Sakurai Chapter 3

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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((3-1))

1. Find the eigenvalues and eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If s_y is measured, what is the probability of the result $\hbar/2$?

We know that

$$|+\rangle_y = \frac{1}{\sqrt{2}}[|+\rangle + i|-\rangle] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \text{and} \qquad |-\rangle_y = \frac{1}{\sqrt{2}}[|+\rangle - i|-\rangle] = \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \text{ where } |\alpha|^2 + |\beta|^2 = 1$$

The probability of finding the state $|+\rangle_y$ is

$$P = \left|{}_y\langle + |\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right|^2 = \frac{1}{2}|\alpha - i\beta|^2 = \frac{1}{2}(\alpha - i\beta)(\alpha^* + i\beta^*)$$

((**3-2**))

2. Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma}\cdot\mathbf{a}}{a_0 - i\boldsymbol{\sigma}\cdot\mathbf{a}},$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

a. Prove that U is unitary and unimodular.

b. In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

(a)

$$\hat{U} = (a_0\hat{1} + i\hat{\boldsymbol{\sigma}}\cdot\mathbf{a})(a_0\hat{1} - i\hat{\boldsymbol{\sigma}}\cdot\mathbf{a})^{-1}$$

$$\hat{U}^+ = (a_0\hat{1} - i\hat{\boldsymbol{\sigma}}\cdot\mathbf{a})(a_0\hat{1} + i\hat{\boldsymbol{\sigma}}\cdot\mathbf{a})^{-1}$$

Then we have

$$\hat{U}^{+}\hat{U} = \hat{U}\hat{U}^{+} = \hat{1}$$

So $\hat{U}$ is the unitary operator.

$$\det\hat{U} = 1$$

from the Mathematica 5.2 (below).

Thus $\hat{U}$ is unimodular.

(b)

$$U = \begin{pmatrix} \frac{a_0{}^2 - \mathbf{a}^2 + 2ia_0a_z}{a_0{}^2 + \mathbf{a}^2} & \frac{2a_0(ia_x + a_y)}{a_0{}^2 + \mathbf{a}^2} \\ \frac{2a_0(ia_x - a_y)}{a_0{}^2 + \mathbf{a}^2} & \frac{a_0{}^2 - \mathbf{a}^2 - 2ia_0a_z}{a_0{}^2 + \mathbf{a}^2} \end{pmatrix}$$

$$\exp(-\frac{i\overline{\boldsymbol{\sigma}}\cdot\mathbf{n}\phi}{2}) = \begin{pmatrix} \cos\frac{\phi}{2} - in_z\sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} - in_z\sin\frac{\phi}{2} \end{pmatrix}$$

$$\cos\frac{\phi}{2} = \frac{a_0{}^2 - \mathbf{a}^2}{a_0{}^2 + \mathbf{a}^2} \quad ,$$

$$\sin^2\frac{\phi}{2} = 1 - \cos^2\frac{\phi}{2} = 1 - \left(\frac{a_0{}^2 - \mathbf{a}^2}{a_0{}^2 + \mathbf{a}^2}\right)^2 = \frac{\left(a_0{}^2 + \mathbf{a}^2\right)^2 - \left(a_0{}^2 - \mathbf{a}^2\right)^2}{\left(a_0{}^2 + \mathbf{a}^2\right)^2} = \frac{4a_0{}^2\mathbf{a}^2}{\left(a_0{}^2 + \mathbf{a}^2\right)^2}$$

