Sakurai
Chapter-1 Solution
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((**1-1**))

1. Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB.$$

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = -\hat{A}\hat{C}\{\hat{D}, \hat{B}\} + \hat{A}\{\hat{C}, \hat{B}\}\hat{D} - \hat{C}\{\hat{D}, \hat{A}\}\hat{B} + \{\hat{C}, \hat{A}\}\hat{D}\hat{B}$$

Proof:

Left had side = $\hat{A}\hat{B}\hat{C}\hat{D} - \hat{C}\hat{D}\hat{A}\hat{B}$

Right hand side = $\hat{A}\hat{C}(\hat{D}\hat{B} + \hat{B}\hat{D}) + \hat{A}(\hat{C}\hat{B} + \hat{B}\hat{C})\hat{D} - \hat{C}(\hat{D}\hat{A} + \hat{A}\hat{D})\hat{B} + (\hat{C}\hat{A} + \hat{A}\hat{C})\hat{D}\hat{B}$

= $\hat{A}\hat{B}\hat{C}\hat{D} - \hat{C}\hat{D}\hat{A}\hat{B}$

Thus we have

Left had side = Right hand side

((**1-2**))

2. Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as

$$X = a_0 + \boldsymbol{\sigma}\cdot\mathbf{a},$$

where a_0 and $a_{1,2,3}$ are numbers.

a. How are a_0 and a_k $(k = 1, 2, 3)$ related to tr(X) and tr$(\sigma_k X)$?

b. Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

$$\hat{X} = a_0\hat{1} + \hat{\sigma}_1 a_1 + \hat{\sigma}_2 a_2 + \hat{\sigma}_3 a_3 = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

with

$$\hat{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X_{11} = a_0 + a_3$, $X_{12} = a_1 - ia_2$, $X_{21} = a_1 + ia_2$, and $X_{22} = a_0 - a_3$

(a)

$$Tr(\hat{1}) = 2\,,\; Tr(\hat{\sigma}_1) = Tr(\hat{\sigma}_2) = Tr(\hat{\sigma}_3) = 0$$

$$Tr(\hat{X}) = Tr(a_0\hat{I} + \hat{\sigma}_1 a_1 + \hat{\sigma}_2 a_2 + \hat{\sigma}_3 a_3) = 2a_0$$

Using the relations

$$\hat{\sigma}_1\hat{\sigma}_2 = -\hat{\sigma}_2\hat{\sigma}_1 = i\hat{\sigma}_3$$

$$\hat{\sigma}_2\hat{\sigma}_3 = -\hat{\sigma}_3\hat{\sigma}_2 = i\hat{\sigma}_1$$

$$\hat{\sigma}_3\hat{\sigma}_1 = -\hat{\sigma}_1\hat{\sigma}_3 = i\hat{\sigma}_2$$

$$\hat{\sigma}_1^2 = \hat{\sigma}_2^2 = \hat{\sigma}_3^2$$

$$\begin{aligned}
Tr(\hat{\sigma}_1\hat{X}) &= Tr[\hat{\sigma}_1(a_0\hat{1} + \hat{\sigma}_1 a_1 + \hat{\sigma}_2 a_2 + \hat{\sigma}_3 a_3)] \\
&= Tr(\hat{\sigma}_1 a_0 + \hat{\sigma}_1^2 a_1 + \hat{\sigma}_1\hat{\sigma}_2 a_2 + \hat{\sigma}_1\hat{\sigma}_3 a_3) \\
&= Tr(\hat{\sigma}_1 a_0 + \hat{\sigma}_1^2 a_1 + i\hat{\sigma}_3 a_2 - i\hat{\sigma}_2 a_3) \\
&= 2a_1
\end{aligned}$$

Similarly, we have

$$Tr(\hat{\sigma}_2\hat{X}) = 2a_2$$

$$Tr(\hat{\sigma}_3\hat{X}) = 2a_3$$

$$\hat{\sigma}_1\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{21} & X_{22} \\ X_{11} & X_{12} \end{pmatrix}$$

$$\hat{\sigma}_2\hat{X} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} -iX_{21} & -iX_{22} \\ iX_{11} & iX_{12} \end{pmatrix}$$

$$\hat{\sigma}_3\hat{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} X_{11} & X_{12} \\ -X_{21} & -X_{22} \end{pmatrix}$$

$$a_0 = \frac{Tr(\hat{X})}{2} = \frac{X_{11} + X_{22}}{2}$$

$$a_1 = \frac{Tr(\hat{\sigma}_1\hat{X})}{2} = \frac{X_{21} + X_{12}}{2}$$

$$a_2 = \frac{Tr(\hat{\sigma}_2\hat{X})}{2} = \frac{-iX_{21} + iX_{12}}{2} = \frac{i}{2}(X_{12} - X_{21})$$

$$a_3 = \frac{Tr(\hat{\sigma}_3\hat{X})}{2} = \frac{X_{11} - X_{22}}{2}$$

((Mathematica 5.2))

```
(*Sakurai 1-2*)
SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :→ Complex[a, -b]}
σ1={{0,1},{1,0}}
```

```
{{0,1},{1,0}}
σ2={{0,-ⅈ},{ⅈ,0}}
{{0,-ⅈ},{ⅈ,0}}
σ3={{1,0},{0,-1}}
{{1,0},{0,-1}}
ϵ={{1,0},{0,1}}
{{1,0},{0,1}}
X=a0 ϵ+σ1 a1 +σ2 a2+σ3 a3
{{a0+a3,a1-ⅈ a2},{a1+ⅈ a2,a0-a3}}
X//MatrixForm
```

$$\begin{pmatrix} a0 + a3 & a1 - i\, a2 \\ a1 + i\, a2 & a0 - a3 \end{pmatrix}$$

```
Tr[X]
2 a0
ϵ//MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Tr[ϵ]
2
Tr[σ1]
0
Tr[σ2]
0
Tr[σ3]
0
Tr[X]
2 a0
Tr[σ1. X]
2 a1
Tr[σ2. X]//Simplify
2 a2
Tr[σ3. X]//Simplify
2 a3

{{a1+ⅈ a2,a0-a3},{a0+a3,a1-ⅈ a2}}
%//MatrixForm
```

$$\begin{pmatrix} a1 + i\, a2 & a0 - a3 \\ a0 + a3 & a1 - i\, a2 \end{pmatrix}$$

```
σ2.X//Simplify
{{-ⅈ a1+a2,-ⅈ (a0-a3)},{ⅈ (a0+a3),ⅈ a1+a2}}
%//MatrixForm
```

$$\begin{pmatrix} -i\, a1 + a2 & -i\,(a0 - a3) \\ i\,(a0 + a3) & i\, a1 + a2 \end{pmatrix}$$

```
σ3.X//Simplify
{{a0+a3,a1-i a2},{-a1-i a2,-a0+a3}}
%//Simplify
{{a0+a3,a1-i a2},{-a1-i a2,-a0+a3}}
```

((**1-3**))

3. Show that the determinant of a 2×2 matrix $\boldsymbol{\sigma}\cdot\mathbf{a}$ is invariant under

$$\boldsymbol{\sigma}\cdot\mathbf{a} \to \boldsymbol{\sigma}\cdot\mathbf{a}' \equiv \exp\left(\frac{i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right)\boldsymbol{\sigma}\cdot\mathbf{a}\exp\left(\frac{-i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\phi}{2}\right).$$

Find a'_k in terms of a_k when $\hat{\mathbf{n}}$ is in the positive z-direction and interpret your result.

$$f_1 = \hat{\sigma}\cdot\mathbf{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix}$$

$$\exp\left(\frac{i\hat{\sigma}_z\phi}{2}\right) = \begin{pmatrix} e^{\frac{i\phi}{2}} & 0 \\ 0 & e^{-\frac{i\phi}{2}} \end{pmatrix}$$

$$\exp\left(-\frac{i\hat{\sigma}_z\phi}{2}\right) = \begin{pmatrix} e^{-\frac{i\phi}{2}} & 0 \\ 0 & e^{\frac{i\phi}{2}} \end{pmatrix}$$

$$\exp\left(\frac{i\hat{\sigma}_z\phi}{2}\right)(\hat{\sigma}\cdot\mathbf{a})\exp\left(-\frac{i\hat{\sigma}_z\phi}{2}\right) = \begin{pmatrix} a_z & (a_x - ia_y)e^{i\phi} \\ (a_x + ia_y)e^{-i\phi} & -a_z \end{pmatrix}$$

which is equal to

$$f_2 = \hat{\sigma}\cdot\mathbf{a}' = \sigma_x a'_x + \sigma_y a'_y + \sigma_z a'_z = \begin{pmatrix} a'_z & a'_x - ia'_y \\ a'_x + ia'_y & -a'_z \end{pmatrix}$$

or

$$a'_x - ia'_y = (a_x - ia_y)e^{i\phi} = (a_x - ia_y)(\cos\phi + i\sin\phi)$$
$$= (a_x\cos\phi + a_y\sin\phi) + i(a_x\sin\phi - a_y\cos\phi)$$

and

$$a'_z = a_z$$

or

$$\begin{pmatrix} a'_x \\ a'_y \\ a'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

