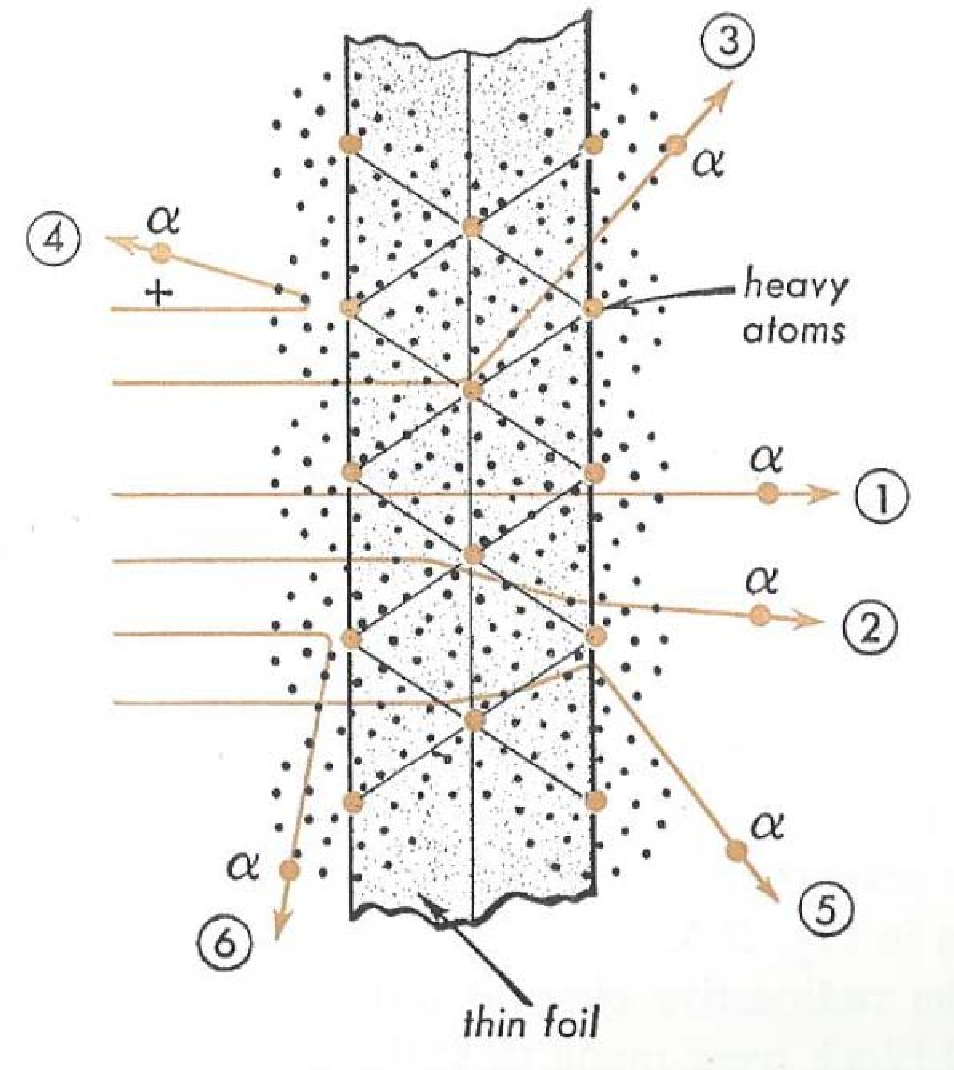

Fig.6 Diagram of the deflection of an α particle by a nucleus: Rutherford scattering (White 1962).

7. Mechanical model for Rutherford scattering by White

H.E. White, Introduction to Atomic and Nuclear Physics (D. Van Norstrand, 1964).

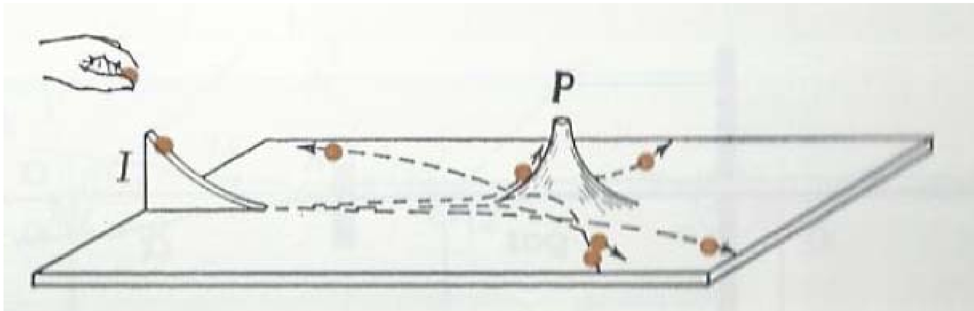

Fig.7 Mechanical model of an atomic nucleus for demonstrating Rutherford scattering (H.E. White, 1964).

Here is an interesting mechanical model for demonstrating Rutherford scattering. Such a model is illustrated in Fig.7, where the circular peak at the right represents the nucleus of an atom and has a form generated by rotating curve of the repulsive potent $V(r)$ vs r about its vertical axis at $r = 0$. Marbles, representing α particles, roll down a chute and along a practically level plane, where they approach the potential hill. Approaching the hill at various angles, the marbles roll up to a certain height and then off to one side or the other, The path they follow, if watched from the above, are hyperbolic in shape. Approaching the hill in a head-on collision, the ball rolls up to a certain point, stops, then roll back again. Thus the potential energy of α particle close to the nucleus is analogous to the potential energy of a marble on the hillside, and the electrostatic force of repulsion is analogous to the component of the downward pull of gravity.

8. Linear momentum for the elastic scattering

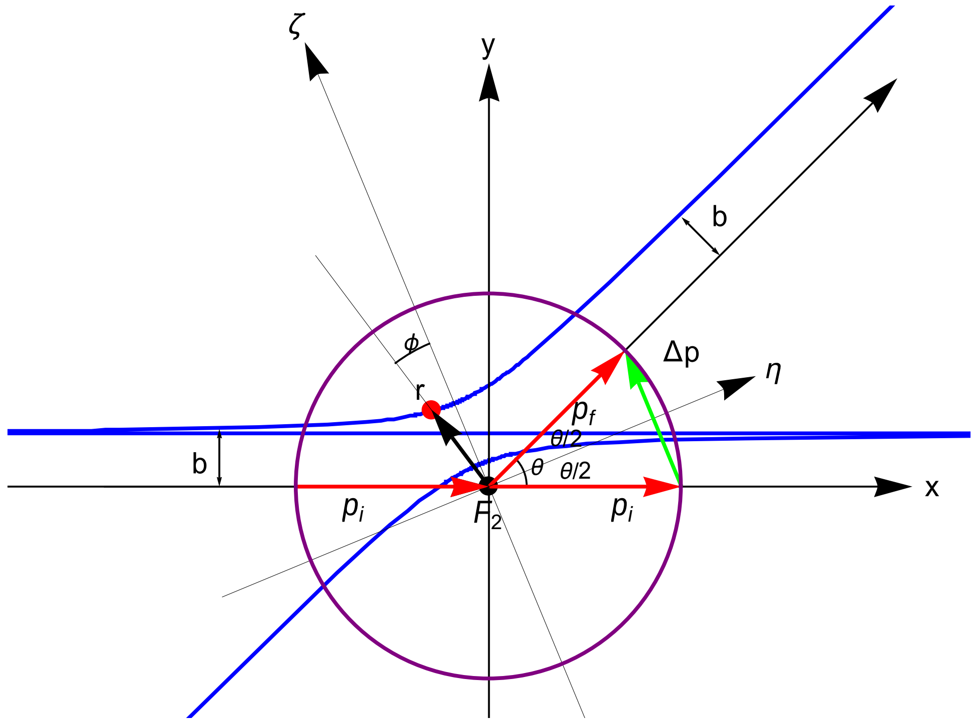

Fig.8 The hyperbolic Rutherford trajectory. The angular momentum is conserved before and after the scattering. The angular momentum: $L = mv_0b$. For the elastic scattering, b is kept constant, where b is the impact parameter. The angle between the initial and the final asymptote of the hyperbola, is related to the impact parameter b.

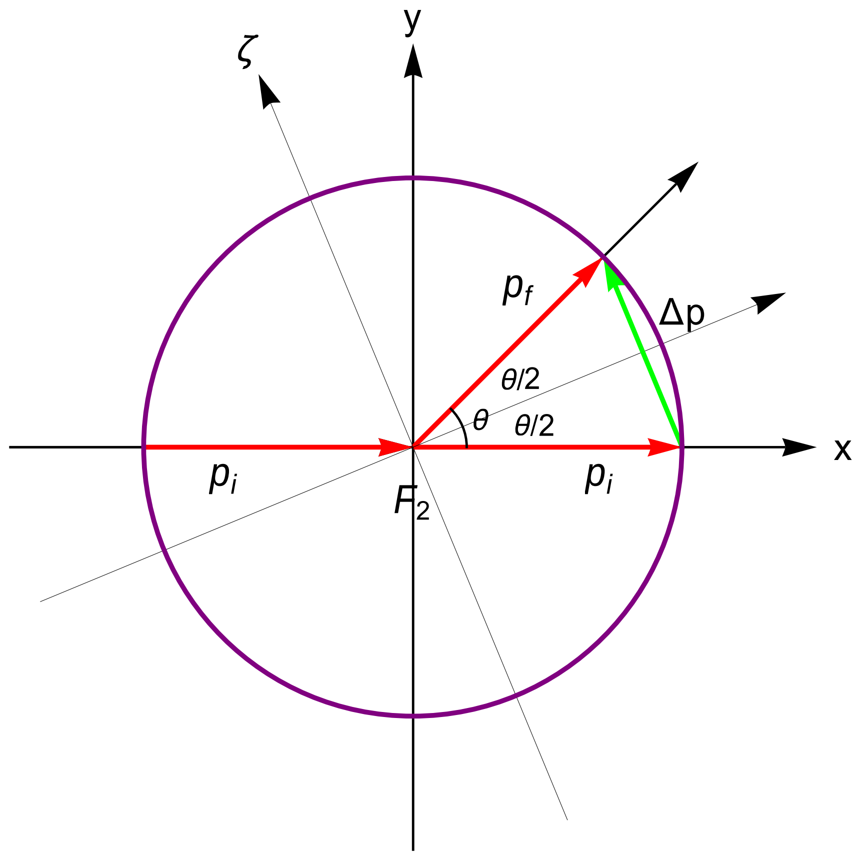

Fig.9 Ewald's sphere for the Rutherford scattering. $\Delta p = 2p_i \sin\frac{\theta}{2} = 2mv_0 \sin\frac{\theta}{2}$.

$$\Delta\boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i = \boldsymbol{Q}, \qquad \text{(Scattering vector)}$$

where

$$|\boldsymbol{p}_f| = |\boldsymbol{p}_i| = p = mv_0 .$$

From the Ewald's sphere, we have

$$Q = \Delta p = 2p\sin\frac{\theta}{2} = 2mv_0\sin\frac{\theta}{2} .$$

9. Newton's second law for rotation: torque and angular momentum

The torque is given by

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \frac{d\boldsymbol{L}}{dt},$$

where $\boldsymbol{\tau}$ is the torque, $\boldsymbol{r}$ is the position vector of the α-particle with charge $2q_e$ (>0) and $\boldsymbol{F}$ is the repulsive Coulomb force (the central force) between the α-particle and the nucleus with charge Zq_e. The direction of the Coulomb force is parallel to that of $\boldsymbol{r}$. In other words, the torque $\boldsymbol{\tau}$ is zero. The angular momentum $\boldsymbol{L}$ is conserved.