or

$$\sin\frac{\phi}{2} = \frac{2a_0|\mathbf{a}|}{a_0{}^2 + \mathbf{a}^2}$$

$$-n_z\sin\frac{\phi}{2} = \frac{2a_0a_z}{a_0{}^2 + \mathbf{a}^2},$$

$$-n_y\sin\frac{\phi}{2} = \frac{2a_0a_y}{a_0{}^2 + \mathbf{a}^2}$$

$$-n_x\sin\frac{\phi}{2} = \frac{2a_0a_x}{a_0{}^2 + \mathbf{a}^2}$$

Then we have

$$n_z = -\frac{a_z}{|\mathbf{a}|},$$

$$n_y = -\frac{a_y}{|\mathbf{a}|}$$

$$n_x = -\frac{a_z}{|\mathbf{a}|}$$

((Mathematica 5.2))

```
(*Sakurai 3-2*)
conjugateRule =
   {Complex[re_, im_] :→ Complex[re, -im]};
Unprotect[SuperStar];
SuperStar /: exp_ * := exp /. conjugateRule;
Protect[SuperStar]

{SuperStar}
σx={{0,1},{1,0}}
{{0,1},{1,0}}
σy={{0,-ⅈ},{ⅈ,0}}
{{0,-ⅈ},{ⅈ,0}}
σz={{1,0},{0,-1}}
{{1,0},{0,-1}}
σI={{1,0},{0,1}}
{{1,0},{0,1}}
P1=a0 σI+ⅈ (σx ax+σy ay+σz az)//Simplify
{{a0+ⅈ az,ⅈ ax+ay},{ⅈ ax-ay,a0-ⅈ az}}
P2=a0 σI-ⅈ (σx ax+σy ay+σz az)//Simplify
{{a0-ⅈ az,-ⅈ ax-ay},{-ⅈ ax+ay,a0+ⅈ az}}
P3=Inverse[P2]//Simplify
```

$$\left\{\left\{\frac{\text{a0}+ⅈ\,\text{az}}{\text{a0}^2+\text{ax}^2+\text{ay}^2+\text{az}^2},\ \frac{ⅈ\,\text{ax}+\text{ay}}{\text{a0}^2+\text{ax}^2+\text{ay}^2+\text{az}^2}\right\},\left\{\frac{ⅈ\,\text{ax}-\text{ay}}{\text{a0}^2+\text{ax}^2+\text{ay}^2+\text{az}^2},\ \frac{\text{a0}-ⅈ\,\text{az}}{\text{a0}^2+\text{ax}^2+\text{ay}^2+\text{az}^2}\right\}\right\}$$

```
U=P1.P3//Simplify
```

$$\left\{\left\{-\frac{-a0^2+ax^2+ay^2-2\,i\,a0\,az+az^2}{a0^2+ax^2+ay^2+az^2},\ \frac{2\,a0\,(i\,ax+ay)}{a0^2+ax^2+ay^2+az^2}\right\},\ \left\{\frac{2\,i\,a0\,(ax+i\,ay)}{a0^2+ax^2+ay^2+az^2},\ -\frac{-a0^2+ax^2+ay^2+2\,i\,a0\,az+az^2}{a0^2+ax^2+ay^2+az^2}\right\}\right\}$$

UH = Transpose[U*] // Simplify

$$\left\{\left\{-\frac{-a0^2+ax^2+ay^2+2\,i\,a0\,az+az^2}{a0^2+ax^2+ay^2+az^2},\ \frac{a0\,(-2\,i\,ax-2\,ay)}{a0^2+ax^2+ay^2+az^2}\right\},\ \left\{\frac{2\,a0\,(-i\,ax+ay)}{a0^2+ax^2+ay^2+az^2},\ -\frac{-a0^2+ax^2+ay^2-2\,i\,a0\,az+az^2}{a0^2+ax^2+ay^2+az^2}\right\}\right\}$$

U.UH//Simplify

{{1,0},{0,1}}

Det[U]//Simplify

1

$$\mathbf{T1} = \mathbf{MatrixExp}\left[-\frac{i}{2}\,\phi\,(nx\,\sigma x + ny\,\sigma y + nz\,\sigma z)\right];$$

T2 =

$$\mathbf{T1}\ /.\ \{nx^2+ny^2+nz^2 \to 1\}\ /.\ \left\{\sqrt{-nx^2-ny^2-nz^2} \to i\right\}\ /.\ \left\{\frac{1}{\sqrt{-nx^2-ny^2-nz^2}} \to -i\right\}\ //\ \mathbf{ExpToTrig}$$

$$\left\{\left\{\mathrm{Cos}\left[\frac{\phi}{2}\right]-i\,nz\,\mathrm{Sin}\left[\frac{\phi}{2}\right],\ -i\,nx\,\mathrm{Sin}\left[\frac{\phi}{2}\right]-ny\,\mathrm{Sin}\left[\frac{\phi}{2}\right]\right\},\ \left\{-i\,nx\,\mathrm{Sin}\left[\frac{\phi}{2}\right]+ny\,\mathrm{Sin}\left[\frac{\phi}{2}\right],\ \mathrm{Cos}\left[\frac{\phi}{2}\right]+i\,nz\,\mathrm{Sin}\left[\frac{\phi}{2}\right]\right\}\right\}$$

((3-3))

3. The spin-dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as

$$H = A\mathbf{S}^{(e^-)}\cdot\mathbf{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right).$$

Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.

a. Is this an eigenfunction of H in the limit $A \to 0$, $eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?

b. Same problem when $eB/mc \to 0$, $A \neq 0$.