((Mathematica 5.2))

```
(*Sakurai 1-3*)
SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :> Complex[a, -b]}
σx={{0,1},{1,0}}
{{0,1},{1,0}}
σy={{0,-ⅈ},{ⅈ,0}}
{{0,-ⅈ},{ⅈ,0}}
σz={{1,0},{0,-1}}
{{1,0},{0,-1}}
f1=σx ax +σy ay+σz az
{{az,ax-ⅈ ay},{ax+ⅈ ay,-az}}
g1 = MatrixExp[ⅈ σz ϕ/2] // ExpToTrig // Simplify
{{Cos[ϕ/2] + ⅈ Sin[ϕ/2], 0}, {0, Cos[ϕ/2] - ⅈ Sin[ϕ/2]}}
g2 = MatrixExp[-ⅈ σz ϕ/2] // ExpToTrig // Simplify
{{Cos[ϕ/2] - ⅈ Sin[ϕ/2], 0}, {0, Cos[ϕ/2] + ⅈ Sin[ϕ/2]}}
h1=g1.f1.g2//PowerExpand//Simplify
{{az,(ax-ⅈ ay) (Cos[ϕ]+ⅈ Sin[ϕ])},{(ax+ⅈ ay) (Cos[ϕ]-ⅈ
Sin[ϕ]),-az}}
f2=σx ax' +σy ay'+σz az'
{{az′, ax′ - ⅈ ay′}, {ax′ + ⅈ ay′, -az′}}
eq1 = (ax - ⅈ ay) (Cos[ϕ] + ⅈ Sin[ϕ]) - (ax′ - ⅈ ay′) == 0
(ax - ⅈ ay) (Cos[ϕ] + ⅈ Sin[ϕ]) - ax′ + ⅈ ay′ == 0
eq2 = (ax + ⅈ ay) (Cos[ϕ] - ⅈ Sin[ϕ]) - (ax′ + ⅈ ay′) == 0
(ax + ⅈ ay) (Cos[ϕ] - ⅈ Sin[ϕ]) - ax′ - ⅈ ay′ == 0
Solve[{eq1, eq2}, {ax′, ay′}]

{{ax′ -> ax Cos[ϕ] + ay Sin[ϕ], ay′ -> ay Cos[ϕ] - ax Sin[ϕ]}}
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((1-4))

4. Using the rules of bra-ket algebra, prove or evaluate the following:
   a. $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, where X and Y are operators;
   b. $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;
   c. $\exp[if(A)] = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;
   d. $\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'')$, where $\psi_{a'}(\mathbf{x}') = \langle \mathbf{x}'|a'\rangle$.

(a)

$$Tr(\hat{X}\hat{Y}) = \sum_{a'}\sum_{a''}\langle a'|\hat{X}|a''\rangle\langle a''|\hat{Y}|a'\rangle = \sum_{a'}\sum_{a''}\langle a''|\hat{Y}|a'\rangle\langle a'|\hat{X}|a''\rangle$$
$$= \sum_{a''}\sum_{a'}\langle a''|\hat{Y}|a'\rangle\langle a'|\hat{X}|a''\rangle = \sum_{a''}\langle a''|\hat{Y}\hat{X}|a''\rangle = Tr(\hat{Y}\hat{X})$$

(b)

$$|\psi\rangle = (\hat{X}\hat{Y})|\alpha\rangle, \qquad \text{or} \qquad |\psi\rangle = \hat{X}(\hat{Y}|\alpha\rangle) = \hat{X}|\beta\rangle,$$

where

$$|\beta\rangle = \hat{Y}|\alpha\rangle \qquad \text{and} \qquad \langle\beta| = \langle\alpha|\hat{Y}^+.$$

Then

$$\langle\psi| = \langle\alpha|(\hat{X}\hat{Y})^+ \qquad \text{or} \qquad \langle\psi| = \langle\beta|\hat{X}^+ = \langle\alpha|\hat{Y}^+\hat{X}^+$$

Thus we have

$$(\hat{X}\hat{Y})^+ = \hat{Y}^+\hat{X}^+$$

(c)

$$A|a'\rangle = a'|a'\rangle \qquad (a' \text{ is real}).$$

$$\exp[if(\hat{A})] = \exp[if(\hat{A})]\sum_{a'}|a'\rangle\langle a'|$$
$$= \sum_{a'}\exp[if(\hat{A})]|a'\rangle\langle a'| = \sum_{a'}\exp[if(a')]|a'\rangle\langle a'|$$

(d)

$$\phi_{a'}(\mathbf{r}') = \langle \mathbf{r}'|a'\rangle$$

$$\sum_{a'}\phi^*_{a'}(\mathbf{r}')\phi_{a'}(\mathbf{r}'') = \sum_{a'}\langle a'|\mathbf{r}'\rangle\langle \mathbf{r}''|a'\rangle = \sum_{a'}\langle \mathbf{r}''|a'\rangle\langle a'|\mathbf{r}'\rangle = \langle \mathbf{r}''|\mathbf{r}'\rangle = \delta(\mathbf{r}' - \mathbf{r}'')$$

((1-5))

5. a. Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle$, $\langle a''|\alpha\rangle, \ldots$ and $\langle a'|\beta\rangle$, $\langle a''|\beta\rangle, \ldots$ are all known, where $|a'\rangle$, $|a''\rangle, \ldots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

b. We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

(a)

$$|\alpha\rangle = \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \langle a_3|\alpha\rangle \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$|\beta\rangle = \begin{pmatrix} \langle a_1|\beta\rangle \\ \langle a_2|\beta\rangle \\ \langle a_3|\beta\rangle \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$\langle\beta| = \begin{pmatrix} \langle a_1|\beta\rangle^* & \langle a_2|\beta\rangle^* & \langle a_3|\beta\rangle^* & \cdot & \cdot & \cdot \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \langle a_3|\alpha\rangle \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \begin{pmatrix} \langle a_1|\beta\rangle^* & \langle a_2|\beta\rangle^* & \langle a_3|\beta\rangle^* & \cdot & \cdot & \cdot \end{pmatrix}$$

$$= \begin{pmatrix} \langle a_1|\alpha\rangle\langle\beta|a_1\rangle & \langle a_1|\alpha\rangle\langle\beta|a_2\rangle & \langle a_1|\alpha\rangle\langle\beta|a_3\rangle & \cdot & \cdot & \cdot \\ \langle a_2|\alpha\rangle\langle\beta|a_1\rangle & \langle a_2|\alpha\rangle\langle\beta|a_2\rangle & \langle a_2|\alpha\rangle\langle\beta|a_3\rangle & \cdot & \cdot & \cdot \\ \langle a_3|\alpha\rangle\langle\beta|a_1\rangle & \langle a_3|\alpha\rangle\langle\beta|a_2\rangle & \langle a_3|\alpha\rangle\langle\beta|a_3\rangle & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

(b)

$$|\alpha\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\beta\rangle = |+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

((1-6))

6. Suppose $|i\rangle$ and $|j\rangle$ are eigenkets of some Hermitian operator A. Under what condition can we conclude that $|i\rangle + |j\rangle$ is also an eigenket of A? Justify your answer.

$$\hat{A}|i\rangle = a_i|i\rangle$$

$$\hat{A}|j\rangle = a_j|j\rangle$$

$$\hat{A}(|i\rangle + |j\rangle) = \lambda(|i\rangle + |j\rangle)$$

Then we have

$$a_i|i\rangle + a_j|j\rangle = \lambda(|i\rangle + |j\rangle)$$

or

$$(a_i - \lambda)|i\rangle + (a_j - \lambda)|j\rangle = 0$$

When $\langle i|j\rangle = 0$, $\lambda = a_i = a_j$ (degenerate case: the same eigenvalues, but different states).

((**1-7**))

7. Consider a ket space spanned by the eigenkets $\{|a'\rangle\}$ of a Hermitian operator A. There is no degeneracy.
   a. Prove that
   $$\prod_{a'}(A-a')$$
   is the null operator.
   b. What is the significance of
   $$\prod_{a''\neq a'}\frac{(A-a'')}{(a'-a'')}?$$
   c. Illustrate (a) and (b) using A set equal to S_z of a spin $\frac{1}{2}$ system.

$$\hat{A}|a_k\rangle = a_k|a_k\rangle$$

(a)

$$\prod_{a_i}(\hat{A}-a_i)|a_k\rangle = (\hat{A}-a_1)(\hat{A}-a_2)\cdots(\hat{A}-a_k)\cdots|a_k\rangle$$
$$= (a_k-a_1)(a_k-a_2)\cdots(a_k-a_k)\cdots|a_k\rangle = 0$$

For any ket $|\beta\rangle$, $|\beta\rangle$ can be described by

$$|\beta\rangle = \sum_k |a_k\rangle\langle a_k|\beta\rangle$$
$$\prod_{a_i}(\hat{A}-a_i)|\beta\rangle = \prod_{a_i}(\hat{A}-a_i)\sum_k |a_k\rangle\langle a_k|\beta\rangle$$
$$= \sum_k \prod_{a_i}(\hat{A}-a_i)|a_k\rangle\langle a_k|\beta\rangle = 0$$

Therefore $\prod_{a_i}(\hat{A}-a_i)$ is a zero operator.

(b)

$$\hat{O} = \prod_{a_i\neq a_j}\frac{(\hat{A}-a_i)}{(a_j-a_i)}|a_k\rangle$$
$$= \frac{(\hat{A}-a_1)(\hat{A}-a_2)(\hat{A}-a_3)\cdots(\hat{A}-a_{j-1})(\hat{A}-a_{j+1})\cdots}{(a_j-a_1)(a_j-a_2)(a_j-a_3)\cdots(a_j-a_{j-1})(a_j-a_{j+1})\cdots}|a_k\rangle$$

We consider the two cases ($k \neq j$ and $k = j$).

For $k \neq j$, $\hat{O} = 0$.

For $k = j$, $\hat{O} = |a_j\rangle$.

Therefore, in general,

$$\hat{O}|a_k\rangle = \prod_{a_i \neq a_j} \frac{(\hat{A} - a_i)}{a_j - a_i} |a_k\rangle = \delta_{j,k} |a_k\rangle$$

For any ket $|\beta\rangle$, $|\beta\rangle$ can be described by

$$|\beta\rangle = \sum_k |a_k\rangle\langle a_k | \beta\rangle$$

$$\hat{O}|\beta\rangle = \hat{O}\sum_k |a_k\rangle\langle a_k|\beta\rangle = \sum_k \hat{O}|a_k\rangle\langle a_k|\beta\rangle = \sum_k \delta_{j,k}|a_k\rangle\langle a_k|\beta\rangle = |a_j\rangle\langle a_j|\beta\rangle$$

Thus $\hat{O}$ is a projection operator on $|a_j\rangle$.

(c)

$$\prod_{a'} (\hat{A} - a') = (\hat{S}_z - \frac{\hbar}{2})(\hat{S}_z + \frac{\hbar}{2})$$

$$|\beta\rangle = \sum_k |a_k\rangle\langle a_k|\beta\rangle = C_+|+\rangle + C_-|-\rangle$$

where

$C_+ = \langle +|\beta\rangle$, and $C_- = \langle -|\beta\rangle$

$$\prod_{a'} (\hat{A} - a')|\beta\rangle = (\hat{S}_z - \frac{\hbar}{2})(\hat{S}_z + \frac{\hbar}{2})(C_+|+\rangle + C_-|-\rangle) = 0$$

(d)

(i) $a_j = \frac{\hbar}{2}$, $\hat{A} = \hat{S}_z$

$$\hat{O} = \prod_{a_i \neq a_j} \frac{(\hat{A} - a_i)}{(a_j - a_i)} = \frac{\hat{S}_z + \frac{\hbar}{2}}{\frac{\hbar}{2} + \frac{\hbar}{2}} = \frac{1}{\hbar}(\hat{S}_z + \frac{\hbar}{2})$$

$$\hat{O}|\beta\rangle = \frac{1}{\hbar}(\hat{S}_z + \frac{\hbar}{2})|\beta\rangle = \frac{1}{\hbar}(\hat{S}_z + \frac{\hbar}{2})(C_+|+\rangle + C_-|-\rangle) = C_+|+\rangle$$

(ii) $a_j = -\frac{\hbar}{2}$, $\hat{A} = \hat{S}_z$

$$\hat{O} = \prod_{a_i \neq a_j} \frac{(\hat{A} - a_i)}{(a_j - a_i)} = \frac{\hat{S}_z - \frac{\hbar}{2}}{-\frac{\hbar}{2} - \frac{\hbar}{2}} = -\frac{1}{\hbar}(\hat{S}_z - \frac{\hbar}{2})$$

$$\hat{O}|\beta\rangle = -\frac{1}{\hbar}(\hat{S}_z - \frac{\hbar}{2})|\beta\rangle = -\frac{1}{\hbar}(\hat{S}_z - \frac{\hbar}{2})(C_+|+\rangle + C_-|-\rangle) = C_-|-\rangle$$

((**1-8**))

8. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k, \qquad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

where

$$S_x = \frac{\hbar}{2}(|+\rangle\langle -| + |-\rangle\langle +|), \qquad S_y = \frac{i\hbar}{2}(-|+\rangle\langle -| + |-\rangle\langle +|),$$

$$S_z = \frac{\hbar}{2}(|+\rangle\langle +| - |-\rangle\langle -|).$$

The proof is given by using the Mathematica 5.2.

```
(*Sakurai Problem 1-8*)
SuperStar /: expr_* :=
 expr /. {Complex[a_, b_] :> Complex[a, -b]}
σx={{0,1},{1,0}}
{{0,1},{1,0}}
σy={{0,-ⅈ},{ⅈ,0}}
General::spell1 : Possible spelling error: new symbol
   name "σy" is similar to existing symbol "σx". More…
{{0,-ⅈ},{ⅈ,0}}
σz={{1,0},{0,-1}}
General::spell : Possible spelling error: new symbol
   name "σz" is similar to existing symbols {σx, σy}. More…
{{1,0},{0,-1}}
σx.σy-σy.σx-2 ⅈ σz
{{0,0},{0,0}}
σy.σz-σz.σy-2 ⅈ σx
{{0,0},{0,0}}
σz.σx-σx.σz-2 ⅈ σy
{{0,0},{0,0}}
σx.σx+σx.σx
{{2,0},{0,2}}
σx.σy+σy.σx
{{0,0},{0,0}}
σx.σz+σz.σx
{{0,0},{0,0}}
σy.σy+σy.σy
{{2,0},{0,2}}
σy.σz+σz.σy
{{0,0},{0,0}}
```

```
σz.σz+σz.σz
{{2,0},{0,2}}
σx.σy-ⅈ σz
{{0,0},{0,0}}
σy.σz-ⅈ σx
{{0,0},{0,0}}
σz.σx-ⅈ σy
{{0,0},{0,0}}
```

((**1-9**))

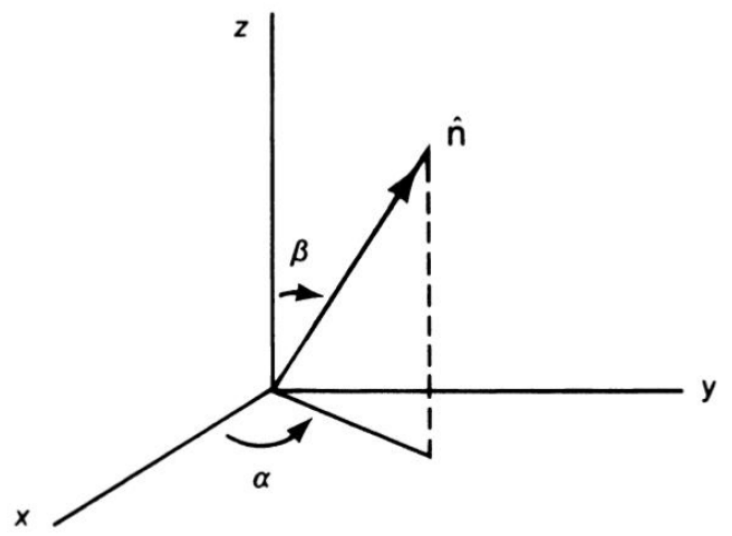
9. Construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that

$$\mathbf{S}\cdot\hat{\mathbf{n}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \left(\frac{\hbar}{2}\right)|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$$

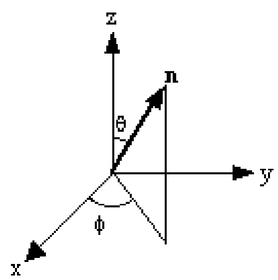
where $\hat{\mathbf{n}}$ is characterized by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$. [*Note*: The answer is

$$\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle.$$

But do not just verify that this answer satisfies the above eigenvalue equation. Rather, treat the problem as a straightforward eigenvalue



problem. Also do not use rotation operators, which we will introduce later in this book.]



$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{S}\cdot\mathbf{n} = \frac{\hbar}{2}(\sigma_x n_x + \sigma_y n_y + \sigma_z n_z) = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

The eigenvalue problem:

$$\hat{S}\cdot\mathbf{n}|\psi\rangle = \lambda|\psi\rangle$$

$$|\psi\rangle = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \lambda\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

or

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \frac{2\lambda}{\hbar}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = \varepsilon\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

where $\varepsilon = \dfrac{2\lambda}{\hbar}$.

Secular equation

$$\begin{vmatrix} \cos\theta - \varepsilon & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - \varepsilon \end{vmatrix} = 0$$

$$(\cos\theta - \varepsilon)(-\cos\theta - \varepsilon) - \sin^2\theta = 0$$

or

$$\varepsilon^2 - 1 = 0 \qquad \text{or} \qquad \varepsilon = \pm 1$$

(i) For $\varepsilon = 1$

$$|\psi\rangle = |+\rangle_n = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix} = 0$$

or

$$(\cos\theta - 1)C_+ + \sin\theta\, e^{-i\phi} C_- = 0 \tag{1}$$

The normalization condition:

$$|C_+|^2 + |C_-|^2 = 1 \tag{2}$$

From Eq. (1),

$$C_- = \frac{1-\cos\theta}{\sin\theta\, e^{-i\phi}} C_+ = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} e^{i\phi} C_+ = \tan\frac{\theta}{2} e^{i\phi} C_+$$

From Eq. (2),

$$|C_+|^2 + \tan^2\frac{\theta}{2}|C_+|^2 = \frac{1}{\cos^2\frac{\theta}{2}}|C_+|^2 = 1$$

or

$$|C_+|^2 = \cos^2\frac{\theta}{2}$$

$$C_- = \tan\frac{\theta}{2} e^{i\phi} C_+$$

We choose

$$C_+ = \cos\frac{\theta}{2} e^{-i\phi/2}, \qquad \text{and} \qquad C_- = \sin\frac{\theta}{2} e^{i\phi/2}$$

$$|+\rangle_n = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

(i) For $\varepsilon = -1$

$$|\psi\rangle = |-\rangle_n = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} = 0$$

or

$$(\cos\theta + 1)C_+ + \sin\theta\, e^{-i\phi} C_- = 0 \qquad (3)$$

The normalization condition:

$$|C_+|^2 + |C_-|^2 = 1 \qquad (4)$$

$$C_- = \frac{1+\cos\theta}{\sin\theta\, e^{-i\phi}} C_+ = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} e^{i\phi} C_+ = -\frac{1}{\tan\frac{\theta}{2}} e^{i\phi} C_+$$

$$|C_+|^2 + \frac{1}{\tan^2\frac{\theta}{2}} |C_+|^2 = \frac{1}{\sin^2\frac{\theta}{2}} |C_+|^2 = 1$$

or

$$|C_+|^2 = \sin^2\frac{\theta}{2}$$

We choose

$$C_+ = -\sin\frac{\theta}{2} e^{-i\phi/2}, \quad \text{and} \quad C_- = \cos\frac{\theta}{2} e^{i\phi/2}$$

$$|-\rangle_n = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\phi/2} \\ \cos\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

For $\theta = 0$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

except for the phase factor.

((**1-10**))

10. The Hamiltonian operator for a two-state system is given by

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$|\psi\rangle = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad \hat{H} = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$$

Eigenvalue equation

$$\begin{pmatrix} a & a \\ a & -a \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = E\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

or

$$\begin{vmatrix} a-E & a \\ a & -a-E \end{vmatrix} = 0$$

$$(a-E)(-a-E) - a^2 = 0, \qquad \text{or} \qquad E^2 = 2a^2, \qquad \text{or} \qquad E = \pm\sqrt{2}a$$

(i) For $E = \sqrt{2}a$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$|\psi_1\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix}$$

(ii) $E = -\sqrt{2}a$

$$|\psi_1\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$$