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v}) = m(r\hat{r}) \times (v_r\hat{r} + v_\phi\hat{\phi}) = mrv_\phi\hat{z} = mr^2\frac{d\phi}{dt}\hat{z}.$$

or

$$mr^2\frac{d\phi}{dt} = mv_0 b$$

or

$$\frac{d\phi}{dt} = \frac{v_0 b}{r^2}$$

where b is the impact parameter.

((**The impulse-momentum theorem**))

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F},$$

or

$$\boldsymbol{Q} = \boldsymbol{p}_f - \boldsymbol{p}_i = \int_{t_i}^{t_f} \boldsymbol{F}dt = \int_{t_i}^{t_f} (F_\varsigma\hat{\varsigma} + F_\eta\hat{\eta})dt = \int_{t_i}^{t_f} F(\cos\phi\hat{\varsigma} + \sin\phi\hat{\eta})dt.$$

Since $\boldsymbol{Q}$ is parallel to the unit vector $\hat{\varsigma}$, we get

$$Q = \int_{t_i}^{t_f} F\cos\phi dt,$$

and

$$\int_{t_i}^{t_f} F \sin\phi dt = 0 .$$

Using the relation $\frac{d\phi}{dt} = \frac{v_0 b}{r^2}$

$$\begin{aligned}
Q &= \int_{t_i}^{t_f} F \cos\phi dt \\
&= \int_{t_i}^{t_f} F \cos\phi \frac{dt}{d\phi} d\phi \\
&= \int_{t_i}^{t_f} \frac{\kappa}{r^2} \cos\phi \frac{r^2}{v_0 b} d\phi \\
&= \frac{\kappa}{v_0 b} \int_{\phi_i}^{\phi_f} \cos\phi d\phi
\end{aligned}$$

where $\kappa = 2Zq_e{}^2$,

$$\phi_i = -\left(\frac{\pi - \theta}{2}\right), \qquad \text{at } t = t_{\mathrm{i}},$$

and

$$\phi_f = \left(\frac{\pi - \theta}{2}\right). \qquad \text{at } t = t_{\mathrm{f}}.$$

Here it should be noted that

$$\begin{aligned}
\int_{t_i}^{t_f} F \sin\phi dt &= \int_{t_i}^{t_f} F \sin\phi \frac{dt}{d\phi} d\phi \\
&= \int_{t_i}^{t_f} \frac{\kappa}{r^2} \sin\phi \frac{r^2}{v_0 b} d\phi \\
&= \frac{\kappa}{v_0 b} \int_{-\phi_f}^{\phi_f} \sin\phi d\phi = 0
\end{aligned}$$

Then we get

$$
\begin{aligned}
2mv_0 \sin\frac{\theta}{2} &= \frac{\kappa}{v_0 b}\int_{\phi_i}^{\phi_f} \cos\phi d\phi \\
&= \frac{\kappa}{v_0 b} 2\int_{0}^{\phi_f} \cos\phi d\phi \\
&= \frac{\kappa}{v_0 b} 2[\sin\phi]_0^{\phi_f} \\
&= \frac{2\kappa}{v_0 b}\sin\phi_f \\
&= \frac{2\kappa}{v_0 b}\cos\frac{\theta}{2}
\end{aligned}
$$

or

$$
b = \frac{\kappa}{m{v_0}^2}\cot\frac{\theta}{2} = \frac{\kappa}{2K_0}\cot\frac{\theta}{2},
$$

and

$$
a = \frac{\kappa}{2K_0},
$$

where K_0 is the kinetic energy of the bombarding α-particle,

$$
K_0 = \frac{1}{2}m{v_0}^2 .
$$

10. Approach from conservation law of angular momentum and energy (Tomonaga)

S. Tomonaga, Quantum Mechanics I: Old Quantum Theory (North Holland, 1962). This book was written in Japanese. The English translation of this book was made by Masatoshi Koshiba. Both Prof. Tomonaga and Prof. Koshiba got Nobel Prize in 1965 (renormalization) and 1987 (observation of neutrino at Kamiokande, Japan), respectively. Here we present a brief summary of the Rutherford scattering based on the Tomonaga's explanation.

Suppose an α particle with a positive charge $2q_e$ is passing by the positive charge Zq_e concentrated in the center of the atom. The particle then moves on a hyperbola with F_1 as the outer focus. We take the x-axis through the focal point F_1 (the line L_1-M-F_1) and parallel to the line, N_2-O-N_2', along which the α particle is approaching the atom from the left. The hyperbola then has this line, N_2-O-N_2', as one of its asymptotes. If we denote the other asymptote by L_1-O-L_1', this gives the direction at infinity after the scattering. The scattering angle is accordingly given by the angle $\angle L_1'ON_2' = \theta$.

The distance from the x-axis of the α particle at infinity when it is approaching the atom, i.e., the distance between the line N_2-O-N_2' and the x-axis is denoted by b. This distance b is a measure of how close the α particle comes to the atom and is an important quantity in this kind of calculation. Hence this distance b is b=given the name of impact parameter. When b is very large, the α particle will pass the atom at a great distance and accordingly suffer hardly any deflection. When, on the contrary, b is zero, the α particle will make a head-on collision with the atom and suffer the maximum deflection which, from symmetry considerations, amounts to 180°. The scattering angle θ is in general a function of b, the form of which we can determine in the following manner.

Let the velocities of the α particle at infinity and at the point of closes approach to the atom, i.e., at the point P, be denoted by v_0 and u, respectively. Then conservation of angular momentum gives the relation,

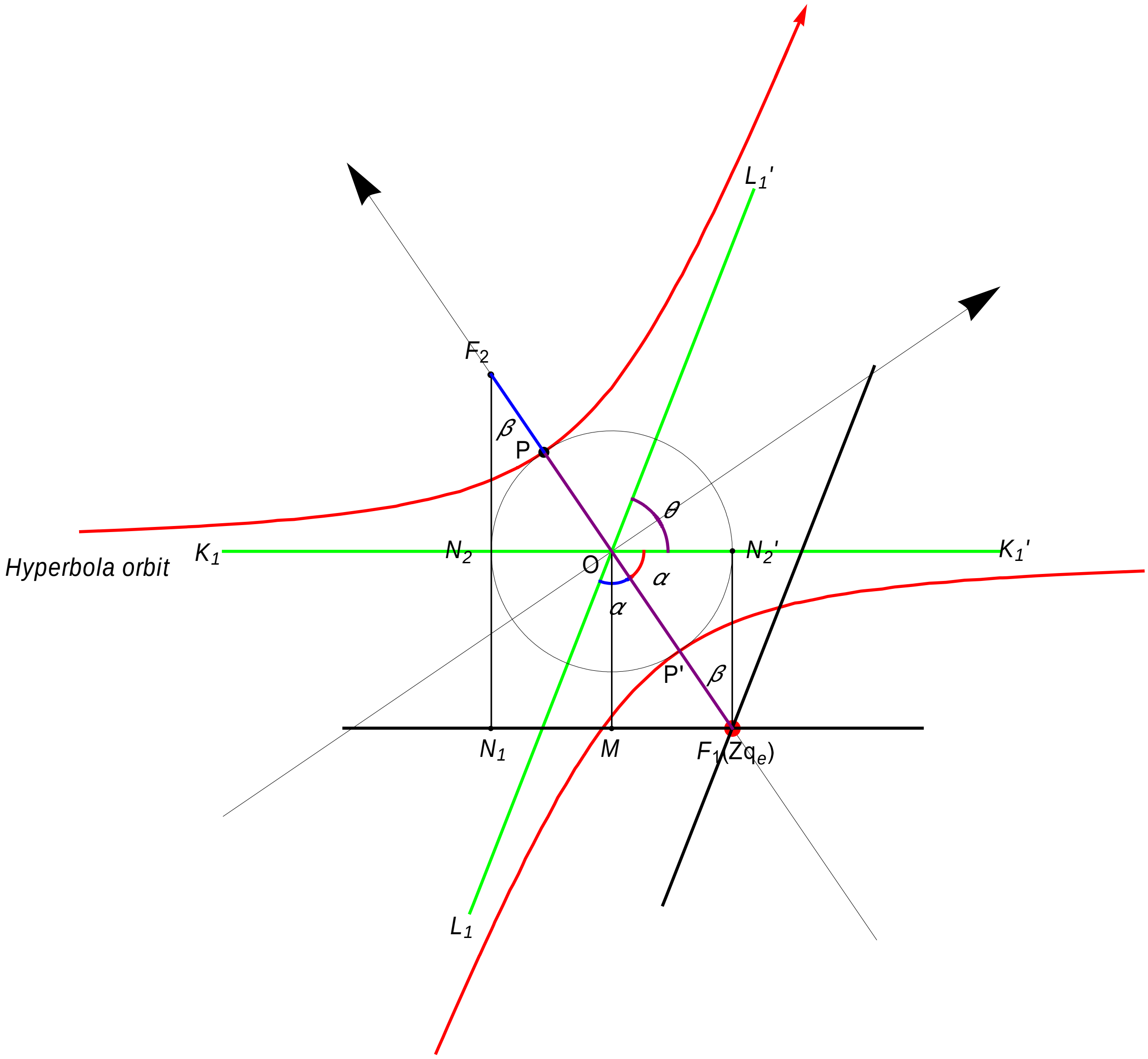

Fig.10 Feature of the hyperbola. P is a point on the hyperbola orbit. $\overline{F_1P} - \overline{F_2P} = 2a$. Since $\overline{P'F_1} = \overline{PF_2}$, the points P, N_2, P', and N_2' are on the circle of radius a centered at the point O. $\overline{N_1F_1} = \overline{N_2N_2'} = 2a$. $\overline{OM} = b$. $\overline{F_2N_1} = 2b$.