We introduce the Dirac spin exchange operator:

$$\hat{P}_{12} = \frac{1}{2}(\hat{1} + \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2)$$

or

$$\hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2 = 2\hat{P}_{12} - \hat{1}$$

The spin Hamiltonian $\hat{H}$ is given by

$$\hat{H} = A\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \frac{eB}{mc}(\hat{S}_{1z} - \hat{S}_{2z}) = \frac{A}{4}\hbar^2 \hat{\boldsymbol{\sigma}}_1 \cdot \hat{\boldsymbol{\sigma}}_2 + \frac{e\hbar B}{2mc}(\hat{\sigma}_{1z} - \hat{S}_{2z})$$

or

$$\hat{H} = \frac{A}{4}\hbar^2(2\hat{P}_{12} - \hat{1}) + \frac{e\hbar B}{2mc}(\hat{\sigma}_{1z} - \hat{\sigma}_{2z})$$

$$\hat{H}|+-\rangle = \frac{A}{4}\hbar^2(2\hat{P}_{12} - \hat{1})|+-\rangle + \frac{e\hbar B}{2mc}(\hat{\sigma}_{1z} - \hat{\sigma}_{2z})|+-\rangle$$

or

$$\hat{H}|+-\rangle = \frac{A}{4}\hbar^2(2\hat{P}_{12} - \hat{1})|+-\rangle + \frac{e\hbar B}{2mc}(\hat{\sigma}_{1z} - \hat{\sigma}_{2z})|+-\rangle$$

$$\hat{H}|+-\rangle = \frac{A}{4}\hbar^2[2|-+\rangle - |+-\rangle + \frac{e\hbar B}{mc}|+-\rangle$$

(a) $A \to 0$

$$\hat{H}|+-\rangle = \hat{H}|+-\rangle$$

Thus the state $|+-\rangle$ is the eigenket of $\hat{H}$ with the eigenvalue $E = \dfrac{e\hbar B}{mc}$

(b) B = 0,

$$\hat{H}|+-\rangle = \frac{A}{4}\hbar^2[2|-+\rangle - |+-\rangle$$

Thus the state $|+-\rangle$ is not the eigenket of $\hat{H}$.

$$\langle +-|\hat{H}|+-\rangle = -\frac{A}{4}\hbar^2$$

((3-4))

4. Consider a *spin* 1 particle. Evaluate the matrix elements of

$$S_z(S_z+\hbar)(S_z-\hbar) \quad \text{and} \quad S_x(S_x+\hbar)(S_x-\hbar).$$

$$f(x) = x(x-\hbar)(x+\hbar)$$

$|m\rangle$ and $|m\rangle_x$ are the eigenkets of $\hat{S}_z$ and $\hat{S}_x$ with the eigenvalues. $\hbar m$ (m = 1, 0, and -1). Note that

$$|m\rangle_x = \hat{U}|m\rangle$$

where $\hat{U}$ is the unitary operator.

$$f(\hat{S}_z)|m\rangle = f(\hbar m)|m\rangle = 0$$

Thus $f(\hat{S}_z)$ is the zero operator.

$$f(\hat{S}_x)|m\rangle = \sum_{m'} f(\hat{S}_x)|m'\rangle_{x\,x}\langle m'|m\rangle = \sum_{m'} f(m'\hbar)|m'\rangle_{x\,x}\langle m'|m\rangle = \sum_{m'} f(m'\hbar)\hat{U}|m'\rangle\langle m'|\hat{U}^+|m\rangle = 0$$

since m' = 1, 0, and -1. Thus the operator is the zero operatoir.

((**3-6**))

6. Let $U = e^{iG_3\alpha} e^{iG_2\beta} e^{iG_3\gamma}$, where (α, β, γ) are the Eulerian angles. In order that U represent a rotation (α, β, γ), what are the commutation rules satisfied by the G_k? Relate $\mathbf{G}$ to the angular momentum operators.

The rotation operator for the Euiler angles (α, β, γ) is given by

$$\hat{R} = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) = \exp(-\frac{i\hat{J}_z\alpha}{\hbar})\exp(-\frac{i\hat{J}y_y\beta}{\hbar})\exp(-\frac{i\hat{J}_z\gamma}{\hbar})$$

When this operator coincides with the unitary operator

$$\hat{U} = \exp(i\hat{G}_3\alpha)\exp(i\hat{G}_2\beta)\exp(i\hat{G}_3\gamma)$$

we find that

$$\hat{G}_i = -\frac{\hat{J}_i}{\hbar}$$

Since $[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k$

$$\hbar^2[\hat{G}_i, \hat{G}_j] = -i\hbar^2\varepsilon_{ijk}\hat{G}_k$$

or

$$[\hat{G}_i, \hat{G}_j] = -i\varepsilon_{ijk}\hat{G}_k$$

((**3-7**))

7. What is the meaning of the following equation:

$$U^{-1}A_kU = \sum R_{kl}A_l,$$

where the three components of **A** are matrices? From this equation show that matrix elements $\langle m|A_k|n\rangle$ transform like vectors.