```
(* Sakurai Problem 1-10*)
 (* The Hamiltonain operator for a two-state system is given
by
        H={{a,a},{a,-a}}

      Find the enrgy eigenvalues and corresponding energy
eigenkets.*)
 SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :→ Complex[a, -b]}
```

```
H1={{a,a},{a,-a}}
{{a,a},{a,-a}}
H1//MatrixForm
```

$$\begin{pmatrix} a & a \\ a & -a \end{pmatrix}$$

```
Eigensystem[H1]
```

$$\left\{\left\{-\sqrt{2}\,a,\ \sqrt{2}\,a\right\},\ \left\{\left\{1-\sqrt{2},\ 1\right\},\ \left\{1+\sqrt{2},\ 1\right\}\right\}\right\}$$

$$\psi 11 = \left\{1+\sqrt{2},\ 1\right\}$$

$$\left\{1+\sqrt{2},\ 1\right\}$$

$$\psi 1 = \frac{\psi 11}{\sqrt{\psi 11^{*}.\psi 11}}\ //\ \mathrm{Simplify}$$

$$\left\{\frac{1+\sqrt{2}}{\sqrt{2\left(2+\sqrt{2}\right)}},\ \frac{1}{\sqrt{2\left(2+\sqrt{2}\right)}}\right\}$$

$$\psi 22 = \left\{1-\sqrt{2},\ 1\right\}$$

$$\left\{1-\sqrt{2},\ 1\right\}$$

$$\psi 2 = \frac{\psi 22}{\sqrt{\psi 22^{*}.\psi 22}}\ //\ \mathrm{Simplify}$$

$$\left\{\frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}},\ \frac{1}{\sqrt{4-2\sqrt{2}}}\right\}$$

```
(*Orthogonality*)
```

$$\psi 1^{*}.\psi 2\ //\ \mathrm{Simplify}$$

```
0
```

((**1-11**))

11. A two-state system is characterized by the Hamiltonian

$$H = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}[|1\rangle\langle 2| + |2\rangle\langle 1|]$$

where H_{11}, H_{22}, and H_{12} are real numbers with the dimension of energy, and $|1\rangle$ and $|2\rangle$ are eigenkets of some observable ($\neq H$). Find the energy eigenkets and corresponding energy eigenvalues. Make sure that your answer makes good sense for $H_{12} = 0$. (You need not solve this problem from scratch. The following fact may be used without proof:

$$(\mathbf{S}\cdot\hat{\mathbf{n}})|\hat{\mathbf{n}};+\rangle = \frac{\hbar}{2}|\hat{\mathbf{n}};+\rangle,$$

with $|\hat{\mathbf{n}};+\rangle$ given by

$$|\hat{\mathbf{n}};+\rangle = \cos\frac{\beta}{2}|+\rangle + e^{i\alpha}\sin\frac{\beta}{2}|-\rangle,$$

where β and α are the polar and azimuthal angles, respectively, that characterize $\hat{\mathbf{n}}$.)

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = a_0\hat{1} + a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z = \begin{pmatrix} a_0 + a_z & a_x - ia_y \\ a_x + ia_y & a_0 - a_z \end{pmatrix}$$

where H_{11}, H_{12}, H_{21}, and H_{22} are real. $H_{12} = H_{21}$.

$$a_0 = \frac{H_{11} + H_{22}}{2}$$

$$a_z = \frac{H_{11} - H_{22}}{2}$$

$$a_x = H_{12}$$

$$a_y = 0$$

$$\hat{H} = \left(\frac{H_{11} + H_{22}}{2}\right)\hat{1} + H_{12}\hat{\sigma}_x + \left(\frac{H_{11} - H_{22}}{2}\right)\hat{\sigma}_z$$

$$= \left(\frac{H_{11} + H_{22}}{2}\right)\hat{1} + \sqrt{H_{12}{}^2 + \left(\frac{H_{11} - H_{22}}{2}\right)^2}$$

$$\times\left(\frac{H_{12}}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}}\hat{\sigma}_x+\frac{\left(\frac{H_{11}-H_{22}}{2}\right)}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}}\hat{\sigma}_z\right)$$

$$=\left(\frac{H_{11}+H_{22}}{2}\right)\hat{1}+\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}\,\hat{\sigma}\cdot\mathbf{n}$$

where

$$\mathbf{n}=\begin{pmatrix}\frac{H_{12}}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}} & 0 & \frac{\left(\frac{H_{11}-H_{22}}{2}\right)}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}}\end{pmatrix}$$

The unit vector ***n*** is in the *xz* plane. The angle between the *z* axis and ***n*** is θ.

$$\cos\theta=\frac{\left(\frac{H_{11}-H_{22}}{2}\right)}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}},\qquad \sin\theta=\frac{H_{12}}{\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}}$$

$\left|\pm\right\rangle_n$ is the eigenket of $\hat{\sigma}\cdot\mathbf{n}$,

$$\hat{\sigma}\cdot\mathbf{n}\left|\pm\right\rangle_n=\pm\left|\pm\right\rangle_n$$

with

$$\left|+\right\rangle_n=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix},\qquad \left|-\right\rangle_n=\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\hat{H}\left|\pm\right\rangle_n=\left(\frac{H_{11}+H_{22}}{2}\right)\hat{1}\left|\pm\right\rangle_n+\sqrt{H_{12}{}^2+\left(\frac{H_{11}-H_{22}}{2}\right)^2}\,\hat{\sigma}\cdot\mathbf{n}\left|\pm\right\rangle_n$$

$$= \left[\left(\frac{H_{11} + H_{22}}{2} \right) \pm \sqrt{H_{12}{}^{2} + \left(\frac{H_{11} - H_{22}}{2} \right)^{2}} \right] \left| \pm \right\rangle_{n}$$

Thus $\left|\pm\right\rangle_n$ is the eigenket of $\hat{H}$ witrh the eigenvalues of

$$\left(\frac{H_{11} + H_{22}}{2} \right) \pm \sqrt{H_{12}{}^{2} + \left(\frac{H_{11} - H_{22}}{2} \right)^{2}}$$

Suppose that $H_{12} = 0$ and $H_{11} \geq H_{22}$.

$\cos\theta = 1$ and $\sin\theta = 0$. or $\theta = 0$.

$$\left|+\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \left|-\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

((Mathematoca 5.2))

```
(*Sakurai 1-11*)
SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :> Complex[a, -b]}
H={{H11,H12},{H12,H22}}
{{H11,H12},{H12,H22}}
σx={{0,1},{1,0}}
{{0,1},{1,0}}
σy={{0,-ⅈ},{ⅈ,0}}
General::spell1 :
 Possible spelling error: new symbol name "σy" is
   similar to existing symbol "σx". More…
{{0,-ⅈ},{ⅈ,0}}
σz={{1,0},{0,-1}}
General::spell :
 Possible spelling error: new symbol name "σz" is
   similar to existing symbols {σx, σy}. More…
{{1,0},{0,-1}}
II={{1,0},{0,1}}
{{1,0},{0,1}}
M=α σx +β σz + γ II
{{β+γ,α},{α,-β+γ}}
eq1=M-H
{{-H11+β+γ,-H12+α},{-H12+α,-H22-β+γ}}
```

```
eq2={-H11+β+γ==0,-H12+α==0,-H22-β+γ==0}
{-H11+β+γ==0,-H12+α==0,-H22-β+γ==0}
rule1=Solve[eq2,{α,β,γ}]//Flatten//Simplify
```

$$\left\{\alpha \to H12,\ \beta \to \frac{H11 - H22}{2},\ \gamma \to \frac{H11 + H22}{2}\right\}$$

$$n = \left\{\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}},\ 0,\ \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right\}$$

$$\left\{\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}},\ 0,\ \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right\}$$

$$M1 = \sqrt{\alpha^2 + \beta^2}\ (\mathrm{Sin}[\theta]\ \sigma x + \mathrm{Cos}[\theta]\ \sigma z)\ + \gamma\ II\ /.\ rule1$$

$$\left\{\left\{\frac{H11 + H22}{2} + \sqrt{H12^2 + \frac{1}{4}(H11 - H22)^2}\ \mathrm{Cos}[\theta],\right.\right.$$
$$\left.\sqrt{H12^2 + \frac{1}{4}(H11 - H22)^2}\ \mathrm{Sin}[\theta]\right\},$$
$$\left\{\sqrt{H12^2 + \frac{1}{4}(H11 - H22)^2}\ \mathrm{Sin}[\theta],\right.$$
$$\left.\left.\frac{H11 + H22}{2} - \sqrt{H12^2 + \frac{1}{4}(H11 - H22)^2}\ \mathrm{Cos}[\theta]\right\}\right\}$$

((**1-12**))

12. A spin $\frac{1}{2}$ system is known to be in an eigenstate of $\mathbf{S}\cdot\hat{\mathbf{n}}$ with eigenvalue $\hbar/2$, where $\hat{\mathbf{n}}$ is a unit vector lying in the xz-plane that makes an angle γ with the positive z-axis.

    a. Suppose S_x is measured. What is the probability of getting $+\hbar/2$?
    b. Evaluate the dispersion in S_x, that is,

    $$\langle(S_x - \langle S_x\rangle)^2\rangle.$$

    (For your own peace of mind check your answers for the special cases $\gamma = 0$, $\pi/2$, and π.)

The unit vector ***n*** is in the *xz* plane. The angle between the *z* axis and n is γ.
The state of the system is

$$|+\rangle_n = \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix}$$

(a) The probability of getting $S_x = \frac{\hbar}{2}$ is given by

$$P = \left| {}_x\langle + | + \rangle_n \right|^2,$$

where

$$|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Since

$${}_x\langle + | + \rangle_n = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\cos\frac{\gamma}{2} + \sin s\frac{\gamma}{2} \right)$$

Therefore we have

$$P = \frac{1}{2}\left(1 + 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\right) = \frac{1}{2}(1 + \sin\gamma)$$

(b)

$$\langle S_x \rangle = {}_n\langle + | \hat{S}_x | + \rangle_n = \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\gamma}{2} \\ \cos\frac{\gamma}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2} 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2} = \frac{\hbar}{2}\sin\gamma$$

$$\left\langle S_x{}^2 \right\rangle = {}_n\langle + | \hat{S}_x{}^2 | + \rangle_n = \frac{\hbar^2}{4} \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} = \frac{\hbar^2}{4}$$

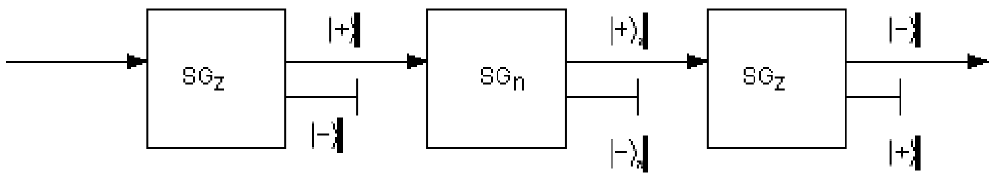
The dispersion in S_x is

$$(\Delta S_x)^2 = \left\langle S_x{}^2 \right\rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4}(1 - \sin^2 \gamma) = \frac{\hbar^2}{4} \cos^2 \gamma$$

((**1-13**))

13. A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach–type measurements as follows:
    a. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.
    b. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\mathbf{S}\cdot\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ making an angle β in the xz-plane with respect to the z-axis.
    c. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

    What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?



After the first measurement, the state of the system is $|+\rangle$ (intensity $I_1 = 1$).

After the second measurement, the state of the system is $|+\rangle_n$.

n is the unit vector in the xz plane. The angle between n and the z axis is β.

$$|+\rangle_n = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix}$$

The probability of finding $|+\rangle_n$ after the second measurement is

$$|{}_n\langle+|+\rangle|^2 = \cos^2\frac{\beta}{2}$$

The intensity is $I_2 = I_1\cos^2\frac{\beta}{2}$.

After the third measurement, the state of the system is $(|-\rangle)$.

The probability of $|-\rangle$ after the third measurement is

$$|\langle-|+\rangle_n|^2 = |{}_n\langle+|-\rangle|^2 = \sin^2\frac{\beta}{2}$$

The resultant intensity is $I_3 = I_2\sin^2\frac{\beta}{2} = I_1\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2} = \frac{1}{4}\sin^2\beta$

When $\beta = \pi/2$, the intensity I_3 takes a maximum (=1/4).

((**1-14**))

14. A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?

b. Give a physical example where all this is relevant.

$$\hat{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a)

$$|\psi\rangle = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

Eigenvalue problem

$$\hat{B}|\psi\rangle = \lambda|\psi\rangle$$

or

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$$

$$M = \begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix}$$

Det(M) should be equal to 0.

$$\det M = \begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} = -\lambda(\lambda-1)(\lambda+1) = 0$$

or

$\lambda = 0, 1, -1$.

(i) $\lambda = 0$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$$

$C_2 = 0$ and $C_3 = -C_1$.

The normalized eigenket:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(ii) $\lambda = 1$

$$\begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$$

$C_2 = \sqrt{2} C_1 = 1$ and $C_3 = C_1 = \frac{1}{\sqrt{2}}$

The normalized eigenket:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

(iii) λ = -1

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0$$

$C_2 = -\sqrt{2}C_1$ and $C_3 = C_1 = \frac{1}{\sqrt{2}}$

The normalized eigenket:

$$|\psi_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

(b) A particle with spin 1

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

((Mathematica 5.2))