$$L = mv_0 b = mur_{\min},$$

or

$$v_0 b = ur_{\min},$$

where $r_{\min}$ is the length of the line $\overline{PF_1}$. On the other hand, the conservation of energy is expressed by

$$\frac{1}{2}mv_0{}^2 = \frac{1}{2}mu^2 + \frac{\kappa}{r_{mjn}},$$

or

$$v_0{}^2 = u^2 + \frac{2\kappa}{mr_{mjn}}, \qquad (1)$$

where

$$\kappa = 2Zq_e{}^2 \quad .$$

From Eqs.(1) and (2), we get

$$v_0{}^2 = \frac{v_0{}^2 b^2}{r_{\min}{}^2} + \frac{2\kappa}{mr_{mjn}},$$

or

$$b^2 = r_{\min}{}^2 - \frac{2\kappa}{mv_0{}^2} r_{mjn}$$

$$= r_{\min}(r_{\min} - \frac{\kappa}{K_0})$$

where

$$K_0 = \frac{1}{2}mv_0{}^2 .$$

Introducing the angle α by

$$\angle \mathrm{F_1ON_2'} = \alpha = \frac{\pi}{2} - \frac{\theta}{2},$$

we get

$$\overline{OF_1} = R = b \csc \alpha \, .$$

Let the normal to the x axis from the other focus F_2 be F_2N_1. Then from the known feature of a hyperbola,

$$\overline{N_1F_1} = 2a = \overline{F_1P} - \overline{F_2P} \, , \qquad \text{(definition of the hyperbola).}$$

or using the length b and the angle $\beta = \dfrac{\pi}{2} - \alpha$, we get

$$\overline{N_1F_1} = 2b \tan \beta = 2b \tan(\frac{\pi}{2} - \alpha) = 2b \cot \alpha \, ,$$

where

$$\overline{F_2N_1} = 2b \, .$$

The distance PF_1 is given by

$$r_{\min} = \overline{PF_1} = \overline{OP} + \overline{OF_1} \, ,$$

or

$$\begin{aligned} r_{\min} &= b(\csc \alpha + \cot \alpha) \\ &= \frac{b}{\sin \alpha}(1 + \cos \alpha) \\ &= \frac{2b \cos^2 \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \\ &= b \cot \frac{\alpha}{2} \end{aligned}$$

Using this value of $r_{\min}$, we have

$$b^2 = r_{\min}{}^2 - \frac{\kappa}{K_0} r_{mjn}$$
$$= b^2 \cot^2\frac{\alpha}{2} - \frac{\kappa}{K_0} b \cot\frac{\alpha}{2}$$

or

$$b = b\cot^2\frac{\alpha}{2} - \frac{\kappa}{K_0}\cot\frac{\alpha}{2},$$

or

$$b(\cot^2\frac{\alpha}{2} - 1) = \frac{\kappa}{K_0}\cot\frac{\alpha}{2},$$

or

$$b\frac{\cos\alpha}{\sin^2\frac{\alpha}{2}} = \frac{\kappa}{K_0}\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}},$$

or

$$b\cos\alpha = \frac{\kappa}{K_0}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \frac{\kappa}{2K_0}\sin\alpha\,.$$

Then we get the relation

$$b = \frac{\kappa}{2K_0}\tan\alpha = \frac{\kappa}{2K_0}\tan(\frac{\pi}{2} - \frac{\theta}{2}) = \frac{\kappa}{2K_0}\cot\frac{\theta}{2}.$$

Since $b = a\tan\alpha = a\tan(\frac{\pi}{2} - \frac{\theta}{2}) = a\cot\frac{\theta}{2}$, we have

$$a = \frac{\kappa}{2K_0}.$$

Note that a depends on the kinetic energy K_0.

11. The Kepler's First Law for the repulsive interaction

We consider the central field problem for the repulsive interaction between the nucleus (Zq_e) and the α particle ($2q_e$).

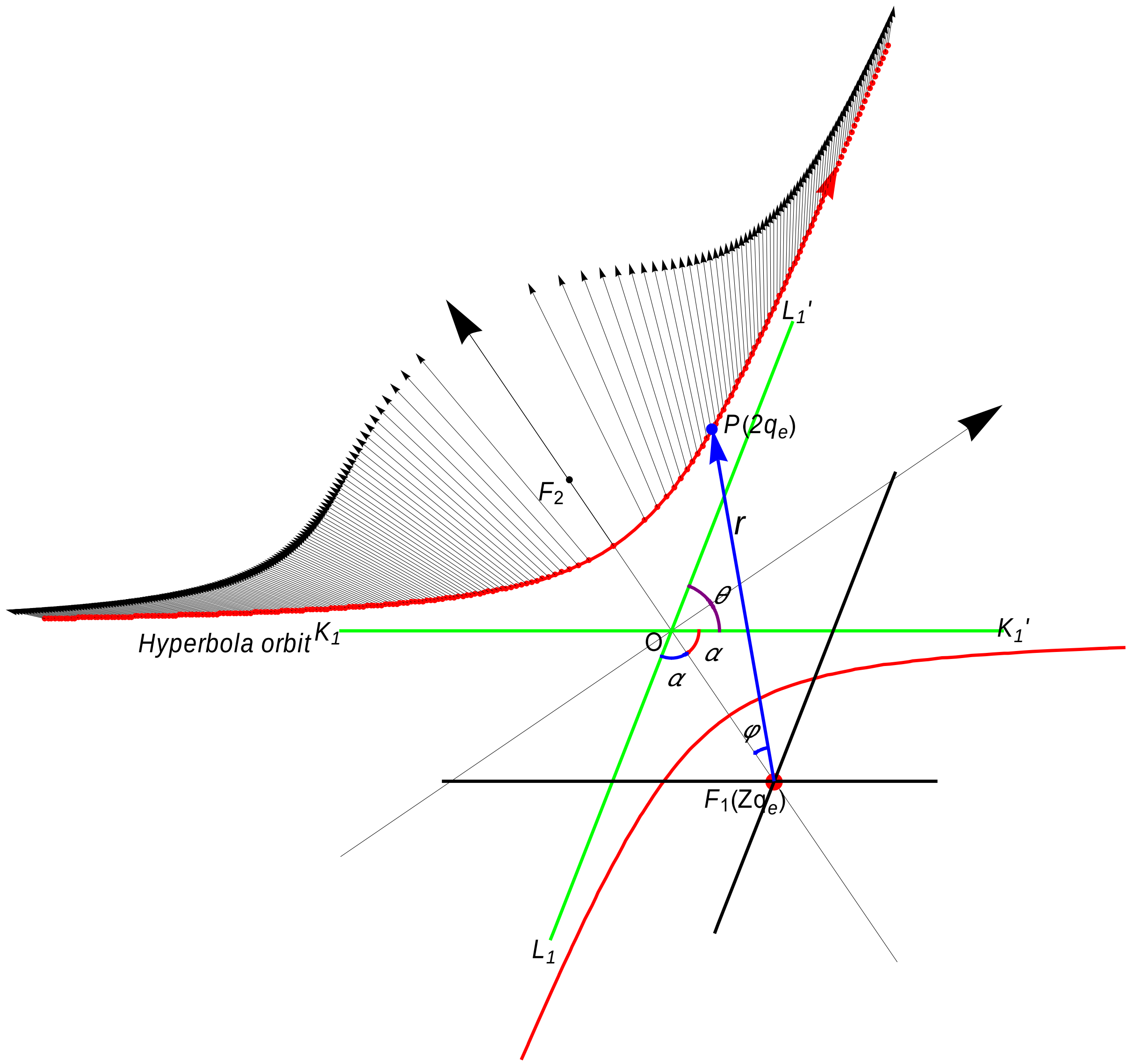

Fig.11 Diagram of the deflection of an *a* particle by a nucleus: Rutherford scattering. Repulsive force between the α particle ($2q_e$) on the hyperbola orbit and the atoms (Zq_e) at the focal point F_1. Two asymptotes: K_1K_1' and L_1L_1'.

The Lagrangian of the system is given by

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{\kappa}{r},$$

where

$$\kappa = 2Zq_e^{\ 2}.$$

The Lagrange equation is obtained as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \left(\frac{\partial L}{\partial r}\right), \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \left(\frac{\partial L}{\partial \varphi}\right),$$

leading to the equations of motion as

$$m\ddot{r} - mr\dot{\varphi}^2 = \frac{\kappa}{r^2},$$

and

$$L_z = mr^2\dot{\varphi} = \text{const.} \qquad \text{(conservation of angular momentum)}$$

Here we have

$$Ldt = mr^2 d\varphi.$$

Note that r depends only on φ.