We assume that the state vector changes from the old states $|n\rangle$ and $|m\rangle$ to the new states $|n'\rangle$ and $|m'\rangle$

$$|n'\rangle = \hat{R}|n\rangle, \quad \text{and} \quad |m'\rangle = \hat{R}|m\rangle$$

or

$$\langle n'| = \hat{R}^{+}\langle n|, \quad \text{and} \qquad \langle m'| = \hat{R}^{+}\langle m|$$

A vector operator $\hat{\mathbf{A}}$ for the system is defined as an operator whose expectation is a vector that rotates together with the physical system.

$$\langle n'|\hat{A}_i|m'\rangle = \langle n|\hat{R}^{+}\hat{A}_i\hat{R}|m\rangle = \sum_j \Re_{ij}\langle n|\hat{A}_j|m\rangle$$

thus the matrix elements transform like vectors.

since

$$\hat{R}^{+}\hat{A}_i\hat{R} = \sum_j \Re_{ij}\hat{A}_j$$

((3-8))

8. Consider a sequence of Euler rotations represented by

$$\mathscr{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

From the Mathematica 5.2

$$\exp(-\frac{i}{2}\hat{\boldsymbol{\sigma}}\cdot\mathbf{n}\theta) = \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & -in_x\sin\frac{\theta}{2} - n_y\sin\frac{\theta}{2} \\ -in_x\sin\frac{\theta}{2} + n_y\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix}$$

where

$$\hat{\boldsymbol{\sigma}} \cdot \mathbf{n} = \hat{\sigma}_x n_x + \hat{\sigma}_y n_y + \hat{\sigma}_z n_z$$

$$\exp(-\frac{i}{2}\sigma_z\alpha)\exp(-\frac{i}{2}\sigma_y\beta)\exp(-\frac{i}{2}\sigma_z\gamma) = \begin{pmatrix} e^{-\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} & -e^{-\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} \\ e^{\frac{i(\alpha-\gamma)}{2}}\sin\frac{\beta}{2} & e^{\frac{i(\alpha+\gamma)}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Then we have

$$\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} = \cos\frac{\theta}{2}$$

$$\cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} = n_z\sin\frac{\theta}{2}$$

$$\sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} = n_y\sin\frac{\theta}{2}$$

$$-\sin\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2} = n_x\sin\frac{\theta}{2}$$

```
(*Matrix representation of Rotaion operator*)
 σx={{0,1},{1,0}}
 {{0,1},{1,0}}
 σy={{0,-ⅈ},{ⅈ,0}}
 {{0,-ⅈ},{ⅈ,0}}
 σz={{1,0},{0,-1}}
 {{1,0},{0,-1}}
 Jn= (σx nx +σy ny +σz nz)
 {{nz,nx-ⅈ ny},{nx+ⅈ ny,-nz}}
 I1=RJn=MatrixExp[-ⅈ Jn θ/2]
```

$$\left\{\left\{-\frac{e^{-\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nz-\sqrt{-nx^2-ny^2-nz^2}\right)}{2\sqrt{-nx^2-ny^2-nz^2}}+\frac{e^{\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nz+\sqrt{-nx^2-ny^2-nz^2}\right)}{2\sqrt{-nx^2-ny^2-nz^2}},\right.\right.$$

$$\left.-\frac{e^{-\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nx-ny\right)}{2\sqrt{-nx^2-ny^2-nz^2}}+\frac{e^{\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nx-ny\right)}{2\sqrt{-nx^2-ny^2-nz^2}}\right\},$$

$$\left\{-\frac{e^{-\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nx+ny\right)}{2\sqrt{-nx^2-ny^2-nz^2}}+\frac{e^{\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(-i\,nx+ny\right)}{2\sqrt{-nx^2-ny^2-nz^2}},\right.$$

$$\left.\left.-\frac{e^{-\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(i\,nz-\sqrt{-nx^2-ny^2-nz^2}\right)}{2\sqrt{-nx^2-ny^2-nz^2}}+\frac{e^{\frac{1}{2}\sqrt{-nx^2-ny^2-nz^2}\,\theta}\left(i\,nz+\sqrt{-nx^2-ny^2-nz^2}\right)}{2\sqrt{-nx^2-ny^2-nz^2}}\right\}\right\}$$

Rule1 $=\left\{\sqrt{-nx^2-ny^2-nz^2}\to i,\ \frac{1}{\sqrt{-nx^2-ny^2-nz^2}}\to -i\right\}$

$$\left\{\sqrt{-nx^2-ny^2-nz^2}\to i,\ \frac{1}{\sqrt{-nx^2-ny^2-nz^2}}\to -i\right\}$$