```
(* Sakurai Problem 1-14*)
SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :→ Complex[a, -b]}
A = 1/√2 {{0, 1, 0}, {1, 0, 1}, {0, 1, 0}}
```

$\left\{\left\{0, \frac{1}{\sqrt{2}}, 0\right\}, \left\{\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{0, \frac{1}{\sqrt{2}}, 0\right\}\right\}$

A//MatrixForm

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Eigensystem[A]

$\{\{-1, 1, 0\},$
$\{\{1, -\sqrt{2}, 1\}, \{1, \sqrt{2}, 1\}, \{-1, 0, 1\}\}\}$

ψ11 = $\{1, \sqrt{2}, 1\}$

$\{1, \sqrt{2}, 1\}$

$\psi 1 = \frac{\psi 11}{\sqrt{\psi 11^{*}.\psi 11}}$ // Simplify

$\left\{\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right\}$

ψ22={-1,0,1}

{-1,0,1}

$\psi 2 = \frac{\psi 22}{\sqrt{\psi 22^{*}.\psi 22}}$ // Simplify

$\left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}$

ψ33 = $\{1, -\sqrt{2}, 1\}$

$\{1, -\sqrt{2}, 1\}$

$\psi 3 = \frac{\psi 33}{\sqrt{\psi 33^{*}.\psi 33}}$ // Simplify

$\left\{\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right\}$

(*Orthogonality*)

ψ1*.ψ2 // Simplify

0

ψ2*.ψ3 // Simplify

0

ψ3*.ψ1 // Simplify

0

```
(*Normalization*)
ψ1*.ψ1 // Simplify
1
ψ2*.ψ2 // Simplify
1
ψ3*.ψ3 // Simplify
1
```

((**1-15**))

15. Let A and B be observables. Suppose the simultaneous eigenkets of A and B $\{|a', b'\rangle\}$ form a *complete* orthonormal set of base kets. Can we always conclude that

$$[A, B] = 0?$$

If your answer is yes, prove the assertion. If your answer is no, give a counterexample.

((**1-16**))

16. Two Hermitian operators anticommute:

$$\{A, B\} = AB + BA = 0.$$

Is it possible to have a simultaneous (that is, common) eigenket of A and B? Prove or illustrate your assertion.

Suppose that $|a,b\rangle$ is the simultaneous eigenket of $\hat{A}$ and $\hat{B}$.

$$\hat{A}|a,b\rangle = a|a,b\rangle, \qquad \hat{B}|a,b\rangle = b|a,b\rangle$$

$$\{\hat{A},\hat{B}\}|a,b\rangle = \left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)|a,b\rangle = (ab+ba)|a,b\rangle = 2ab|a,b\rangle$$

So the simultaneous eigenket exists only when $ab = 0$.

((**1-17**))

17. Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,

$$[A_1, A_2] \neq 0,$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = 0, \qquad [A_2, H] = 0.$$

Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = \mathbf{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

((**1-18**))

18. a. The simplest way to derive the Schwarz inequality goes as follows. First, observe

$$(\langle\alpha| + \lambda^*\langle\beta|)\cdot(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

for any complex number λ; then choose λ in such a way that the preceding inequality reduces to the Schwarz inequality.

b. Show that the equality sign in the generalized uncertainty relation holds if the state in question satisfies

$$\Delta A|\alpha\rangle = \lambda\, \Delta B|\alpha\rangle$$

with λ purely *imaginary*.

c. Explicit calculations using the usual rules of wave mechanics show that the wave function for a Gaussian wave packet given by

$$\langle x'|\alpha\rangle = (2\pi d^2)^{-1/4} \exp\left[\frac{i\langle p\rangle x'}{\hbar} - \frac{(x' - \langle x\rangle)^2}{4d^2}\right]$$

satisfies the minimum uncertainty relation

$$\sqrt{\langle(\Delta x)^2\rangle}\sqrt{\langle(\Delta p)^2\rangle} = \frac{\hbar}{2}.$$

Prove that the requirement

$$\langle x'|\Delta x|\alpha\rangle = (\text{imaginary number})\langle x'|\Delta p|\alpha\rangle$$

is indeed satisfied for such a Gaussian wave packet, in agreement with (b).

(a) We consider

$$\left|\chi\right\rangle = \left|\alpha\right\rangle + \lambda\left|\beta\right\rangle$$

$\left\langle\chi|\chi\right\rangle \geq 0$ for any complex number λ.

or

$$\left\langle\chi|\chi\right\rangle = (\left\langle\alpha\right| + \lambda^*\left\langle\beta\right|)(\left|\alpha\right\rangle + \lambda\left|\beta\right\rangle \geq 0$$

or

$$\left\langle\alpha|\alpha\right\rangle + \lambda\left\langle\alpha|\beta\right\rangle + \lambda^*\left\langle\beta|\alpha\right\rangle + \left|\lambda\right|^2\left\langle\beta|\beta\right\rangle \geq 0$$

The best inequality is obtained if λ is chosen so as to minimize the left-hand side. By differentiation, the value of λ that accomplishes this is found to be

$$\lambda = -\frac{\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle} = -\frac{\left\langle\alpha|\beta\right\rangle^*}{\left\langle\beta|\beta\right\rangle}$$

Then we have

$$(\left\langle\alpha\right| - \frac{\left\langle\alpha|\beta\right\rangle}{\left\langle\beta|\beta\right\rangle}\left\langle\beta\right|)(\left|\alpha\right\rangle - \frac{\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle}\left|\beta\right\rangle \geq 0$$

or

$$\left\langle\alpha|\alpha\right\rangle - 2\frac{\left\langle\alpha|\beta\right\rangle\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle} + \frac{\left\langle\alpha|\beta\right\rangle\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle} \geq 0$$

or

$$\left\langle\alpha|\alpha\right\rangle - \frac{\left|\left\langle\alpha|\beta\right\rangle\right|^2}{\left\langle\beta|\beta\right\rangle} \geq 0$$

or

$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

The equality holds if and only if

$|\alpha\rangle + \lambda|\beta\rangle = 0$

In this case, λ should be equal to

$\langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle = 0$, and $\langle\beta|\alpha\rangle + \lambda\langle\beta|\beta\rangle = 0$

$$\lambda = -\frac{\langle\alpha|\alpha\rangle}{\langle\alpha|\beta\rangle} = -\frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle},$$

In other words, $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle = \langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = |\langle\alpha|\beta\rangle|^2$

((Mathematica 5.2))

```
(*Schwartz's inequality*)
f =
  A^2 + (x^2 + y^2) B^2 +
    (x + ⅈ y) (a + ⅈ b) +
    (x - ⅈ y) (a - ⅈ b) // Simplify
```

$A^2 + 2\,a\,x + B^2 x^2 - 2\,b\,y + B^2 y^2$