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{\partial}{\partial\theta}\varphi = \frac{L}{mr^2}\frac{d}{d\varphi},$$

$$\frac{d}{dt}(\frac{d}{dt}) = \frac{L}{mr^2}\frac{d}{d\varphi}(\frac{L}{mr^2}\frac{d}{d\varphi}),$$

or

$$\frac{L}{mr^2}\frac{d}{d\theta}(\frac{L}{mr^2}\frac{dr}{d\theta}) = \frac{L^2}{m^2r^3} + \frac{\kappa}{mr^2}.$$

We define u as $u = \dfrac{1}{r}$,

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d}{d\theta}(\frac{1}{r}) = -\frac{du}{d\theta}.$$

Then we have

$$\frac{L^2}{m^2r^2}\frac{d}{d\varphi}(-\frac{du}{d\varphi}) = \frac{L^2}{m^2r^3} + \frac{\kappa}{mr^2},$$

or

$$\frac{d^2u}{d\varphi^2} + u = -\frac{m\kappa}{L^2}.$$

where

$$\frac{m\kappa}{L^2} = \frac{m\kappa}{(mv_0b)^2} = \frac{m\kappa}{2K_0b^2}.$$

Then we have

$$u = A\cos\varphi - \frac{m\kappa}{L^2},$$

since u is an even function of φ. So there is no term of $\sin\varphi$. When $\varphi = 0$, $u = \frac{1}{r_{\min}} = \frac{1}{a(1+e)}$.

When $\varphi = \alpha = \frac{\pi - \theta}{2}$, $u = 0$. Then we get

$$u = \frac{1}{r} = \frac{m\kappa}{L^2}(e\cos\varphi - 1),$$

or

$$r = \frac{1}{u} = \frac{a(e^2-1)}{e\cos\varphi - 1} = \frac{l}{e\cos\varphi - 1},$$

where

$$\cos\alpha = \cos(\frac{\pi-\theta}{2}) = \sin\frac{\theta}{2} = \frac{1}{e}.$$

We note that l is called the semi-latus rectum,

$$l = a(e^2-1) = \frac{L^2}{m\kappa} = \frac{2K_0 b^2}{\kappa}.$$

12. Differential cross section: $\frac{d\sigma}{d\Omega}$ (classical case)

Let us consider all those particles that approach the target with impact parameters between b and b +db. These are incident on the annulus (the shaded ring shape). This annulus has cross sectional area

$$d\sigma = 2\pi b db .$$

These same particles emerge between angles θ and θ+ dθ in a solid angle given by

$$d\Omega = 2\pi \sin\theta d\theta .$$

The differential cross section $\frac{d\sigma}{d\Omega}$ is defined as follows.

$$d\sigma = \frac{d\sigma}{d\Omega} d\Omega = 2\pi b db ,$$

or

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b db}{d\Omega} = \frac{2\pi b db}{2\pi\sin\theta d\theta} = \frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{1}{2\sin\theta}\frac{d}{d\theta}b^2 .$$

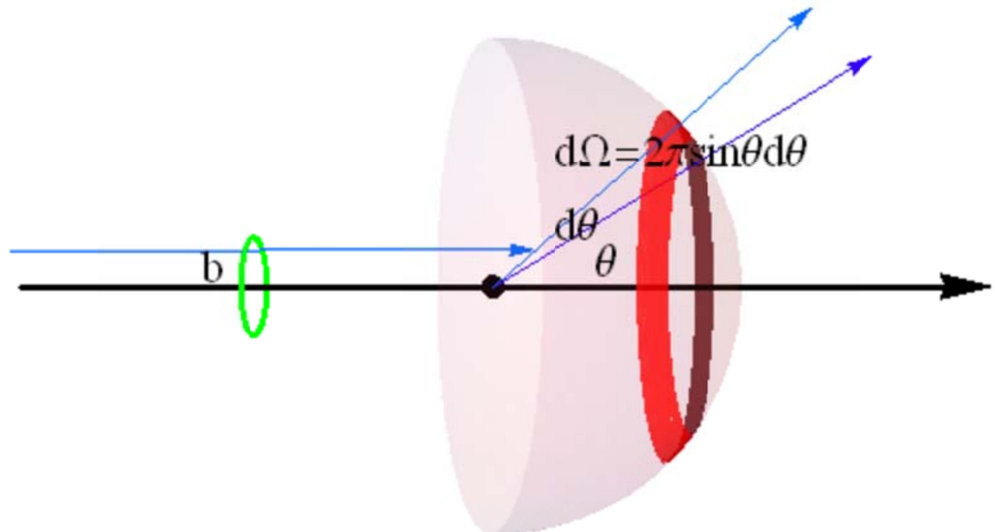

Fig.12(a)

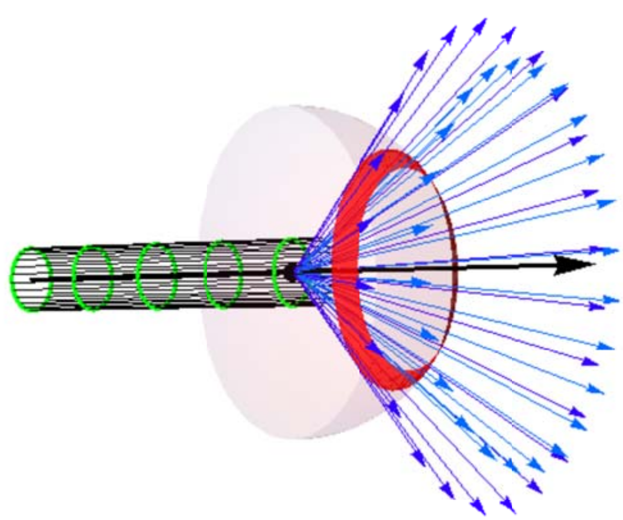

Fig.12(b)

Note that

$$\frac{db^2}{d\theta} = \left(\frac{\kappa}{2K_0}\right)^2 \frac{d}{d\theta}\cot^2\frac{\theta}{2} = -\left(\frac{\kappa}{2K_0}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}.$$

Then we get

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2\sin\theta}\frac{d}{d\theta}b^2 \\
&= \frac{\left(\frac{\kappa}{2K_0}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}d\theta}{2\sin\theta d\theta} \\
&= \frac{\left(\frac{\kappa}{2K_0}\right)^2 \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}}{4\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \\
&= \frac{\kappa^2}{16K_0^{\ 2}}\frac{1}{\sin^4\frac{\theta}{2}}
\end{aligned}$$

This is the celebrated Rutherford scattering formula. It gives the differential cross section for scattering of α particle ($2q_e$), with kinetic energy K_0, off a fixed target of charge (Zq_e)

((**Mathematica**))

```
Clear["Global`*"]; b1 = k Cot[θ/2];

f1 = D[b1, θ]

-1/2 k Csc[θ/2]^2

f2 = 1/(2 Sin[θ]) D[b1^2, θ] // TrigFactor

-1/4 k^2 Csc[θ/2]^4
```

13. Quantum mechanics (scattering due to the Yukawa potential

The Yukawa potential is given by

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r},$$

where V_0 is independent of r. $1/\mu$ corresponds to the range of the potential. The scattering amplitude (from the first order Born Approximation)

$$f^{(1)}(\theta) = -\frac{2\mu_m}{\hbar^2}\frac{1}{Q}\int_0^\infty dr' r' \frac{V_0}{\mu r'} e^{-\mu r'} \sin(Qr')$$
$$= -\frac{2\mu_m}{\hbar^2}\frac{1}{Q}\frac{V_0}{\mu}\int_0^\infty dr' e^{-\mu r'} \sin(Qr')$$

Here we use μ_m for the reduced mass in order to avoid the confusion of the co-efficient μ for the Yukawa potential with the reduced mass.

$$\mu_m = \frac{mM}{m+M} \approx m,$$

where m is the mass of a particle and M is the mass of atom; $M >> m$

Note that

$$\int_0^\infty dr' e^{-\mu r'} \sin(Qr') = \frac{Q}{Q^2+\mu^2}, \qquad \text{(Laplace transformation)}$$

$$f^{(1)}(\theta) = -\frac{2\mu_0}{\hbar^2}\frac{V_0}{\mu}\frac{1}{Q^2+\mu^2}.$$

Since

$$Q^2 = 4k^2 \sin^2(\frac{\theta}{2}) = 2k^2(1-\cos\theta).$$

so, in the first Born approximation,

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta)\right|^2 = \left(\frac{2\mu_m V_0}{\mu\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)+\mu^2]^2}.$$

Note that as $\mu \to 0$, the Yukawa potential is reduced to the Coulomb potential, provided the ratio V_0/μ is fixed.

$$\frac{V_0}{\mu} = 2Zq_e^{\ 2} = \kappa,$$

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta)\right|^2 = \frac{(2\mu_m)^2(2Zq_e^{\ 2})^2}{\hbar^4}\frac{1}{16k^4\sin^4(\theta/2)}.$$

Using $K_0 = \dfrac{\hbar^2 k^2}{2\mu_m}$, we have

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 = \frac{1}{16}\frac{\kappa^2}{K_0^{\ 2}}\frac{1}{\sin^4(\theta/2)},$$

which is the Rutherford scattering cross section (that can be obtained classically).
The total cross section can be obtained as follows.