I2=I1/.Rule1//ExpToTrig

$$\left\{\left\{\mathrm{Cos}\left[\frac{\theta}{2}\right]-i\,nz\,\mathrm{Sin}\left[\frac{\theta}{2}\right],\ -i\,nx\,\mathrm{Sin}\left[\frac{\theta}{2}\right]-ny\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\right\},\right.$$

$$\left.\left\{-i\,nx\,\mathrm{Sin}\left[\frac{\theta}{2}\right]+ny\,\mathrm{Sin}\left[\frac{\theta}{2}\right],\ \mathrm{Cos}\left[\frac{\theta}{2}\right]+i\,nz\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\right\}\right\}$$

MatrixExp[-i σz α/2]. MatrixExp[-i σy β/2].MatrixExp[-i σz γ/2]//Simplify

$$\left\{\left\{e^{-\frac{1}{2} i(\alpha+\gamma)} \mathrm{Cos}\left[\frac{\beta}{2}\right], -e^{-\frac{1}{2} i(\alpha-\gamma)} \mathrm{Sin}\left[\frac{\beta}{2}\right]\right\}, \left\{e^{\frac{1}{2} i(\alpha-\gamma)} \mathrm{Sin}\left[\frac{\beta}{2}\right], e^{\frac{1}{2} i(\alpha+\gamma)} \mathrm{Cos}\left[\frac{\beta}{2}\right]\right\}\right\}$$

((3-12))

12. An angular-momentum eigenstate $|j, m = m_{\max} = j\rangle$ is rotated by an infinitesimal angle ε about the y-axis. Without using the explicit form of the $d^{(j)}_{m'm}$ function, obtain an expression for the probability for the new rotated state to be found in the original state up to terms of order ε^2.

$$\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$$

$$\hat{J}_y{}^2 = -\frac{1}{4}(\hat{J}_+{}^2 + \hat{J}_-{}^2 - \hat{J}_+\hat{J}_- - \hat{J}_-\hat{J}_+)$$

$$\langle j, m = j|\hat{J}_y|j, m = j\rangle = 0$$

$$\langle j, m = j|\hat{J}_y{}^2|j, m = j\rangle = \frac{1}{4}\langle j, m = j|\hat{J}_+\hat{J}_-|j, m = j\rangle = \frac{1}{4}(j + j)(j - j + 1)\hbar^2 = \frac{1}{2}j\hbar^2$$

The new state after the rotation is

$$|\psi'\rangle = \exp(-\frac{i\hat{J}_y}{\hbar}\varepsilon)|j, m = j\rangle = [1 + (-\frac{i\hat{J}_y}{\hbar}\varepsilon) + \frac{1}{2!}(-\frac{i\hat{J}_y}{\hbar}\varepsilon)^2 + ...]|j, m = j\rangle$$

The function

$$d^{(j)}_{m=j,m'=j}(\varepsilon) = \langle j, m = j|\psi'\rangle = \langle j, m = j|\exp(-\frac{i\hat{J}_y}{\hbar}\varepsilon)|j, m = j\rangle$$

$$= \langle j, m = j|\hat{1} + (-\frac{i\hat{J}_y}{\hbar}\varepsilon) + \frac{1}{2!}(-\frac{i\hat{J}_y}{\hbar}\varepsilon)^2|j, m = j\rangle$$

$$= 1 + \frac{1}{2}(\frac{i}{\hbar})^2\varepsilon^2\frac{1}{2}j\hbar^2 = 1 - \frac{1}{4}j\varepsilon^2$$

Then the probability $P(\varepsilon)$ of finding the original state $\left| j,m=j \right\rangle$ in the system is

$$P(\varepsilon) = \left| d^{(j)}_{m=j,m'=j}(\varepsilon) \right|^2 = (1-\frac{1}{4}j\varepsilon^2)^2 = 1-\frac{1}{2}j\varepsilon^2$$

((3-13))

13. Show that the 3×3 matrices G_i $(i=1,2,3)$ whose elements are given by

$$(G_i)_{jk} = -i\hbar\varepsilon_{ijk},$$

where j and k are the row and column indices, satisfy the angular momentum commutation relations. What is the physical (or geometric) significance of the transformation matrix that connects G_i to the more usual 3×3 representations of the angular-momentum operator J_i with J_3 taken to be diagonal? Relate your result to

$$\mathbf{V} \to \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V}$$

under infinitesimal rotations. (*Note*: This problem may be helpful in understanding the photon spin.)