```
k1=Flatten[Solve[D[f,x]==0,x]]
```

$\left\{x \to -\frac{a}{B^2}\right\}$

```
k2=Flatten[Solve[D[f,y]==0,y]]
```

$\left\{y \to \frac{b}{B^2}\right\}$

```
x+ ⅈ y/.k1/.k2
```

$-\frac{a}{B^2} + \frac{i\,b}{B^2}$

```
f/.k1/.k2//Simplify
```

$-\frac{a^2 + b^2 - A^2 B^2}{B^2}$

(b)

$\hat{A}$ and $\hat{B}$ are two Hermitian operators with the condition $[\hat{A},\hat{B}] = i\hat{C}$. Then we have a Heisenberg's principle of uncertainty:

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle C\rangle\right|.$$

Uncertainty product in a normalized state $|\psi\rangle$.

$$(\Delta A)^2 = \langle\psi|(\hat{A}-\langle A\rangle)^2|\psi\rangle$$

$$(\Delta B)^2 = \langle\psi|(\hat{B}-\langle B\rangle)^2|\psi\rangle$$

where

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$$

and

$$\langle B\rangle = \langle\psi|\hat{B}|\psi\rangle$$

Let us define

$$\delta\hat{A} = \hat{A}-\langle A\rangle, \qquad \delta\hat{B} = \hat{B}-\langle B\rangle$$

$$\delta\hat{A}^+ = \delta\hat{A}, \qquad \delta\hat{B}^+ = \delta\hat{B}$$

$$(\Delta A)^2(\Delta B)^2 = \langle\psi|(\delta\hat{A})^2|\psi\rangle\langle\psi|(\delta\hat{B})^2|\psi\rangle = \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq \left|\langle\alpha|\beta\rangle\right|^2$$

where

$$|\alpha\rangle = \delta\hat{A}|\psi\rangle, \quad \text{and} \quad |\beta\rangle = \delta\hat{B}|\psi\rangle$$

The equality sign holds if and only if

$$\left|\alpha\right\rangle + \lambda\left|\beta\right\rangle = 0$$

or

$$\delta\hat{A}\left|\psi\right\rangle = -\lambda\delta\hat{B}\left|\psi\right\rangle$$

Here

$$\left\langle\alpha\middle|\beta\right\rangle = \left\langle\psi\right|\delta\hat{A}\delta\hat{B}\left|\psi\right\rangle$$

$$\delta\hat{A}\delta\hat{B} = \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A}) + \frac{1}{2}\left[\delta\hat{A},\delta\hat{B}\right]$$

$$[\delta\hat{A},\delta\hat{B}] = [\hat{A} - \langle A\rangle, \hat{B} - \langle B\rangle] = [\hat{A},\hat{B}] = i\hat{C}$$

Thus we have

$$\delta\hat{A}\delta\hat{B} = \hat{G} + \frac{1}{2}i\hat{C}$$

where

$$\hat{G} = \frac{1}{2}(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A})$$

$$(\Delta A)^2(\Delta B)^2 \geq \left|\left\langle\psi\right|\hat{G} + \frac{1}{2}i\hat{C}\left|\psi\right\rangle\right|^2 = \left|\left\langle\psi\right|\hat{G}\left|\psi\right\rangle + \frac{1}{2}i\left\langle\psi\right|\hat{C}\left|\psi\right\rangle\right|^2$$

Note that $\left\langle\psi\right|\hat{G}\left|\psi\right\rangle$ and $\left\langle\psi\right|\hat{C}\left|\psi\right\rangle$ are real since $\hat{G}$ and $\hat{C}$ are Hermitian.

$$\hat{C} = -i[\hat{A},\hat{B}] = -i(\hat{A}\hat{B} - \hat{B}\hat{A}),$$

$$\hat{C}^+ = -(\hat{A}\hat{B} - \hat{B}\hat{A})^+ i^+ = -(\hat{B}\hat{A} - \hat{A}\hat{B})i^* = i(\hat{B}\hat{A} - \hat{A}\hat{B}) = \hat{C}^+$$

Then we have

$$(\Delta A)^2(\Delta B)^2 \geq \left|\langle\psi|\hat{G}|\psi\rangle + \frac{1}{2}i\langle\psi|\hat{C}|\psi\rangle\right|^2 = \left|\langle\psi|\hat{G}|\psi\rangle\right|^2 + \frac{1}{4}\left|\langle\psi|\hat{C}|\psi\rangle\right|^2 \geq \frac{1}{4}\left|\langle\psi|\hat{C}|\psi\rangle\right|^2$$

or

$$(\Delta A)(\Delta B) \geq \frac{1}{2}\left|\langle\psi|\hat{C}|\psi\rangle\right|$$

The last inequality holds if and only if

$$\langle\psi|\hat{G}|\psi\rangle = 0$$

or

$$\langle\psi|(\delta\hat{A}\delta\hat{B} + \delta\hat{B}\delta\hat{A})|\psi\rangle = 0$$

Noting that

$$\delta\hat{A}|\psi\rangle = -\lambda\delta\hat{B}|\psi\rangle \qquad \text{or} \qquad \delta\hat{B}|\psi\rangle = -\frac{1}{\lambda}\delta\hat{A}|\psi\rangle$$

$$\frac{1}{\lambda}\langle\psi|(\delta\hat{A}\delta\hat{A}|\psi\rangle + \lambda\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle = 0 \qquad (1)$$

From the definition,

$$[\delta\hat{A}, \delta\hat{B}] = i\hat{C}$$

$$\langle\psi|(\delta\hat{A}\delta\hat{B} - \delta\hat{B}\delta\hat{A})|\psi\rangle = i\langle\psi|\hat{C}|\psi\rangle$$

or

$$-\frac{1}{\lambda}\langle\psi|\delta\hat{A}\delta\hat{A}|\psi\rangle + \lambda\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle = i\langle\psi|\hat{C}|\psi\rangle$$

Thus we have

$$2\lambda\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle = i\langle\psi|\hat{C}|\psi\rangle$$

or

$$\lambda = \frac{i\langle\psi|\hat{C}|\psi\rangle}{2\langle\psi|\delta\hat{B}\delta\hat{B})|\psi\rangle} = \frac{i\langle C\rangle}{2(\Delta B)^2}$$

or

$$\frac{1}{\lambda} = \frac{2(\Delta B)^2}{i\langle C\rangle} = \frac{2(\Delta A)^2(\Delta B)^2}{i\langle C\rangle(\Delta A)^2} = \frac{\frac{1}{2}\langle C\rangle^2}{i\langle C\rangle(\Delta A)^2} = -\frac{i\langle C\rangle}{2(\Delta A)^2}$$

Thus λ is a pure imaginary.

Gaussian wave packet

$\hat{A} = \hat{p}$, and $\hat{B} = \hat{x}$

$[\hat{p}, \hat{x}] = i\hat{C} = -i\hbar\hat{1}$

$$(\Delta x)(\Delta p) \geq \frac{1}{2}\left|\langle\psi|i\hbar\hat{1}|\psi\rangle\right| = \frac{\hbar}{2}$$

(Heisenberg's principle of uncertainty).

We now consider the case of $(\Delta x)(\Delta p) = \frac{\hbar}{2}$.

$$\lambda = -\frac{i\hbar}{2(\Delta x)^2}$$

$$\delta\hat{A}|\psi\rangle = -\lambda\delta\hat{B}|\psi\rangle$$

or

$$(\hat{p}-\langle p\rangle)|\psi\rangle=\frac{i\hbar}{2(\Delta x)^2}(\hat{x}-\langle x\rangle)|\psi\rangle$$

$$\langle x|\hat{p}-\langle p\rangle|\psi\rangle=\frac{i\hbar}{2(\Delta x)^2}\langle x|\hat{x}-\langle x\rangle|\psi\rangle$$

$$(\frac{\hbar}{i}\frac{\partial}{\partial x}-\langle p\rangle)\langle x|\psi\rangle=\frac{i\hbar}{2(\Delta x)^2}(x-\langle x\rangle)\langle x|\psi\rangle$$

or

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle=[\frac{i\hbar}{2(\Delta x)^2}(x-\langle x\rangle)+\langle p\rangle]\langle x|\psi\rangle$$

The normalized solution is

$$\langle x|\psi\rangle=\frac{1}{[2\pi(\Delta x)^2]^{1/4}}\exp[-\frac{1}{4(\Delta x)^2}(x-\langle x\rangle)^2+\frac{i}{\hbar}\langle p\rangle x]$$

This is a minimum uncertainty wave packet (Gaussian wave packet). This state represents a plane wave that is modulated by a Gaussian amplitude function. Since l is imaginary, this equation is an eigenfunction of the non-Hermitian operator $\hat{A}+\lambda\hat{B}=\hat{p}+\lambda\hat{x}$.

((**1-19**))

19. a. Compute

$$\left\langle(\Delta S_x)^2\right\rangle\equiv\langle S_x^2\rangle-\langle S_x\rangle^2,$$

where the expectation value is taken for the S_z+ state. Using your result, check the generalized uncertainty relation

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle\geq\frac{1}{4}|\langle[A,B]\rangle|^2,$$

with $A\rightarrow S_x, B\rightarrow S_y$.

b. Check the uncertainty relation with $A\rightarrow S_x, B\rightarrow S_y$ for the S_x+ state.

$$\hat{S}_x=\frac{\hbar}{2}\hat{\sigma}_x,\qquad \hat{S}_y=\frac{\hbar}{2}\hat{\sigma}_y\qquad \hat{S}_z=\frac{\hbar}{2}\hat{\sigma}_z$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_x{}^2 = \hat{\sigma}_y{}^2 = \hat{\sigma}_z{}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(\Delta S_x)^2 = \langle + | \hat{S}_x{}^2 | + \rangle - \langle + | \hat{S}_x | + \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$$(\Delta S_y)^2 = \langle + | \hat{S}_y{}^2 | + \rangle - \langle + | \hat{S}_y | + \rangle^2 = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

$$\langle + | [\hat{S}_x, \hat{S}_y] | + \rangle = \langle + | i\hbar \hat{S}_z | + \rangle = \frac{i\hbar^2}{2}$$

or

$$\frac{1}{4} \left| \langle + | [\hat{S}_x, \hat{S}_y] | + \rangle \right|^2 = \frac{\hbar^4}{16}$$

Then we have the relation

$$(\Delta S_x)^2 (\Delta S_y)^2 = \frac{1}{4} \left| \langle + | [\hat{S}_x, \hat{S}_y] | + \rangle \right|^2$$

(b)

$$(\Delta S_x)^2 = {}_x\langle + | \hat{S}_x{}^2 | + \rangle_x - {}_x\langle + | \hat{S}_x | + \rangle_x{}^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0$$

$${}_x\langle + | [\hat{S}_x, \hat{S}_y] | + \rangle_x = {}_x\langle + | i\hbar \hat{S}_z | + \rangle_x = 0$$

Here

$${}_x\langle +|\hat{S}_z|+\rangle_x = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

Thus we have

$$(\Delta S_x)^2(\Delta S_y)^2 = \frac{1}{4}\left|\langle +|[\hat{S}_x, \hat{S}_y]|+\rangle\right|^2 = 0$$

((**1-20**))

20. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle (\Delta S_x)^2\rangle \langle (\Delta S_y)^2\rangle .$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

We assume that the state is given by

$$|\psi\rangle = |+\rangle_n = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\langle\psi|\hat{S}_x{}^2|\psi\rangle = \frac{\hbar^2}{4}$$

$$\langle\psi|\hat{S}_x|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix} = \frac{\hbar}{2}\sin\theta\cos\phi$$

$$(\Delta S_x)^2 = \langle\psi|\hat{S}_x{}^2|\psi\rangle - \langle\psi|\hat{S}_x|\psi\rangle^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\theta\cos^2\phi$$

Similarly,

$$\langle\psi|\hat{S}_y{}^2|\psi\rangle = \frac{\hbar^2}{4}$$

$$\langle\psi|\hat{S}_y|\psi\rangle = \frac{\hbar}{2}\begin{pmatrix} e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix} = \frac{\hbar}{2}\sin\theta\sin\phi$$

$$(\Delta S_y)^2 = \langle\psi|\hat{S}_y{}^2|\psi\rangle - \langle\psi|\hat{S}_y|\psi\rangle^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}\sin^2\theta\sin^2\phi$$

Therefore we have

$$f = (\Delta S_x)^2(\Delta S_y)^2 = \left(\frac{\hbar^2}{4}\right)^2\left(1-\sin^2\theta\cos^2\phi\right)\left(1-\sin^2\theta\sin^2\phi\right)$$

or

$$f = \left(\frac{\hbar^2}{4}\right)^2\left(1-\sin^2\theta + \frac{1}{4}\sin^4\theta\sin^2 2\phi\right)$$

$$\langle\psi|[\hat{S}_x,\hat{S}_y]|\psi\rangle = \langle\psi|i\hbar\hat{S}_z|\psi\rangle = \frac{i\hbar^2}{2}\langle\psi|\hat{\sigma}_z|\psi\rangle = \frac{i\hbar^2}{2}\begin{pmatrix} e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= \frac{i\hbar^2}{2}\begin{pmatrix} e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} & e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ -e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix} = \frac{i\hbar^2}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = \frac{i\hbar^2}{2}\cos\theta$$

or

$$\langle\psi|[\hat{S}_x,\hat{S}_y]|\psi\rangle = \frac{i\hbar^2}{2}\cos\theta$$

For $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

f becomes maximum for $\theta = 0$ or π, where ϕ is arbitrary (see Mathematica 5.2)

$$|\psi\rangle = \begin{pmatrix} e^{-i\frac{\phi}{2}} \\ 0 \end{pmatrix} = e^{-i\frac{\phi}{2}}|+\rangle \qquad \text{for } \theta = 0.$$

$$|\psi\rangle = \begin{pmatrix} 0 \\ e^{i\frac{\phi}{2}} \end{pmatrix} = e^{i\frac{\phi}{2}}|-\rangle \qquad \text{for } \theta = \pi.$$

Thus $|\psi\rangle$ corresponds to $|+\rangle$ or $|-\rangle$ with a phase factor.

In this case,

$$(\Delta S_x)^2 = \langle\psi|\hat{S}_x{}^2|\psi\rangle - \langle\psi|\hat{S}_x|\psi\rangle^2 = \frac{\hbar^2}{4}$$

$$(\Delta S_y)^2 = \langle\psi|\hat{S}_y{}^2|\psi\rangle - \langle\psi|\hat{S}_y|\psi\rangle^2 = \frac{\hbar^2}{4}$$

$$\langle\psi|[\hat{S}_x, \hat{S}_y]|\psi\rangle = \pm\frac{i\hbar^2}{2}$$

Then we have

$$(\Delta S_x)^2(\Delta S_y)^2 = \frac{1}{4}\left|\langle\psi|[\hat{S}_x, \hat{S}_y]|\psi\rangle\right|^2 = \frac{\hbar^4}{16}$$

which does not violate the uncertainty relation.

$$(\Delta S_x)^2(\Delta S_y)^2 \ge \frac{1}{4}\left|\langle\psi|[\hat{S}_x, \hat{S}_y]|\psi\rangle\right|^2$$

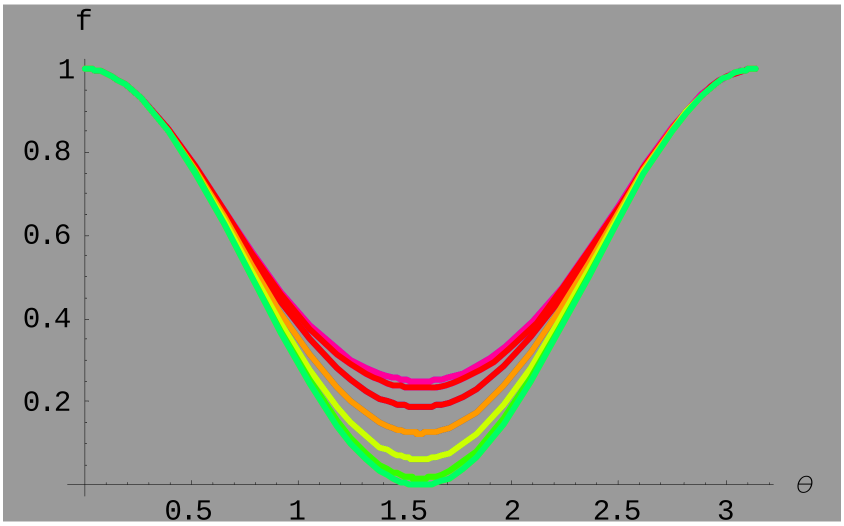
((Mathematica 5.2))

```
(*Problem Sakurai 1-20*)
```

$$f = 1 - \text{Sin}[\theta]^2 + \frac{1}{4}\text{Sin}[\theta]^4\,\text{Sin}[2\,\phi]^2$$

$$1 - \text{Sin}[\theta]^2 + \frac{1}{4}\text{Sin}[\theta]^4\,\text{Sin}[2\,\phi]^2$$