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^2}{16K_0^{\ 2}}\frac{1}{(\sin^2\frac{\theta}{2}+\frac{\mu^2}{4k^2})^2}.$$

The total cross section is

$$\sigma = 2\pi\int\frac{d\sigma}{d\Omega}\sin\theta d\theta = \frac{\kappa^2}{16K_0^{\ 2}}\int_0^\pi \frac{\sin\theta d\theta}{(\sin^2\frac{\theta}{2}+\frac{\mu^2}{4k^2})^2}.$$

The change of variable $x=\frac{2k}{\mu}\sin\frac{\theta}{2}$ leads to $\sin\theta d\theta=\frac{\mu^2}{k^2}xdx$. Then

$$
\begin{aligned}
\sigma &= \frac{\kappa^2 k^2}{\mu^2 K_0{}^2}\int_0^{2k/\mu}\frac{xdx}{(x^2+1)^2} \\
&= \frac{\kappa^2 k^2}{2\mu^2 K_0{}^2}\left(\frac{\frac{4k^2}{\mu^2}}{1+\frac{4k^2}{\mu^2}}\right) \\
&= \frac{2\kappa^2 k^4}{\mu^2 K_0{}^2}\frac{1}{\mu^2+4k^2}
\end{aligned}
$$

14. Schematic diagram for the Rutherford scattering

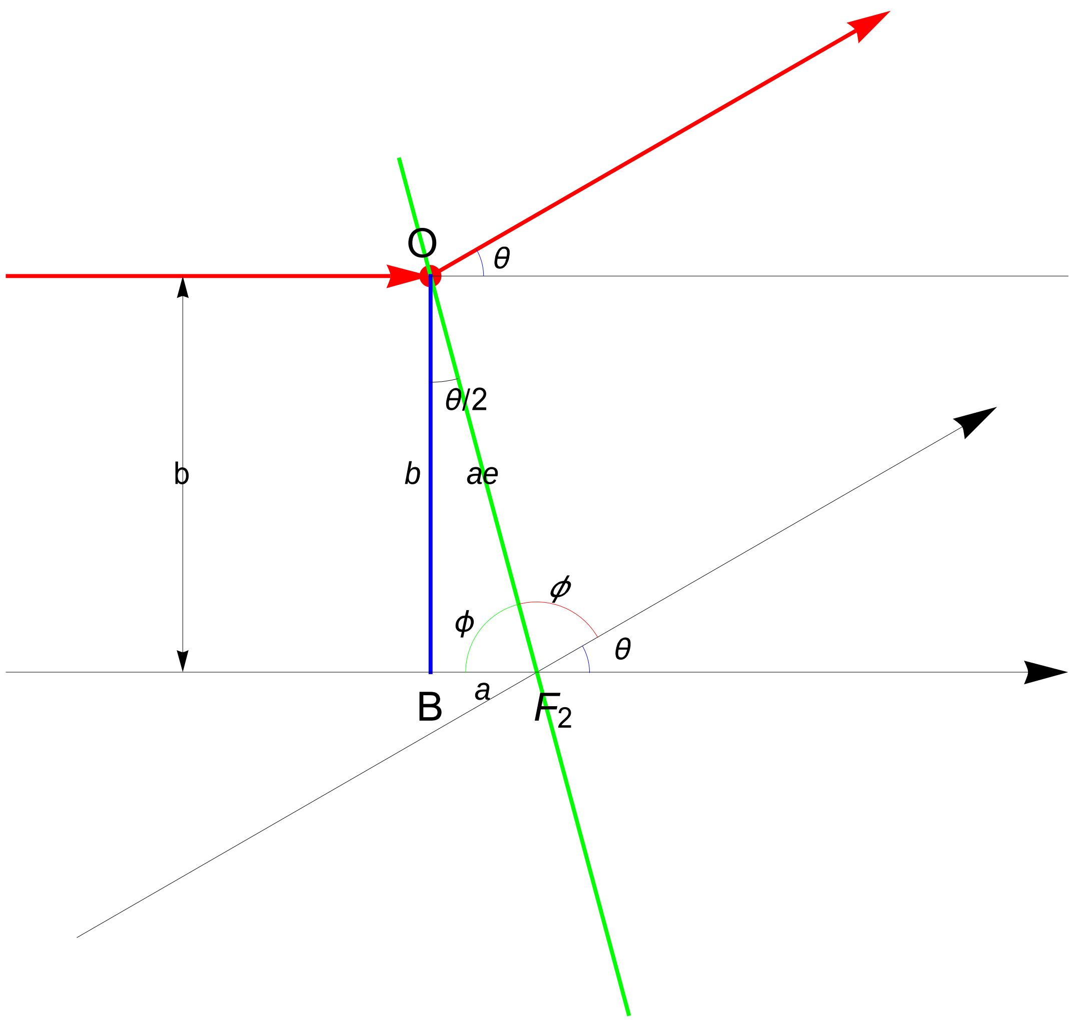

Fig.13 Schematic diagram for the Rutherford scattering. b is the impact parameter and θ is the scattering angle. The hyperbolic orbit near the target (at the point F_2) is simplified by a straight line. $\overline{OF_1} = ae$. The point A is the intersection of the initial and final asymptotes of the hyperbola. $b = a\cot\frac{\theta}{2}$. Geometry for the Rutherford scattering. $b = ae\cos\frac{\theta}{2}$. θ is the scattering angle.

As shown in the above figure, the impact parameter b is given by

$$b = ae\sin\phi = ae\sin(\frac{\pi}{2} - \frac{\theta}{2}) = ae\cos\frac{\theta}{2} = a\cot\frac{\theta}{2}.$$

The impact parameter b is also expressed by

$$b = \frac{\kappa}{2K_0}\cot\frac{\theta}{2},$$

where

$$a = \frac{\kappa}{2K_0}. \qquad \text{(units of length)}$$

The differential cross section can be expressed by

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\kappa^2}{16K_0^{\ 2}}\frac{1}{\sin^4\frac{\theta}{2}} \\ &= \frac{a^2}{4}\frac{1}{\sin^4\frac{\theta}{2}} \\ &= \frac{(ae)^4}{4a^2}\end{aligned}$$

where

K_0 is the kinetic energy. $L = mv_0 b$ (angular momentum). $\sin\frac{\theta}{2} = \frac{1}{e}$,

$$L^2 = m^2 v_0{}^2 b^2 = 2mb^2 K_0 = 2mb^2 \frac{\kappa}{2a} = m\kappa \frac{b^2}{a} = m\kappa \frac{a^2}{a}(e^2 - 1) = m\kappa a(e^2 - 1) .$$

or

$$L^2 = m\kappa l .$$

where l is called the semi-latus rectum,

$$l = \frac{b^2}{a} = a(e^2 - 1) .$$

In general, a particle with impact parameters smaller than a particular value of b will have scattering angles larger than the corresponding value of b will have scattering angles larger than the corresponding value of θ. The area πb^2 is called the cross section for scattering with angles greater than θ.

((**Note**))

Here we discuss how to draw the diagram for the simplified Rutherford scattering.

In Fig.13, the length $\overline{BF_2} = a$ is given. The scattering angle is changed as a parameter. The impact parameter b is $b = a\cot\frac{\theta}{2}$. The length $\overline{OF_2}$ is ae. The point O is expressed by

$$\overrightarrow{OA} = (-ae\cos\phi, ae\sin\phi) = ae(-\sin\frac{\theta}{2}, \cos\frac{\theta}{2}) ,$$

where $\phi = (\pi - \theta)/2$.

So we make a plot of the above diagram when the scattering angle θ is changed as a parameter with the value of a kept fixed. The diagram consists of the initial and the final asymptotes of the hyperbola. For simplicity, the hyperbola is replaced by the two asymptotes. The point O is the intersection of two asymptotes. Because of the angular momentum conservation, the impact parameter b remains unchanged for both initial and final asymptotes.

The eccentricity ε of the hyperbola is

$$e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2} \ (>1).$$

So we have

$$\sin\frac{\theta}{2} = \frac{a}{ae} = \frac{1}{e}.$$

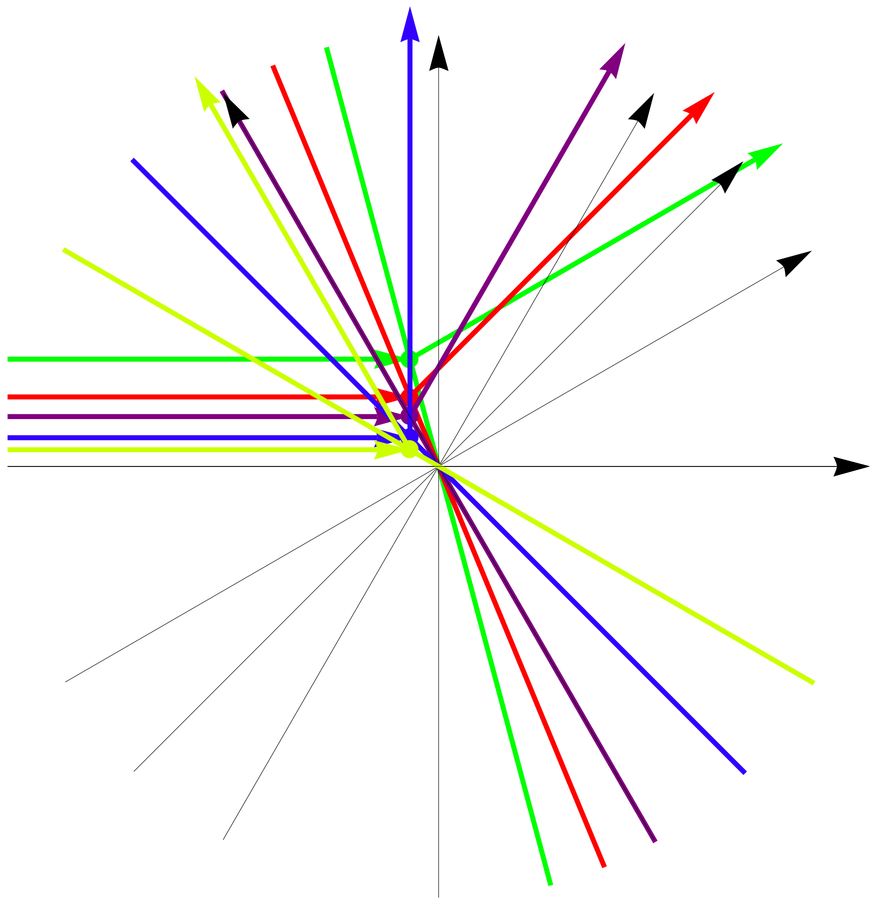

Fig.14(a) Schematic diagram for the Rutherford scattering where θ is varied as a parameter. The relation between the impact parameter b and the scattering angle θ. As b increases, the angle θ decreases (smaller angle).