The matrices

$$\hat{G}_1 = -i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\hat{G}_2 = -i\hbar\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\hat{G}_3 = -i\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Mathematica 5.2, we can show that

$$[\hat{G}_i,\hat{G}_j] = i\hbar\varepsilon_{ijk}\hat{G}_k$$

((Mathematica 5.2))

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(*Sakurai 3-13*)
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```
G1=-ⅈ ℏ {{0,0,0},{0,0,1},{0,-1,0}}
{{0,0,0},{0,0,-ⅈ ℏ},{0,ⅈ ℏ,0}}
G2=-ⅈ ℏ {{0,0,-1},{0,0,0},{1,0,0}}
{{0,0,ⅈ ℏ},{0,0,0},{-ⅈ ℏ,0,0}}
G3=-ⅈ ℏ{{0,1,0},{-1,0,0},{0,0,0}}
{{0,-ⅈ ℏ,0},{ⅈ ℏ,0,0},{0,0,0}}
G1//MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}$$

```
G2//MatrixForm
```

$$\begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix}$$

```
G3//MatrixForm
```

$$\begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
G1.G2-G2.G1-ⅈ ℏ G3//Simplify
{{0,0,0},{0,0,0},{0,0,0}}
G2.G3-G3.G2-ⅈ ℏ G1//Simplify
{{0,0,0},{0,0,0},{0,0,0}}
G3.G1-G1.G3-ⅈ ℏ G2//Simplify
{{0,0,0},{0,0,0},{0,0,0}}
Eigensystem[G3]
{{0,-ℏ,ℏ},{{0,0,1},{ⅈ,1,0},{-ⅈ,1,0}}}
```

We consider the unitary operator $\hat{U}$

$|1\rangle = \hat{U}|\varphi_1\rangle$, $\qquad |1,0\rangle = \hat{U}|\varphi_2\rangle$, $\qquad |-1\rangle = \hat{U}|\varphi_3\rangle$

where

$$\hat{G}_3|1\rangle = \hbar|1\rangle$$

$$\hat{G}_3|0\rangle = 0$$

$$\hat{G}_3|-1\rangle = -\hbar|-1\rangle$$

or

$|1\rangle, |0\rangle, |-1\rangle$ are the eigenkets of $\hat{G}_3$.

Then we have

$$\hat{U}^{+}\hat{G}_3\hat{U}|\varphi_1\rangle = \hbar|\varphi_1\rangle$$

$$\hat{U}^{+}\hat{G}_3\hat{U}|\varphi_0\rangle = 0$$

$$\hat{U}^{+}\hat{G}_3\hat{U}|\varphi_{-1}\rangle = -\hbar|\varphi_{-1}\rangle$$

where

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{U}^{+}\hat{G}_3\hat{U} = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under the basis of $|\varphi_i\rangle$

$$\hat{G}_3|1\rangle = \hbar|1\rangle$$
$$\hat{G}_3|0\rangle = 0$$
$$\hat{G}_3|-1\rangle = -\hbar|-1\rangle$$

$$\hat{G}_3 = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under the basis of $|1\rangle, |0\rangle, |-1\rangle$

We consider the rotation operator $\hat{R}$

$$|\psi'\rangle = \hat{R}|\psi\rangle$$

$$\hat{R}_3(\varepsilon) = \exp(-\frac{i}{\hbar}\varepsilon \hat{G}_3)$$

$$\hat{U}^+\hat{R}_3(\varepsilon)\hat{U} = \exp[-\frac{i}{\hbar}\varepsilon \hat{U}^+\hat{G}_3(\varepsilon)\hat{U}] = \begin{pmatrix} e^{-i\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\varepsilon} \end{pmatrix} = \begin{pmatrix} 1-i\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+i\varepsilon \end{pmatrix}$$

$$\hat{R}_3(\varepsilon) = \hat{U}\begin{pmatrix} 1-i\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+i\varepsilon \end{pmatrix}\hat{U}^+ = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$|\varphi_1'\rangle = \hat{R}_3(\varepsilon)|\varphi_1\rangle = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix} = |\varphi_1\rangle + \varepsilon|\varphi_2\rangle$$