```
Plot[Evaluate[Table[f,{φ,0,2 π,
π/24}]],{θ,0,π},PlotStyle→Table[Hue[0.1
i],{i,0,10}],Prolog→AbsoluteThickness[2],AxesLabel→{"θ","f"
},Background→GrayLevel[0.6]]
```



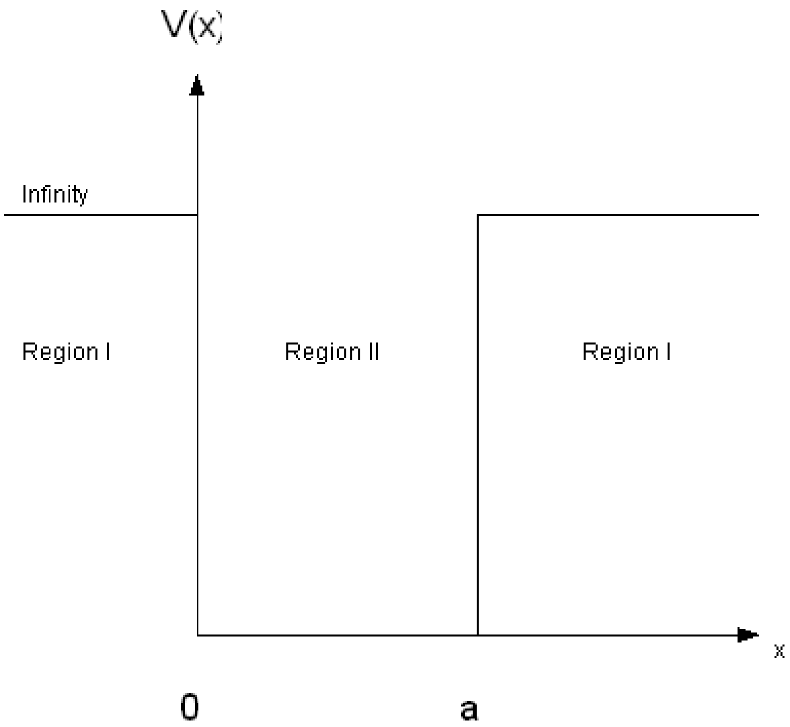
-Graphics-

((**1-21**))

21. Evaluate the x-p uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for a one-dimensional particle confined between two rigid walls

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases}$$

Do this for both the ground and excited states.



$$\hat{H} = \frac{\hat{p}^2}{2m}$$

$$H\varphi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi(x) = E\varphi(x) = \frac{\hbar^2 k^2}{2m}\varphi(x)$$

The solution of this equation is

$$\varphi(x) = A\sin(kx) + B\cos(kx)$$

where

$$E = \frac{\hbar^2 k^2}{2m}$$

Using the boundary condition:

$$\varphi(x=0) = \varphi(x=a) = 0$$

we have

$B = 0$ and $A \neq 0$.

$$\sin(ka) = 0$$

$$ka = n\pi \ \ (n = 1, 2, \ldots)$$

Note that $n = 0$ is not included in our solution because the corresponding wave function becomes zero.

The wave function is given by

$$\varphi_n(x) = \langle x|\varphi_n\rangle = A_n \sin(\frac{n\pi x}{a}) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$$

with

$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$$

$$\langle\varphi_n|\hat{x}|\varphi_n\rangle = \int_0^a dx\varphi_n^*(x)x\varphi_n(x) = \frac{a}{2}$$

$$\langle\varphi_n|\hat{x}^2|\varphi_n\rangle = \int_0^a dx\varphi_n^*(x)x^2\varphi_n(x) = \frac{a^2}{6}(2 - \frac{3}{n^2\pi^2})$$

$$(\Delta x)^2 = \langle\varphi_n|\hat{x}^2|\varphi_n\rangle - \langle\varphi_n|\hat{x}|\varphi_n\rangle^2 = \frac{a^2}{6}(2 - \frac{3}{n^2\pi^2}) - \frac{a^2}{4} = \frac{a^2}{n^2\pi^2}\left(\frac{n^2\pi^2}{12} - \frac{1}{2}\right)$$

$$\langle\varphi_n|\hat{p}|\varphi_n\rangle = \int_0^a dx\varphi_n^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\varphi_n(x) = 0$$

$$\langle\varphi_n|\hat{p}^2|\varphi_n\rangle = \int_0^a dx\varphi_n^*(x)\left(\frac{\hbar}{i}\right)^2\frac{\partial^2}{\partial x^2}\varphi_n(x) = \left(\frac{n\pi\hbar}{a}\right)^2$$

$$(\Delta p)^2 = \langle \varphi_n | \hat{p}^2 | \varphi_n \rangle - \langle \varphi_n | \hat{p} | \varphi_n \rangle^2 = \left(\frac{n\pi\hbar}{a} \right)^2$$

$$(\Delta x)^2 (\Delta p)^2 = \frac{a^2}{n^2\pi^2}\left(\frac{n^2\pi^2}{12} - \frac{1}{2}\right)\frac{n^2\pi^2\hbar^2}{a^2} = \hbar^2\left(\frac{n^2\pi^2}{12} - \frac{1}{2}\right) > \frac{\hbar^2}{4}$$

((Mathematica 5.2))

(*Sakurai Problem 1-21*)

$$\psi[x\_] := \sqrt{\frac{2}{a}}\ \mathrm{Sin}\left[\frac{n\,\pi\,x}{a}\right]$$

$$\mathrm{avx} = \int_0^a x\,\psi[x]^2\,dx\ //\ \mathrm{Simplify}$$

$$-\frac{a\,(-1 - 2\,n^2\,\pi^2 + \mathrm{Cos}[2\,n\,\pi] + 2\,n\,\pi\,\mathrm{Sin}[2\,n\,\pi])}{4\,n^2\,\pi^2}$$

avx1=Simplify[avx,n∈Integers]

$$\frac{a}{2}$$

$$\mathrm{avsqx} = \int_0^a x^2\,\psi[x]^2\,dx\ //\ \mathrm{Simplify}$$

$$\frac{a^2\,(4\,n^3\,\pi^3 - 6\,n\,\pi\,\mathrm{Cos}[2\,n\,\pi] + (3 - 6\,n^2\,\pi^2)\,\mathrm{Sin}[2\,n\,\pi])}{12\,n^3\,\pi^3}$$

avsqx1=Simplify[avsqx,n∈Integers]

$$\frac{1}{6}\,a^2\left(2 - \frac{3}{n^2\,\pi^2}\right)$$

$$\mathrm{avp} = \frac{\hbar}{i}\int_0^a \psi[x]\,D[\psi[x],\,x]\,dx\ //\ \mathrm{Simplify}$$

$$-\frac{i\,\hbar\,\mathrm{Sin}[n\,\pi]^2}{a}$$

avp1=Simplify[avp,n∈Integers]

0

$$\mathrm{avsqp} = \left(\frac{\hbar}{i}\right)^2\int_0^a \psi[x]\,D[\psi[x],\,\{x,\,2\}]\,dx\ //\ \mathrm{Simplify}$$

$$\frac{n\,\pi\,\hbar^2\left(n\,\pi - \frac{1}{2}\,\mathrm{Sin}[2\,n\,\pi]\right)}{a^2}$$

avsqp1=Simplify[avsqp,n∈Integers]

$$\frac{n^2\,\pi^2\,\hbar^2}{a^2}$$

```
(avsqx1 – avx1^2) (avsqp1 – avp1^2) // Simplify
```

$$\frac{1}{12}\left(-6+n^2\pi^2\right)\hbar^2$$

```
%/.n→1//N
```

$0.322467\ \hbar^2$

((**1-22**))

22. Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations which do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it *in seconds*.

((**1-23**))

23. Consider a three-dimensional ket space. If a certain set of orthonormal kets—say, $|1\rangle$, $|2\rangle$, and $|3\rangle$—are used as the base kets, the operators A and B are represented by

$$A \doteq \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad B \doteq \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

with a and b both real.

a. Obviously A exhibits a degenerate spectrum. Does B also exhibit a degenerate spectrum?
b. Show that A and B commute.
c. Find a new set of orthonormal kets which are simultaneous eigenkets of both A and B. Specify the eigenvalues of A and B for each of the three eigenkets. Does your specification of eigenvalues completely characterize each eigenket?

$$\hat{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

or

$\hat{A}|1\rangle = a|1\rangle$, $\hat{A}|2\rangle = -a|2\rangle$, $\hat{A}|3\rangle = -a|3\rangle$

$\hat{B}|1\rangle = b|1\rangle$, $\hat{B}|2\rangle = ib|3\rangle$ $\hat{B}|3\rangle = -ib|2\rangle$

$[\hat{A}, \hat{B}] = 0$

The eigenkets of $\hat{B}$ should be the eigenkets of $\hat{A}$, and vice versa.
The operators $\hat{A}$ and $\hat{B}$ are the Hermitian operators.

Eigenkets of $\hat{A}$:

$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, (eigenvalue: *a*)

$|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, (eigenvalue: -*a*)

$|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, (eigenvalue: -*a*)

Eigenkets of $\hat{B}$:

$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle$, (eigenvalue: *b*)

$\hat{B}|2\rangle = ib|3\rangle$ $\hat{B}|3\rangle = -ib|2\rangle$

In the subspace spanned by $|2\rangle$ and $|3\rangle$, we consider the eigenvalue problem

$$\hat{B}_{sub} = \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix}$$

$$\hat{B}_{sub}|\phi\rangle = \lambda|\phi\rangle$$

with

$$|\phi\rangle = \begin{pmatrix} C_2 \\ C_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = \lambda\begin{pmatrix} C_2 \\ C_3 \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & -ib \\ ib & -\lambda \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0$$

$$M = \begin{pmatrix} -\lambda & -ib \\ ib & -\lambda \end{pmatrix}$$

Det[M]=0:

$$\begin{vmatrix} -\lambda & -ib \\ ib & -\lambda \end{vmatrix} = \lambda^2 - b^2 = 0$$

(i) $\lambda = b$

$$\begin{pmatrix} -b & -ib \\ ib & -b \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0$$

$$C_3 = iC_2$$

$$|C_2|^2 + |C_3|^3 = 1$$

Then we have $C_2 = -\frac{i}{\sqrt{2}}$ and $C_3 = \frac{1}{\sqrt{2}}$

or

$$|\psi_2\rangle = \begin{pmatrix} 0 \\ -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{i}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle: \quad \text{(eigenvalue } b\text{)}$$

(ii) λ = -*b*

$$\begin{pmatrix} b & -ib \\ ib & b \end{pmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = 0$$

$$C_3 = iC_2$$

$$|C_2|^2 + |C_3|^3 = 1$$

Then we have $C_2 = -\frac{i}{\sqrt{2}}$ and $C_3 = \frac{1}{\sqrt{2}}$

or

$$|\psi_3\rangle = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{i}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle: \qquad \text{(eigenvalue } -b\text{)}$$

Since any combinations of $|2\rangle$ and $|3\rangle$ are the eigenkets of $\hat{A}$ with an eigenvalue (-*a*). Then

$$\hat{A}|\psi_2\rangle = -a|\psi_2\rangle, \qquad \text{and} \qquad \hat{A}|\psi_3\rangle = -a|\psi_3\rangle$$

In conclusion, $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$ are the simultaneous eigenkets of $\hat{A}$ and $\hat{B}$.

((Mathematica 5.2))

```
(*Sakurai 1-23
      Simultaneous eigenkets
  *)
SuperStar /: expr_* :=
  expr /. {Complex[a_, b_] :> Complex[a, -b]}
A={{a,0,0},{0,-a,0},{0,0,-a}}
{{a,0,0},{0,-a,0},{0,0,-a}}
B={{b,0,0},{0,0,-ⅈ b},{0,ⅈ b,0}}
{{b,0,0},{0,0,-ⅈ b},{0,ⅈ b,0}}
A//MatrixForm
```

$$\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}$$

```
B//MatrixForm
```

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -i\,b \\ 0 & i\,b & 0 \end{pmatrix}$$

```
Eigensystem[B]
{{-b,b,b},{{0,ⅈ,1},{0,-ⅈ,1},{1,0,0}}}
A.B-B.A//Simplify
{{0,0,0},{0,0,0},{0,0,0}}
(*Thus [A,B] = 0, commutable*)
ψ1={1,0,0}
{1,0,0}
ψ222={0,-ⅈ,1}
{0,-ⅈ,1}
```

$$\psi 2 = \frac{\psi 222}{\sqrt{\psi 222^{*}.\psi 222}} \;//\; \text{Simplify}$$

$$\left\{0,\ -\frac{i}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\}$$

```
ψ333={0,ⅈ,1}
{0,ⅈ,1}
```

$$\psi 3 = \frac{\psi 333}{\sqrt{\psi 333^{*}.\psi 333}} \;//\; \text{Simplify}$$

$$\left\{0,\ \frac{i}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\}$$

```
A.ψ1-a ψ1
{0,0,0}
A.ψ2+a ψ2
{0,0,0}
A.ψ3+a ψ3
{0,0,0}
```

((**1-24**))

24. a. Prove that $(1/\sqrt{2})(1+i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

b. Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

$$\hat{R} = \frac{1}{\sqrt{2}}(\hat{1} + i\hat{\sigma}_x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since

$$|+\rangle_n = e^{-i\alpha/2}\cos\frac{\beta}{2}|+\rangle + e^{i\alpha/2}\sin\frac{\beta}{2}|-\rangle$$

$$|-\rangle_n = e^{-i\alpha/2}\sin\frac{\beta}{2}|+\rangle - e^{i\alpha/2}\cos\frac{\beta}{2}|-\rangle$$

When the $\boldsymbol{n}$ is along the y-axis, $\alpha = \beta = \pi/2$

$$|+\rangle_y = e^{-i\pi/4}\cos\frac{\pi}{4}|+\rangle + e^{i\pi/4}\sin\frac{\pi}{4}|-\rangle = \frac{1}{\sqrt{2}}(e^{-i\pi/4}|+\rangle + e^{i\pi/4}|-\rangle) = \frac{1}{\sqrt{2}}e^{-i\pi/4}(|+\rangle + i|-\rangle)$$

The phase factor is arbitrary. So we have

$$|+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

and

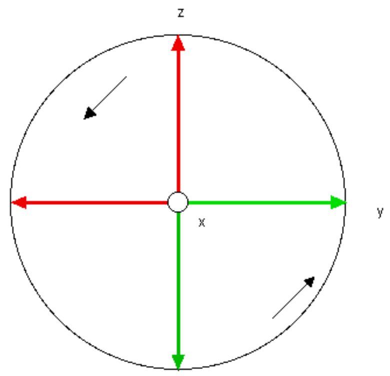
$$|-\rangle_y = e^{-i\pi/4}\sin\frac{\pi}{4}|+\rangle - e^{i\pi/4}\cos\frac{\pi}{4}|-\rangle = \frac{1}{\sqrt{2}}(e^{-i\pi/4}|+\rangle - e^{i\pi/4}|-\rangle) = \frac{1}{\sqrt{2}}e^{-i\pi/4}(|+\rangle - i|-\rangle)$$

or

$$|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$\hat{R}|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = |+\rangle_y$$

$$\hat{R}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix} = i\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = i|-\rangle_y$$



So we can conclude that $\hat{R} = \frac{1}{\sqrt{2}}(\hat{1} + i\hat{\sigma}_x)$ is the rotation operator around the *x* axis.

(b)

We need to calculate the matix elements

$$ {}_y\langle +|\hat{S}_z|+\rangle_y = \frac{\hbar}{2}{}_y\langle +|\hat{\sigma}_z|+\rangle_y = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = 0, $$

$$ {}_y\langle +|\hat{S}_z|-\rangle_y = \frac{\hbar}{2}{}_y\langle +|\hat{\sigma}_z|-\rangle_y = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2} $$

$$ {}_y\langle -|\hat{S}_z|+\rangle_y = \frac{\hbar}{2}{}_y\langle -|\hat{\sigma}_z|+\rangle_y = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2} $$

$$ {}_y\langle -|\hat{S}_z|-\rangle_y = \frac{\hbar}{2}{}_y\langle -|\hat{\sigma}_z|-\rangle_y = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = 0 $$

((Mathemnatica))