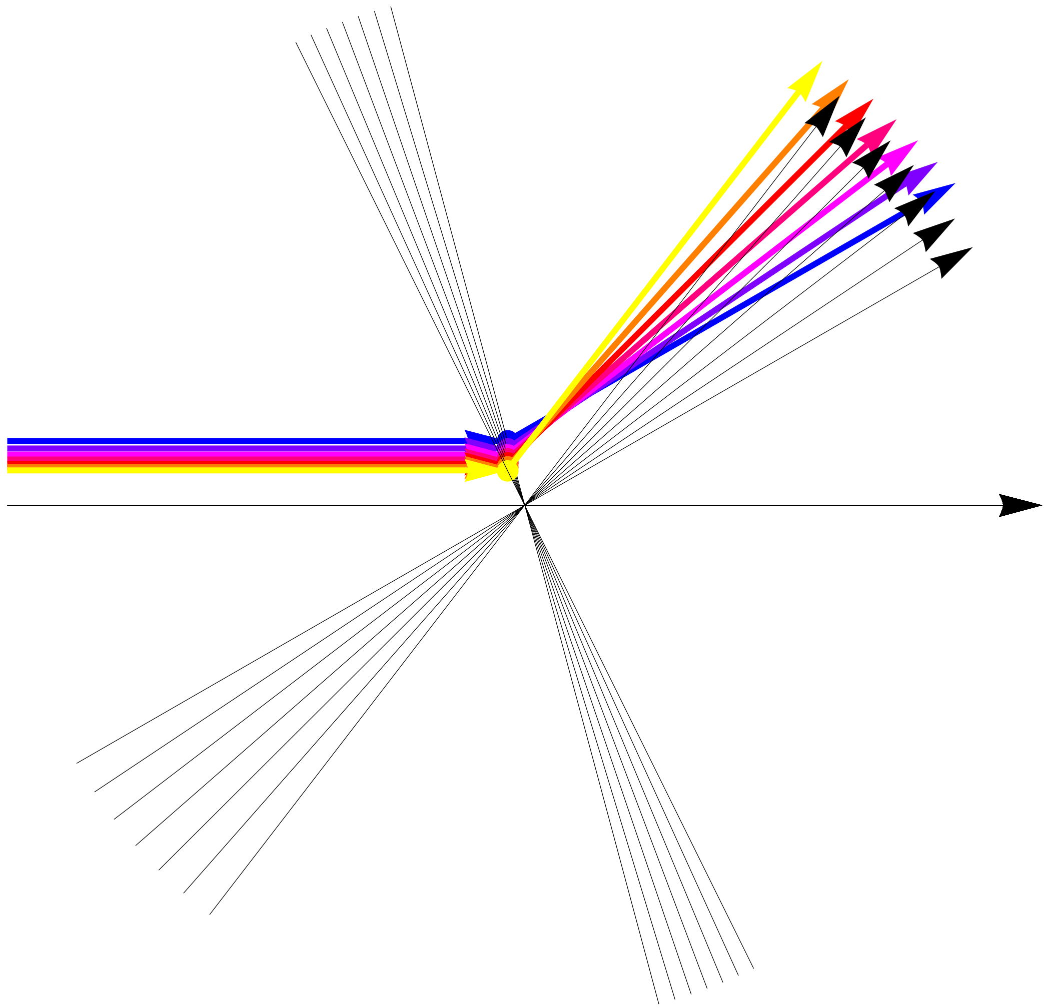

Fig.14(b) The α particles with impact parameters between b and b + db are scattered into the angular range between θ and θ+ dθ.

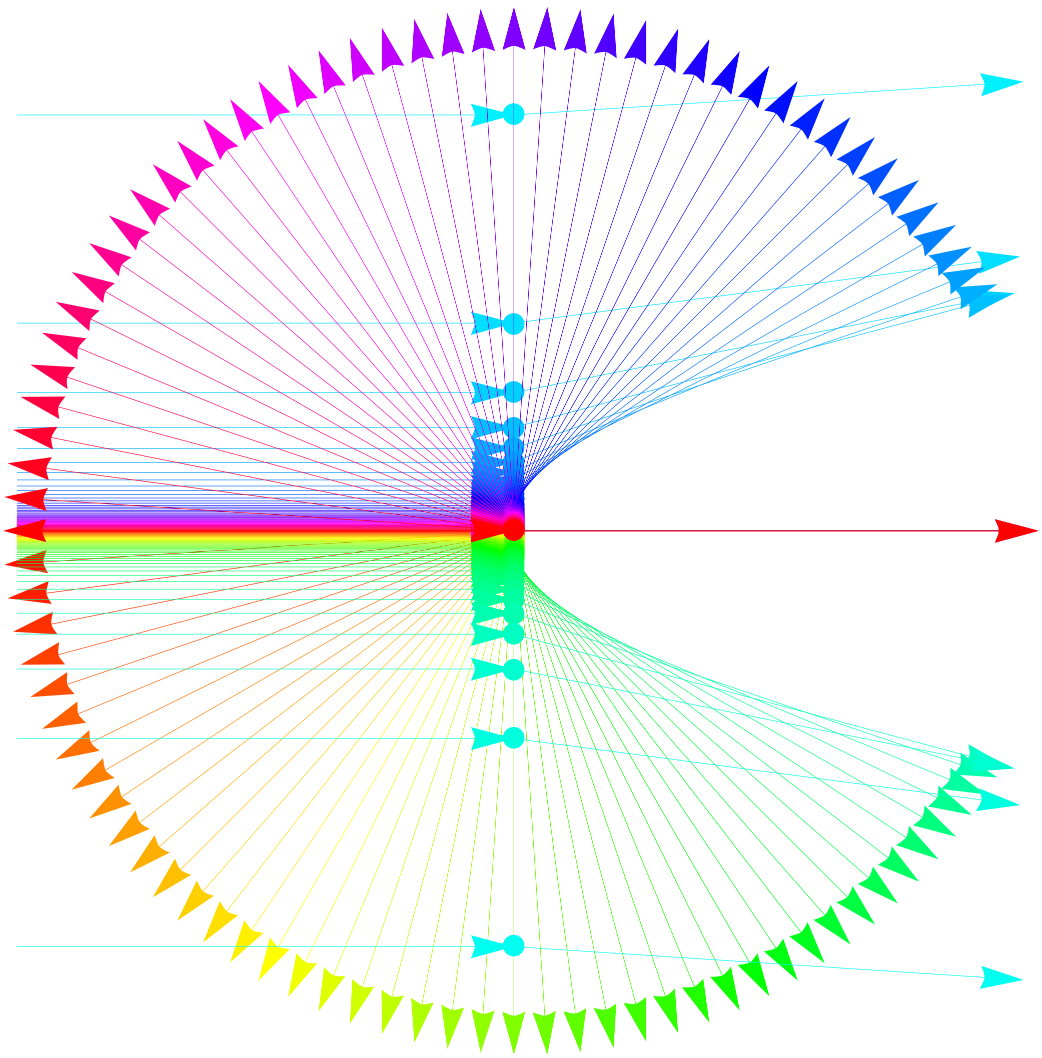

Fig.14(c) Rutherford scattering of α particles. The hyperbolic orbit near the target (at the point O) is simplified by a straight line. $\overline{OA} = R$. The point denoted by $\overrightarrow{OA}$ is shown in the figure. The value of a (related to the kinetic energy of the particle) is kept constant.

15. The use of Mathematica for drawing the hyperbola

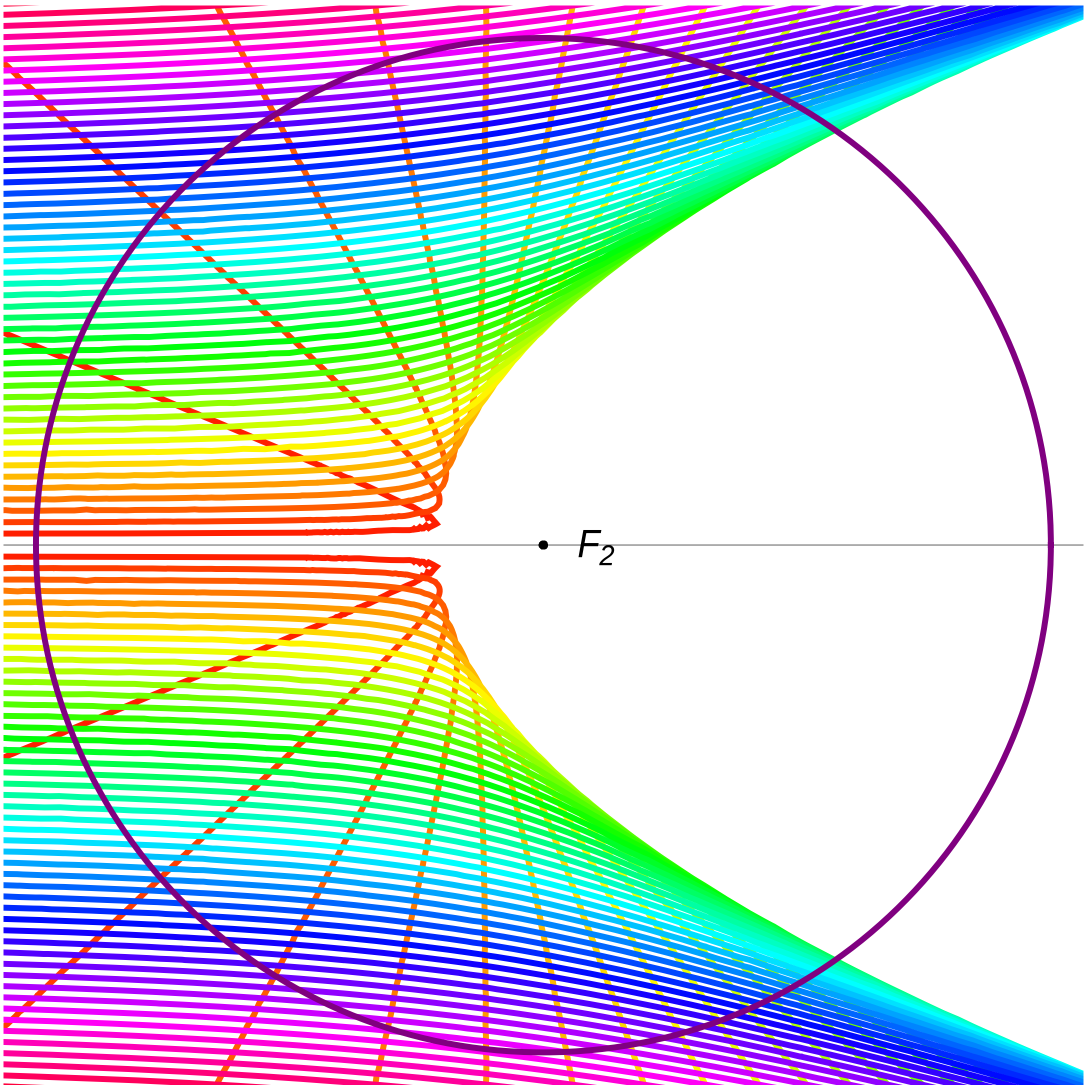

Fig.15 a = 0.5 (fixed). b is changed as a parameter ($0.1 \le b \le 5$ with $\Delta b = 0.1.$). F_1 is the focal point (scatterer). The center of circle is at the point F_2.The detector is on the circle.