$$|\varphi_2'\rangle = \hat{R}_3(\varepsilon)|\varphi_2\rangle = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\varepsilon \\ 1 \\ 0 \end{pmatrix} = -\varepsilon|\varphi_1\rangle + |\varphi_2\rangle$$

$$|\varphi_3'\rangle = \hat{R}_3(\varepsilon)|\varphi_3\rangle = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |\varphi_3\rangle$$

These relations are similar to the rotation of vectors around the z axis by angle ε.

$$\mathbf{V} \to \mathbf{V}' = \mathbf{V} + \varepsilon\mathbf{n} \times \mathbf{V}$$

$$\begin{pmatrix} V_x' \\ V_y'' \\ V_z \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\mathbf{n} \times \mathbf{V} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ V_x & V_y & V_z \end{vmatrix} = (-V_y, V_x, 0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

---

14. a. Let $\mathbf{J}$ be angular momentum. It may stand for orbital $\mathbf{L}$, spin $\mathbf{S}$, or $\mathbf{J}_{\text{total}}$.) Using the fact that J_x, J_y, $J_z (J_{\pm} \equiv J_x \pm iJ_y)$ satisfy the usual angular-momentum commutation relations, prove

$$\mathbf{J}^2 = J_z^2 + J_+ J_- - \hbar J_z .$$

b. Using (a) (or otherwise), derive the "famous" expression for the coefficient c_- that appears in

$$J_- \psi_{jm} = c_- \psi_{j,m-1} .$$

(a)

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y$$

$$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^{\,2} + \hat{J}_y^{\,2} - i[\hat{J}_x, \hat{J}_y]$$

Since $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$

$$\hat{J}_+\hat{J}_- = \hat{J}_x^{\,2} + \hat{J}_y^{\,2} + \hbar\hat{J}_z = \hat{\mathbf{J}}^2 - \hat{J}_z^{\,2} + \hbar\hat{J}_z$$

or

$$\hat{\mathbf{J}}^2 = \hat{J}_z^{\,2} + \hat{J}_+\hat{J}_- - \hbar\hat{J}_z$$

(b)

$$\hat{\mathbf{J}}^2 \left| j,m \right\rangle = \hat{J}_z^{\,2} \left| j,m \right\rangle + \hat{J}_+\hat{J}_- \left| j,m \right\rangle - \hbar\hat{J}_z \left| j,m \right\rangle$$

$$\hbar^2 j(j+1) \left| j,m \right\rangle = m^2\hbar^2 \left| j,m \right\rangle + \hat{J}_+\hat{J}_- \left| j,m \right\rangle - m\hbar^2 \left| j,m \right\rangle$$

or

$$\left\langle j,m \right| \hat{J}_+\hat{J}_- \left| j,m \right\rangle = \hbar^2 [\, j(j+1) - m^2 + m\,]$$

We assume that

$$\hat{J}_- \left| j,m \right\rangle = c_- \left| j,m-1 \right\rangle$$

or

$$\left\langle j,m \right| \hat{J}_+ = c_-^{\,*} \left\langle j.m-1 \right|$$

From these we have

$$|c_-| = \hbar\sqrt{j(j+1) - m^2 + m}$$

When we choose a real $c\_$ then we have

$$c_- = \hbar\sqrt{j(j+1) - m^2 + m} = \sqrt{(j+m)(j-m+1)}$$

((3-15))

15. The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z)f(r).$$

a. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?
b. What are the probabilities for the particle to be found in various m_l states?
c. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

(a)

$$\psi(\mathbf{r}) = (x + y + 3z) f(r)$$

$$\frac{x}{r} = \sqrt{\frac{2\pi}{3}}[Y_1^{-1}(\theta,\phi) - Y_1^{1}(\theta,\phi)]$$

$$\frac{y}{r} = i\sqrt{\frac{2\pi}{3}}[Y_1^{1}(\theta,\phi) + Y_1^{-1}(\theta,\phi)]$$