```
(*Sakurai 1-24*)
(*a*)
Clear["Global`"]
σx={{0,1},{1,0}}
{{0,1},{1,0}}
σy={{0,-ⅈ},{ⅈ,0}}
{{0,-ⅈ},{ⅈ,0}}
σz={{1,0},{0,-1}}
{{1,0},{0,-1}}
conjugateRule =
  {Complex[re_, im_] :→ Complex[re, -im]};
Unprotect[SuperStar];
SuperStar /: exp_ * := exp /. conjugateRule;
Protect[SuperStar]

{SuperStar}
II={{1,0},{0,1}}
{{1,0},{0,1}}
II//MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A = \frac{1}{\sqrt{2}} (II + i\,\sigma X)}$$

$$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}$$

$$\psi\mathbf{py} = \left\{\frac{e^{-i(\pi/4)}}{\sqrt{2}}, \frac{e^{i(\pi/4)}}{\sqrt{2}}\right\} \mathbin{//} \mathbf{ExpToTrig}$$

$$\left\{\frac{1}{2} - \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\right\}$$

ψpy//MatrixForm

$$\begin{pmatrix} \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \end{pmatrix}$$

$$\psi\mathbf{my} = \left\{\frac{-e^{-i(\pi/4)}}{\sqrt{2}}, \frac{e^{i(\pi/4)}}{\sqrt{2}}\right\} \mathbin{//} \mathbf{ExpToTrig}$$

$$\left\{-\frac{1}{2} + \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\right\}$$

ψ1={1,0}

{1,0}

ψ1

{1,0}

ψ2={0,1}

{0,1}

ψ2

{0,1}

A.ψ1//Simplify

$$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$$

A.ψ2//Simplify

$$\left\{\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$\phi\mathbf{1} = e^{i(\pi/4)}\,\psi\mathbf{py} \mathbin{//} \mathbf{Simplify}$$

$$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$$

$$\phi\mathbf{2} = e^{i(-\pi/4)}\,\psi\mathbf{my} \mathbin{//} \mathbf{Simplify}$$

```
{ i/√2 , 1/√2 }
σx
{{0,1},{1,0}}
(*Rotation operator*)
MatrixExp[-i θ/2 σx] /. θ → -π/2
{{1/√2, i/√2}, {i/√2, 1/√2}}
A
{{1/√2, i/√2}, {i/√2, 1/√2}}
(* b*)
ϕ1*.σz.ϕ1
0
ϕ1*.σz.ϕ2
i
ϕ2*.σz.ϕ1
-i
ϕ2*.σz.ϕ2
0
```

((**1-25**))

25. Some authors define an *operator* to be real when every member of its matrix elements $\langle b'|A|b''\rangle$ is real in some representation ($\{|b'\rangle\}$ basis in this case). Is this concept representation independent, that is, do the matrix elements remain real even if some basis other than $\{|b'\rangle\}$ is used? Check your assertion using familiar operators such as S_y and S_z (see Problem 24) or x and p_x.

((**1-26**))

26. Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|.$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \hat{U}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}U_{11}\\U_{21}\end{pmatrix}$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \hat{U}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12}\\U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}U_{12}\\U_{22}\end{pmatrix}$$

Thus the Unitary operator is obtained by

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

Note that

$$|+\rangle\langle+| + |-\rangle\langle-| = \hat{1} \qquad \text{(closure relation).}$$

Thus $\hat{U}$ can be also expressed by

$$\hat{U} = \hat{U}(|+\rangle\langle+| + |-\rangle\langle-|) = \hat{U}|+\rangle\langle+| + \hat{U}|-\rangle\langle-| = (|+\rangle_x\langle+|) + (|-\rangle_x\langle-|)$$

((**1-27**))

27. a. Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A|a'\rangle = a'|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.

b. Using the continuum analogue of the result obtained in (a), evaluate

$$\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle.$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, and z are *operators*.

(a)

$$|b'\rangle = \hat{U}|a'\rangle, \qquad |b''\rangle = \hat{U}|a''\rangle$$

or

$$\langle b'| = \langle a'|\hat{U}^+ , \qquad \langle b''| = \langle a''|\hat{U}^+$$

$$\langle b''|f(\hat{A})|b'\rangle = \langle a''|\hat{U}^+ f(\hat{A})\hat{U}|a'\rangle = \sum_{a'''}\langle a''|\hat{U}^+ f(\hat{A})|a'''\rangle\langle a'''|\hat{U}|a'\rangle$$

$$= \sum_{a'''}\langle a''|\hat{U}^+|a'''\rangle f(a''')\langle a'''|\hat{U}|a'\rangle$$

$$= \sum_{a'''}\langle a'''|\hat{U}|a''\rangle^* f(a''')\langle a'''|\hat{U}|a'\rangle$$

where

$$\langle a''|\hat{U}^+|a'''\rangle = \langle a'''|\hat{U}|a''\rangle^*$$

(b)

$$\langle \mathbf{p}''|F(\hat{\mathbf{r}})|\mathbf{p}'\rangle = \int d\mathbf{r}'\int d\mathbf{r}''\langle \mathbf{p}''|\mathbf{r}''\rangle\langle \mathbf{r}''|F(\hat{\mathbf{r}})|\mathbf{r}'\rangle\langle \mathbf{r}'|\mathbf{p}'\rangle$$

$$= \int d\mathbf{r}'\int d\mathbf{r}''\langle \mathbf{p}''|\mathbf{r}''\rangle F(\mathbf{r}')\delta(\mathbf{r}'-\mathbf{r}'')\langle \mathbf{r}'|\mathbf{p}'\rangle$$

$$= \int d\mathbf{r}'\langle \mathbf{p}''|\mathbf{r}'\rangle F(\mathbf{r}')\langle \mathbf{r}'|\mathbf{p}'\rangle$$

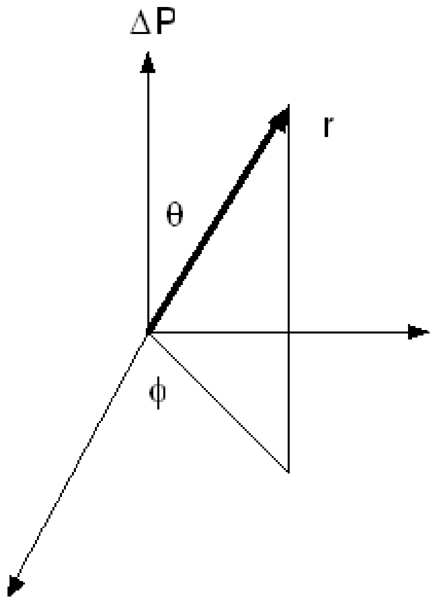
Using the transformation function

$$\langle \mathbf{r}|\mathbf{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\exp(\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar})$$

$$\langle \mathbf{p}''|F(\hat{\mathbf{r}})|\mathbf{p}'\rangle = \frac{1}{(2\pi\hbar)^3}\int d\mathbf{r}'\exp(-\frac{i\mathbf{p}''\cdot\mathbf{r}'}{\hbar})F(\mathbf{r}')\exp(\frac{i\mathbf{p}'\cdot\mathbf{r}'}{\hbar})$$

$$= \frac{1}{(2\pi\hbar)^3}\int d\mathbf{r}'\exp[\frac{i(\mathbf{p}'-\cdot\mathbf{p}'')\cdot\mathbf{r}'}{\hbar}]F(\mathbf{r}')$$

Here we use the spherical co-ordinate (r, θ, ϕ). The direction of $\Delta\mathbf{p} = \mathbf{p}'-\cdot\mathbf{p}''$ is the z axis.



$$d\mathbf{r}' = r'^2 \sin\theta dr' d\theta d\phi$$

$$(\mathbf{p}' - \mathbf{p}'')\cdot\mathbf{r}' = |\mathbf{p}' - \mathbf{p}''| r' \cos\theta$$

Suppose that $F(\mathbf{r})$ is a function of the magnitude of $\boldsymbol{r}$.

$$\langle \mathbf{p}''|F(\hat{\mathbf{r}})|\mathbf{p}'\rangle = \frac{2\pi}{(2\pi\hbar)^3}\int_0^\infty F(r')r'^2 dr' \int_0^\pi d\theta \sin\theta \exp[\frac{i|\mathbf{p}'-\mathbf{p}''|r'\cos\theta}{\hbar}]$$

Note that

$$\int_0^\pi d\theta \sin\theta \exp[\frac{i|\mathbf{p}'-\mathbf{p}''|r'\cos\theta}{\hbar}] = \frac{2\hbar}{|\mathbf{p}'-\mathbf{p}''|r'}\sin\left(\frac{|\mathbf{p}'-\mathbf{p}''|r'}{\hbar}\right)$$

Then

$$\langle \mathbf{p}''|F(\hat{\mathbf{r}})|\mathbf{p}'\rangle = \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty dr' r'^2 F(r') \frac{\sin\left(\frac{|\mathbf{p}'-\mathbf{p}''|r'}{\hbar}\right)}{\frac{|\mathbf{p}'-\mathbf{p}''|r'}{\hbar}}$$

((**1-28**))

28. a. Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{\text{classical}}.$$

b. Let x and p_x be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$\left[x, \exp\left(\frac{ip_x a}{\hbar}\right)\right].$$

c. Using the result obtained in (b), prove that

$$\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle, \qquad (x|x'\rangle = x'|x'\rangle)$$

is an eigenstate of the coordinate operator x. What is the corresponding eigenvalue?

(a)

$$[x, F(p_x)]_{classical} = \frac{\partial x}{\partial x}\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F(p_x)}{\partial x} = \frac{\partial F(p_x)}{\partial p_x}$$

(b)

We use the Gottfried's result

$$[\hat{x}, \exp(\frac{i\hat{p}_x a}{\hbar})] = i\hbar\frac{\partial}{\partial \hat{p}_x}\exp(\frac{i\hat{p}_x a}{\hbar}) = -a\exp(\frac{i\hat{p}_x a}{\hbar})$$

(c)

$$\hat{x}\exp(\frac{i\hat{p}_x a}{\hbar}) = \exp(\frac{i\hat{p}_x a}{\hbar})\hat{x} - a\exp(\frac{i\hat{p}_x a}{\hbar})$$

$$\hat{x}\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle = \exp(\frac{i\hat{p}_x a}{\hbar})\hat{x}|x'\rangle - a\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle = (x'-a)\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle$$

The ket $\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle$ is the eigenket of $\hat{x}$ with an eigenvalue (x'-a). Therefore $\exp(\frac{i\hat{p}_x a}{\hbar})$ is a translation operator.

((**1-29**))

29. a. On page 247, Gottfried (1966) states that

$$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \qquad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$

can be "easily derived" from the fundamental commutation relations for all functions of F and G that can be expressed as power series in their arguments. Verify this statement.

b. Evaluate $[x^2, p^2]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{\text{classical}}$.