((**Mathematica**))

```
Clear["Global`*"];

Hy1[a_, b_] := Module[{e1, ϕ1, J1, J2, J3, J4}, e1 = Sqrt[a^2 + b^2]/a;
  ϕ1 = ArcTan[b/a];
  J1 = ContourPlot[x^2/a^2 - y^2/b^2 == 1, {x, -6, 0}, {y, -6, 6},
    ContourStyle → {Hue[b / 5], Thick}, Axes → False];
  J2 = Graphics[{Translate[J1[[1]], {-a e1, 0}],
     Point[{0, 0}]}];
  J3 = Graphics[{Rotate[J2[[1]], -ϕ1, {0, 0}]}];
  J4 = Graphics[{Rotate[J2[[1]], ϕ1, {0, 0}]}]; Show[J3, J4]];

G1 = Graphics[{Black, Thin, Line[{{-5, 0}, {5, 0}}],
   Purple, Thick, Circle[{0, 0}, 4.7],
   Text[Style["F2", Black, Italic, 12], {0.5, 0}] }];
G2 = Show[ Table[Hy1[0.5, b], {b, 0.1, 5, 0.1}], G1,
  PlotRange → {{-5, 5}, {-5, 5}}]
```

16. Experimental results

If the gold foil were 1 micrometer thick, then using the diameter of the gold atom from the periodic table suggests that the foil is about 2800 atoms thick.

Density of Au

$$\rho = 19.30 \text{ g/cm}^3.$$

Atomic mass of Au;

$$M_g = 196.96654 \text{ g/mol}.$$

The number of Au atoms per cm^3;

$$n = \frac{\rho(g/cm^3)}{M_g(g/mol)} N_A,$$

where N_A is the Avogadro number. Then we get the number of target nuclei in the volume At (cm^3) as

$$N_s = ntA\,.$$

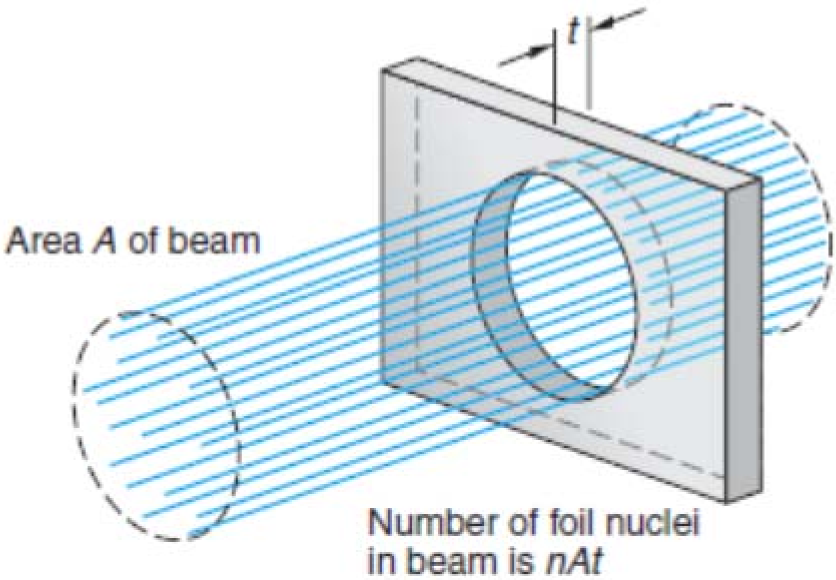

Fig.16 The total number of nuclei of foil atoms in the area covered by the beam is ntA, where n is the number of foil atoms per unit volume, A is the area of the beam, and t is the thickness of the foil.
[P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.8)]

If $\sigma(=\pi b^2)$ is the cross section for each nucleus, $ntA\sigma$ is the total area exposed by the target nuclei. The fraction of incident particles scattered by an angle of θ or *greater* is

$$f = \frac{ntA\sigma}{A} = nt\sigma = nt\pi b^2 = nt\pi\left(\frac{Zq_e^{\,2}}{K_0}\right)^2 \cot^2\frac{\theta}{2}.$$

The number of α particles which can be compared with measurements, is defined by

$$N(\theta) = I_0 nt\frac{d\sigma}{d\Omega}\Delta\Omega_{sc} = \left(\frac{I_0 A_{sc} nt}{r^2}\right)\frac{Z^2 q_e^{\,4}}{4K_0^{\,2}}\frac{1}{\sin^4\dfrac{\theta}{2}},$$

where r is the distance between the target and the detector, I_0 is the intensity of incident α particles, n is the number density of the target, and the solid angle $\Delta\Omega_{sc}$ is defined by

$$\Delta\Omega_{sc} = \frac{A_{sc}}{r^2}.$$

((**Experimental results**))

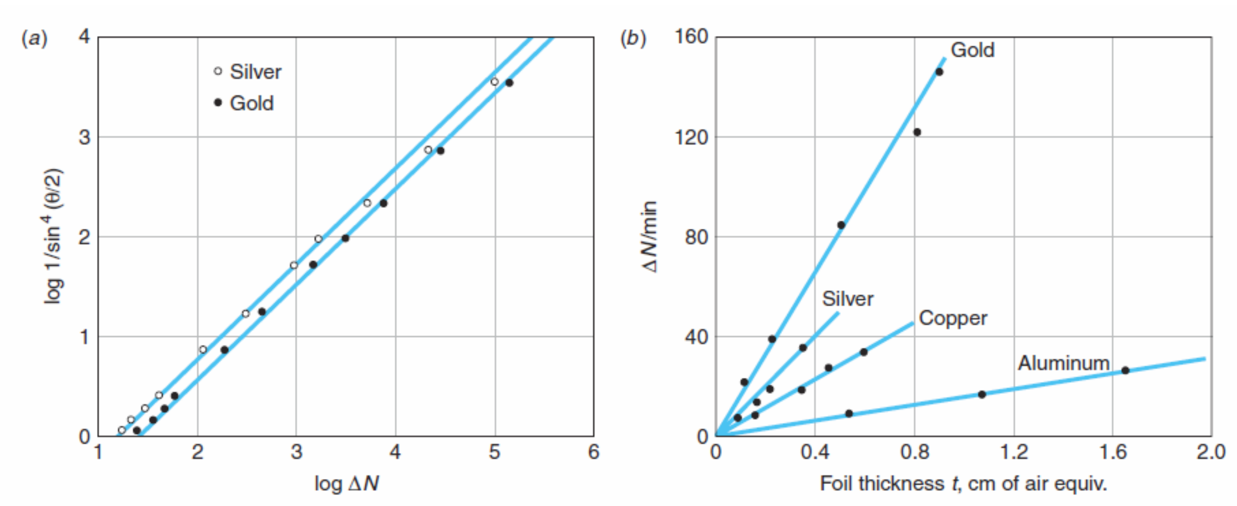

Fig.17 (a) Geiger and Marsden's data for α scattering from thin gold and silver foils. The graph is a log-log plot to show the data over several orders of magnitude. Note that scattering angle increases downward along the vertical axis. (b) Geiger and Marsden also measured the dependence of ΔN on t predicted by $N(\theta) = I_0 nt \frac{d\sigma}{d\Omega}\Delta\Omega_{sc} = \left(\frac{I_0 A_{sc} nt}{r^2}\right)\frac{Z^2 q_e^{\ 4}}{4K_0^{\ 2}}\frac{1}{\sin^4\frac{\theta}{2}}$ for foils made from a wide range of elements, this being an equally critical test. Results for four of the elements used are shown. $Z = 79$ for Au. $Z = 47$ for Ag, $Z = 29$ for Cu and $Z = 13$ for Al.
P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (Fig.4.9).

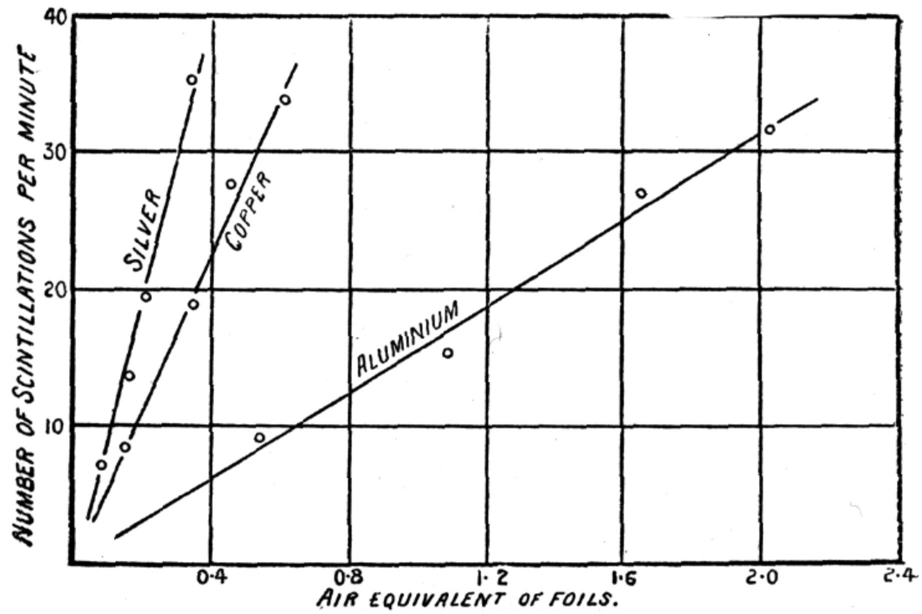

Fig.18 Original data presented by by H. Geiger and E. Marsden [Pjil. Mag. 24, 604, 1913]

Using the value of $N(\theta = \pi)$, we have

$$\frac{N(\theta)}{N(\theta = \pi)} = \frac{1}{\sin^4 \dfrac{\theta}{2}},$$

where

$$N(\theta = \pi) = \frac{I_0 A_{sc} n t Z^2 q_e{}^4}{4 r^2 K_0{}^2}.$$

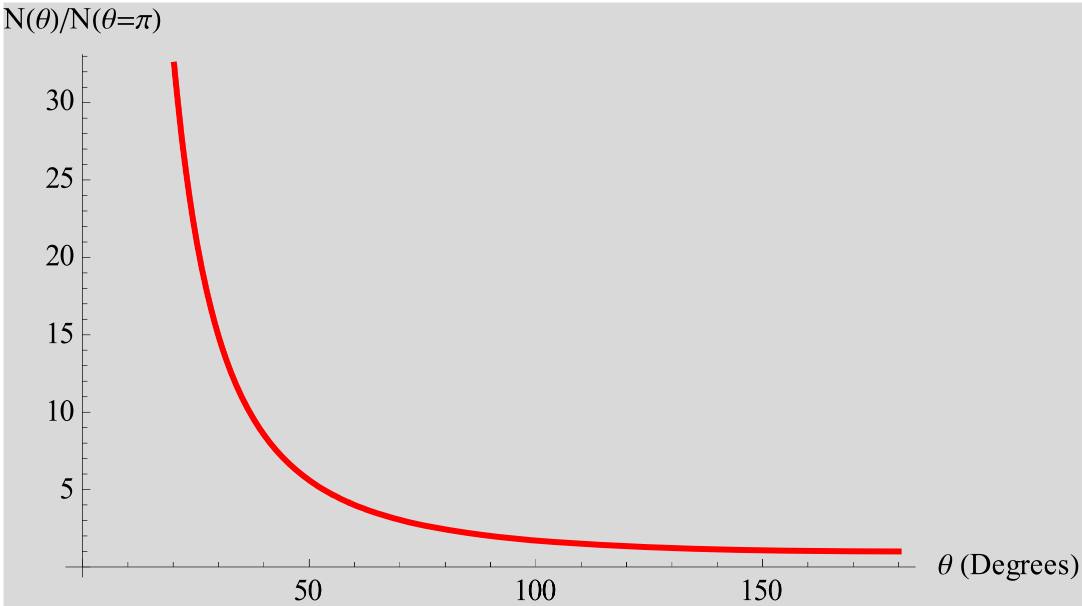

Fig.19 Plot of $\dfrac{N(\theta)}{N(\theta=\pi)} = \dfrac{1}{\sin^4 \dfrac{\theta}{2}}$ as a function of scattering angle θ.

17. Rough evaluation for the size of nucleus

We use the energy conservation law, we have

$$E_{tot} = K + U = \frac{1}{2}mv^2 + \frac{\kappa}{r} = const\,,$$

where K is the kinetic energy and U is the potential energy. At $r = \infty$,

$$E_{tot} = \frac{1}{2}mv_0{}^2 = K_0\,.$$

At $r = r_0$ (size of nucleus)

$$E_{tot} = \frac{\kappa}{r_0}\,,$$

when $v = 0$. Then we have

$$\frac{\kappa}{r_0} = K_0\,,$$

or

$$r_0 = \frac{\kappa}{K_0}.$$

We note that r_0 is related to a as

$$a = \frac{1}{2} r_0.$$

Note that when $r_0 = 2a$, α particle undergoes a head-on collision, during which the velocity of the α particle becomes zero.

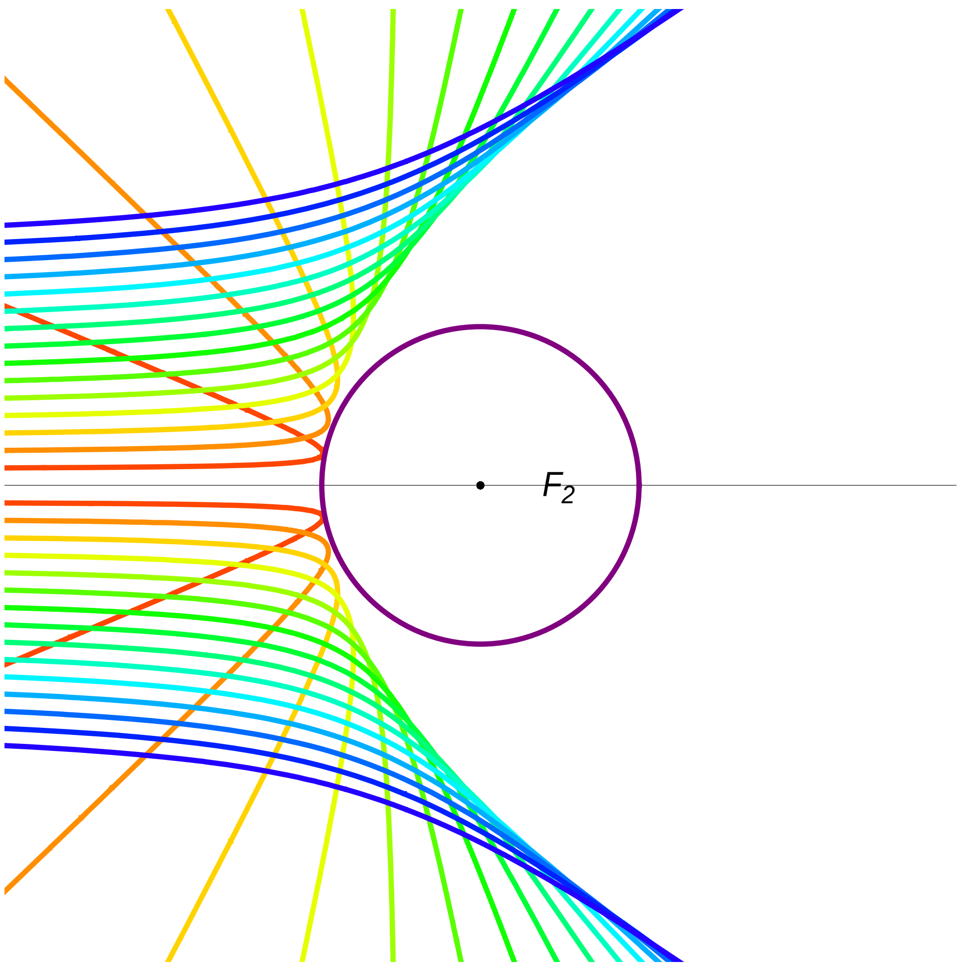

Fig.20 Rutherford scattering. $a = 0.05$ (fixed). b is changed as a parameter between $b = 0.01$ and 0.15, ($\Delta b = 0.01$). The circle centered at F_2 has a radius $r_0 = 2a$.

((**Example**)) Z = 79 for Au. K = 7.7 MeV.

$r_0 = 2.955 \times 10^{-14}$ m = 29.5474 fermi

In conclusion, most of the mass and all of the positive charge of an atom, $+Zq_e$, are concentrated in a minute volume of the atom with a diameter of about 10^{-14} m.

1 fermi = 10^{-15} m

18. Summary: From Rutherford scattering to Bohr model of hydrogen atom
M. Longair, Quantum Concepts in Physics (Cambridge, 2013).

Rutherford attended the First Solvay Conference in 1911, but made no mention of his remarkable experiments, which led directly to his nuclear model of the atom. Remarkably, this key result for understanding the nature of atoms made little impact upon the physics community at the time and it was not until 1914 that Rutherford was thoroughly convinced of the necessity of adopting his nuclear model of the atom. Before that time, however, Niels Bohr, the first theorist to apply successfully quantum concepts to the structure of atoms. Niels Bohr spent four months with Rutherford in Manchester. Bohr was immediately struck by the significance of Rutherford's model of the nuclear structure of the atom and began to devote all his energies to understanding atomic structure on that basis. In the summer of 1912, Bohr wrote an unpublished memorandum for Rutherford, in which he made his first attempt at quantizing the energy levels of the electrons in atoms (Bohr, 1912).

In 1913 Niels Bohr proposed a model of the hydrogen atom that combined the work of Planck, Einstein, and Rutherford and was remarkably successful in predicting the observed spectrum of hydrogen. The Rutherford model assigned charge and mass to the nucleus but was silent regarding the distribution of the charge and mass of the electrons. Bohr made the assumption that the electron in the hydrogen atom moved in an orbit about the positive nucleus, bound by the electrostatic attraction of the nucleus. Classical mechanics allows circular or elliptical orbits in this system, just as in the case of the planets orbiting the Sun. For simplicity, Bohr chose to consider circular orbits. Such a model is mechanically stable

REFERENCES

E. Rutherford, Philosophical Magazine 21, 669-688 (1911). "The Scattering of α and β particles by Matter and the Structure of the Atom."

H. Geiger and E. Marsden, *Philosophical Magazine*, **25**, 604–623 (1913). "The laws of deflexion of α-particles through large angles."

D. Bohm, Quantum Theory (Prentice-Hall, 1951). (Dover, 1989).

S. Tomonaga, Quantum Mechanics I: Old Quantum Theory (North Holland, 1962).

S. Wright, Classical Scientific Papers Physics (Mills & Boon, 1964).

H.E. White, Introduction to Atomic and Nuclear Physics (D. Van Nostrand Company, Inc., Princeton, NJ, 1964).

D. ter Haar, The Old Quantum Theory (Pergamon. 1967).

A. Beiser, Perspectives of Modern Physics (McGraw-Hill, 1969).

J.B. Marion, Classical Dynamics of Particles and Systems, second edition (Academic Press, 1970).

A.P. French, Newtonian Mechanics (W.W. Norton &Company. Inc., 1971).

H. Goldstein, Classical Mechanics (Addison-Wesley, 1980).

E. Segre, From X-rays to Quarks: Modern Physicists and Their Discoveries (W.H. Freeman and Company, 1980).

A.D. Davis, Classical mechanics (Academic Press, 1986).

A.L. Fetter and J.D. Walecka, Theoretical Mechanics of Particles and Continua (Dover, 2003).

P.A. Tipler and R.A. Llewellyn, Modern Physics 5-th edition (W.H. Freeman, 2008).

V. Barger and M. Olsson, Classical Mechanics: A Modern Perspective (McGraw-Hill, 2011)

K. Krane, Modern Physics. Third edition (John Wiley & Sons, 2012).

M. Longair, Quantum Concepts in Physics (Cambridge, 2013).

J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, second edition (Addison-Wesley, 2011).