(a)

(i)

$$\langle \mathbf{p}|[\hat{x}_i, G(\hat{\mathbf{p}})]|\alpha\rangle = [i\hbar\frac{\partial}{\partial p_i}G(\mathbf{p}) - G(\mathbf{p})i\hbar\frac{\partial}{\partial p_i}]\langle \mathbf{p}|\alpha\rangle$$

$$= i\hbar\frac{\partial}{\partial p_i}[G(\mathbf{p})\langle \mathbf{p}|\alpha\rangle] - i\hbar G(\mathbf{p})\frac{\partial}{\partial p_i}\langle \mathbf{p}|\alpha\rangle$$

$$= i\hbar\left(\frac{\partial}{\partial p_i}G(\mathbf{p})\right)\langle \mathbf{p}|\alpha\rangle + i\hbar G(\mathbf{p})\frac{\partial}{\partial p_i}\langle \mathbf{p}|\alpha\rangle - i\hbar G(\mathbf{p})\frac{\partial}{\partial p_i}\langle \mathbf{p}|\alpha\rangle$$

$$= i\hbar\left(\frac{\partial}{\partial p_i}G(\mathbf{p})\right)\langle \mathbf{p}|\alpha\rangle$$

$$= \langle \mathbf{p}|i\hbar\frac{\partial}{\partial \hat{p}_i}G(\hat{\mathbf{p}})|\alpha\rangle$$

Thus we have the final result

$$[\hat{x}_i, G(\hat{\mathbf{p}})] = i\hbar\frac{\partial}{\partial \hat{p}_i}G(\hat{\mathbf{p}})$$

(ii)

$$\langle \mathbf{r}|[\hat{p}_i, F(\hat{\mathbf{r}})]|\alpha\rangle = [\frac{\hbar}{i}\frac{\partial}{\partial x_i}F(\mathbf{r}) - F(\mathbf{r})\frac{\hbar}{i}\frac{\partial}{\partial x_i}]\langle \mathbf{r}|\alpha\rangle$$

$$= \frac{\hbar}{i}\frac{\partial}{\partial x_i}[F(\mathbf{r})\langle \mathbf{r}|\alpha\rangle] - \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle$$

$$= \frac{\hbar}{i}\left(\frac{\partial}{\partial x_i}F(\mathbf{r})\right)\langle \mathbf{r}|\alpha\rangle + \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle] - \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle$$

$$= \frac{\hbar}{i}\left(\frac{\partial}{\partial x_i}F(\mathbf{r})\right)\langle \mathbf{r}|\alpha\rangle = \langle \mathbf{r}|\frac{\hbar}{i}\frac{\partial}{\partial \hat{x}_i}F(\hat{\mathbf{r}})|\alpha\rangle$$

or

$$[\hat{p}_i, F(\hat{\mathbf{r}})] = \frac{\hbar}{i}\frac{\partial}{\partial \hat{x}_i}F(\hat{\mathbf{r}})$$

(b)

$$[\hat{x}^2, \hat{p}^2] = \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x} = \hat{x}i\hbar\frac{\partial}{\partial \hat{p}}\hat{p}^2 + i\hbar\left(\frac{\partial}{\partial \hat{p}}\hat{p}^2\right)\hat{x} = 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})$$

The classical Poisson bracket is defined by

$$[x^2, p^2]_{classic} = \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x} = 4xp = 2(xp + px)$$

((**1-30**))

30. The translation operator for a finite (spatial) displacement is given by

$$\mathscr{T}(\mathbf{l}) = \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right),$$

where **p** is the momentum *operator*.

a. Evaluate

$$[x_i, \mathscr{T}(\mathbf{l})].$$

b. Using (a) (or otherwise), demonstrate how the expectation value $\langle \mathbf{x} \rangle$ changes under translation.

(a)

The translation operator is defined by

$$\hat{T}(\mathbf{l}) = \exp(-\frac{i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar})$$

$$[\hat{x}_i, \hat{T}(\mathbf{l})] = i\hbar\frac{\partial}{\partial \hat{p}_i}\hat{T}(\mathbf{l}) = l_i \exp(-\frac{i\hat{\mathbf{p}}\cdot\mathbf{l}}{\hbar}) = l_i\hat{T}(\mathbf{l})\ ]$$

or

$$[\hat{\mathbf{r}}, \hat{T}(\mathbf{l})] = \mathbf{l}\hat{T}(\mathbf{l})$$

(b)

$$|\alpha'\rangle = \hat{T}(\mathbf{l})|\alpha\rangle$$

$$\langle\alpha'|\hat{\mathbf{r}}|\alpha'\rangle = \langle\alpha|\hat{T}^+(\mathbf{l})\hat{\mathbf{r}}\hat{T}(\mathbf{l})|\alpha'\rangle = \langle\alpha|\hat{T}^+(\mathbf{l})[\hat{T}(\mathbf{l})\hat{\mathbf{r}} + \mathbf{l}\hat{T}(\mathbf{l})]|\alpha'\rangle = \langle\alpha|\hat{\mathbf{r}} + \mathbf{l}|\alpha'\rangle$$

or

$$\langle\alpha'|\hat{\mathbf{r}}|\alpha'\rangle = \langle\alpha|\hat{\mathbf{r}}|\alpha\rangle + \mathbf{l}$$

((1-31))

31. In the main text we discussed the effect of $\mathscr{T}(d\mathbf{x}')$ on the position and momentum eigenkets and on a more general state ket $|\alpha\rangle$. We can also study the behavior of expectation values $\langle\mathbf{x}\rangle$ and $\langle\mathbf{p}\rangle$ under infinitesimal translation. Using (1.6.25), (1.6.45), and $|\alpha\rangle \rightarrow \mathscr{T}(d\mathbf{x}')|\alpha\rangle$ only, prove $\langle\mathbf{x}\rangle \rightarrow \langle\mathbf{x}\rangle + d\mathbf{x}', \langle\mathbf{p}\rangle \rightarrow \langle\mathbf{p}\rangle$ under infinitesimal translation.

We use the commutation relations

$$[\hat{\mathbf{r}}, \hat{T}(d\mathbf{r})] = d\mathbf{r}\hat{T}(d\mathbf{r})$$

and

$$[\hat{\mathbf{p}}, \hat{T}(d\mathbf{r})] = 0$$

We have

$$\langle\alpha|\hat{T}^{+}(d\mathbf{r})\hat{\mathbf{r}}\hat{T}(d\mathbf{r})|\alpha\rangle = \langle\alpha|\hat{T}^{+}(d\mathbf{r})[\hat{T}(d\mathbf{r})\hat{\mathbf{r}} + d\mathbf{r}\hat{T}(d\mathbf{r})|\alpha\rangle = \langle\alpha|\hat{\mathbf{r}} + d\mathbf{r}|\alpha\rangle$$

or

$$\langle\alpha'|\hat{\mathbf{r}}|\alpha'\rangle = \langle\alpha|\hat{\mathbf{r}}|\alpha\rangle + d\mathbf{r}$$

Similarly

$$\langle\alpha|\hat{T}^{+}(d\mathbf{r})\hat{\mathbf{p}}\hat{T}(d\mathbf{r})|\alpha\rangle = \langle\alpha|\hat{T}^{+}(d\mathbf{r})\hat{T}(d\mathbf{r})\hat{\mathbf{p}}|\alpha\rangle = \langle\alpha|\hat{\mathbf{p}}|\alpha\rangle$$

((**1-32**))

32. a. Verify (1.7.39a) and (1.7.39b) for the expectation value of p and p^2 from the Gaussian wave packet (1.7.35).
    b. Evaluate the expectation value of p and p^2 using the momentum-space wave function (1.7.42).

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\exp(ikx' - \frac{x'^2}{2d^2})$$

$$\langle p\rangle = \langle\alpha|\hat{p}|\alpha\rangle = \int dx'\langle\alpha|x'\rangle\langle x'|\hat{p}|\alpha\rangle = \int dx'\langle\alpha|x'\rangle\frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$= \int dx'\frac{1}{\pi^{1/2}d}\exp(-ikx' - \frac{x'^2}{2d^2})\frac{\hbar}{i}\frac{\partial}{\partial x'}\exp(ikx' - \frac{x'^2}{2d^2})$$

$$= \int dx'\frac{1}{\pi^{1/2}d}\left(\hbar k + i\hbar\frac{x'}{d^2}\right)\exp(-\frac{x'^2}{2d^2})$$

Since the second term is an odd function,

$$\langle p\rangle = \int dx'\frac{\hbar k}{\pi^{1/2}d}\exp(-\frac{x'^2}{2d^2}) = \frac{\hbar k}{\pi^{1/2}d}\pi^{1/2}d = \hbar k$$

$$\langle p^2\rangle = \langle\alpha|\hat{p}^2|\alpha\rangle = \int dx'\langle\alpha|x'\rangle\langle x'|\hat{p}^2|\alpha\rangle = \int dx'\langle\alpha|x'\rangle\left(\frac{\hbar}{i}\right)^2\frac{\partial^2}{\partial x'^2}\langle x'|\alpha\rangle$$

$$= -\frac{\hbar^2}{\pi^{1/2} d}\int dx'(-\frac{1}{d^2} - k^2 - \frac{2ik}{d^2}x' + \frac{x'^2}{d^4})\exp(-\frac{x'^2}{d^2})$$

$$= -\frac{\hbar^2}{\pi^{1/2} d}\left[-(\frac{1}{d^2} + k^2)\pi^{1/2} d + \frac{d^2}{2d^4}\pi^{1/2} d\right] = \hbar^2(\frac{1}{2d^2} + k^2)$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar^2}{2d^2}$$

(b)

$$\langle p' | \alpha \rangle = \frac{1}{\pi^{1/4}}\sqrt{\frac{d}{\hbar}}\exp\left[-\frac{(p' - \hbar k)^2 d^2}{2\hbar^2}\right]$$

$$\langle p \rangle = \langle \alpha | \hat{p} | \alpha \rangle = \int dp' \langle \alpha | p' \rangle \langle p' | \hat{p} | \alpha \rangle = \int p' dp' \langle \alpha | p' \rangle \langle p' | \alpha \rangle = \frac{d}{\hbar\sqrt{\pi}}\int p' dp' \exp[-\frac{(p' - \hbar k)^2 d^2}{\hbar^2}]$$

We put

$$p'' = p' - \hbar k$$

$$\langle p \rangle = \frac{d}{\hbar\sqrt{\pi}}\int (p'' + \hbar k) dp'' \exp[-\frac{p''^2 d^2}{\hbar^2}] = \frac{dk}{\sqrt{\pi}}\int dp'' \exp[-\frac{p''^2 d^2}{\hbar^2}] = \hbar k$$

$$\langle p^2 \rangle = \frac{d}{\hbar\sqrt{\pi}}\int (p'' + \hbar k)^2 dp'' \exp[-\frac{p''^2 d^2}{\hbar^2}] = \frac{dk}{\sqrt{\pi}}\int (p''^2 + \hbar^2 k^2) dp'' \exp[-\frac{p''^2 d^2}{\hbar^2}] = \hbar^2 k^2 + \frac{\hbar^2}{2d^2}$$

((Mathematica 5.2))

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(* Sakurai Problem 1-32*)
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$$f = \frac{k\,d}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Exp}\left[-\frac{p^2 d^2}{\hbar^2}\right] dp$$

$$\frac{1}{\sqrt{\pi}}\left(\text{d k If}\left[\text{Re}\left[\frac{d^2}{\hbar^2}\right]>0,\ \frac{\sqrt{\pi}}{\sqrt{\frac{d^2}{\hbar^2}}},\right.\right.$$

$$\left.\left.\text{Integrate}\left[e^{-\frac{d^2 p^2}{\hbar^2}},\ \{p,\ -\infty,\ \infty\},\ \text{Assumptions}\ \to \text{Re}\left[\frac{d^2}{\hbar^2}\right]\le 0\right]\right]\right)$$

$$\mathbf{g} = \frac{\mathbf{d}}{\hbar\ \sqrt{\pi}}\int_{-\infty}^{\infty}\mathbf{Exp}\left[-\frac{\mathbf{p^2\,d^2}}{\hbar^2}\right]\ (\mathbf{p^2} + \hbar^2\,\mathbf{k^2})\ d\mathbf{p}$$

$$\frac{1}{\sqrt{\pi}\ \hbar}\left(\text{d If}\left[\text{Re}\left[\frac{d^2}{\hbar^2}\right]>0,\ \frac{(1+2\,d^2\,k^2)\sqrt{\pi}}{2\left(\frac{d^2}{\hbar^2}\right)^{3/2}},\right.\right.$$

$$\text{Integrate}\left[e^{-\frac{d^2 p^2}{\hbar^2}}\ (p^2 + k^2\,\hbar^2),\ \{p,\ -\infty,\ \infty\},\ \text{Assumptions}\ \to \text{Re}\left[\frac{d^2}{\hbar^2}\right]\le 0\right]\right]$$

((**1-33**))

33. a. Prove the following:

(i) $$\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle,$$

(ii) $$\langle \beta|x|\alpha\rangle = \int dp'\,\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p'),$$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

b. What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right),$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.

(a)

(i)

$$\langle p'|\hat{x}|\alpha\rangle = \int dx'\langle p'|x'\rangle\langle x'|\hat{x}|\alpha\rangle = \int dx'x'\langle p'|x'\rangle\langle x'|\alpha\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx'x'e^{-\frac{ip'x'}{\hbar}}\langle x'|\alpha\rangle$$

$$= i\hbar\frac{\partial}{\partial p'}\frac{1}{\sqrt{2\pi\hbar}}\left(\int dx'e^{-\frac{ip'x'}{\hbar}}\langle x'|\alpha\rangle\right) = i\hbar\frac{\partial}{\partial p'}\int dx'\langle p'|x'\rangle\langle x'|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

(ii)

$$\langle\beta|\hat{x}|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|\hat{x}|\alpha\rangle = \int dp'\langle\beta|p'\rangle\langle p'|\hat{x}|\alpha\rangle = \int dp'\langle\beta|p'\rangle i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

$$= \int dp'\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$$

(b)

$$\hat{p}\exp(\frac{i\xi\hat{x}}{\hbar})|p'\rangle = \left\{\left[\hat{p},\exp(\frac{i\xi\hat{x}}{\hbar})\right] + \exp(\frac{i\xi\hat{x}}{\hbar})\hat{p}\right\}|p'\rangle = \{\xi\exp(\frac{i\xi\hat{x}}{\hbar}) + \exp(\frac{i\xi\hat{x}}{\hbar})p'\}|p'\rangle$$

or

$$\hat{p}\exp(\frac{i\xi\hat{x}}{\hbar})|p'\rangle = (\xi + p')\exp(\frac{i\xi\hat{x}}{\hbar})|p'\rangle$$

Therefore $\exp(\frac{i\xi\hat{x}}{\hbar})|p'\rangle$ is the eigenket of $\hat{p}$ with an eigenvalue of (p' + ξ).;