$$\frac{z}{r} = \sqrt{\frac{4\pi}{3}}Y_1^{0}(\theta,\phi)$$

$$\frac{x + y + 3z}{r} = \sqrt{\frac{2\pi}{3}}[(1+i)Y_1^{-1}(\theta,\phi) - (1-i)Y_1^{1}(\theta,\phi) + 3\sqrt{2}Y_1^{0}(\theta,\phi)]$$

$$\psi(x,y,z) = \sqrt{\frac{2\pi}{3}}[(1+i)Y_1^{-1}(\theta,\phi) - (1-i)Y_1^{1}(\theta,\phi) + 3\sqrt{2}Y_1^{0}(\theta,\phi)]rf(r)$$

$$|\psi\rangle = \frac{1}{\sqrt{11}}[-\frac{(1-i)}{\sqrt{2}}|1,1\rangle + 3|1,0\rangle + \frac{(1+i)}{\sqrt{2}}|1,-1\rangle]$$

(b)

$$P(l=1, m=1) = \frac{1}{11},$$
$$P(l=1, m=0) = \frac{9}{11},$$
$$P(l=1, m=-1) = \frac{1}{22},$$

(c)

Schrödinger equation

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r) = \frac{1}{2m}(p_r{}^2 + \frac{\mathbf{L}^2}{r^2})$$

$$p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r$$

$$[-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial}{\partial r} r) + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)]rf(r) = Erf(r)$$

with $l = 1$.

Then we have

$$V(r) = E + \frac{\hbar^2}{2m}\left(\frac{rf''(r) + 4f'(r)}{rf(r)}\right)$$

$$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$$
$$\hat{L}_x = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-)$$

$$\hat{L}_x{}^2 = \frac{1}{4}(\hat{L}_+ + \hat{L}_-)(\hat{L}_+ + \hat{L}_-) = \frac{1}{4}(\hat{L}_+{}^2 + \hat{L}_-{}^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+)$$

$$\hat{L}_x{}^2 = \frac{1}{4}(\hat{L}_+ + \hat{L}_-)(\hat{L}_+ + \hat{L}_-) = \frac{1}{4}(\hat{L}_+{}^2 + \hat{L}_-{}^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+)$$

$$\hat{L}_+|lm\rangle = \hbar\sqrt{(l-m)(l+m+1)}\,|l, m-1\rangle$$

$$\langle lm|\hat{L}_-\hat{L}_+|lm\rangle = \hbar^2(l-m)(l+m+1)$$

$$\hat{L}_-|lm\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m-1\rangle$$

$$\langle lm|\hat{L}_+\hat{L}_-|lm\rangle = \hbar^2(l+m)(l-m+1)$$

$$\langle lm|\hat{L}_z|lm\rangle = m\hbar$$

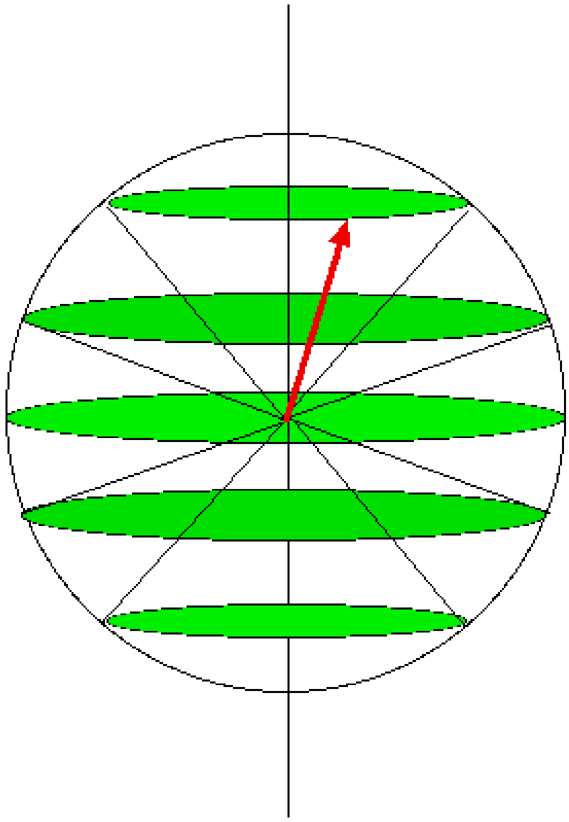
$$\langle lm|\hat{L}_x|lm\rangle = \langle lm|\hat{L}_y|lm\rangle = 0$$

$$\langle lm|\hat{L}_x{}^2|lm\rangle = \langle lm|\hat{L}_y{}^2|lm\rangle = \frac{1}{4}\langle lm|\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+|lm\rangle = \frac{1}{4}\langle lm|\hat{L}_+\hat{L}_-|lm\rangle + \frac{1}{4}\langle lm|\hat{L}_-\hat{L}_+|lm\rangle$$

or

$$\langle lm|\hat{L}_x{}^2|lm\rangle = \langle lm|\hat{L}_y{}^2|lm\rangle = \frac{\hbar^2}{2}[l(l+1)-m^2]$$

Consequently, the angular momentum of a particle in the state $|l,m\rangle$ behaves, in so far as the mean values of its components and their squares are concerned, like a classical angular momentum of magnitude $\hbar\sqrt{l(l+1)}$ having a projection $m\hbar$ along the z axis, but the angle ϕ is a random variable evenly distributed between 0 and 2π.



((3-17))

17. Suppose a half-integer l-value, say $\frac{1}{2}$, were allowed for orbital angular momentum. From

$$L_+ Y_{1/2,1/2}(\theta,\phi) = 0,$$

we may deduce, as usual,

$$Y_{1/2,1/2}(\theta,\phi) \propto e^{i\phi/2}\sqrt{\sin\theta}\,.$$

Now try to construct $Y_{1/2,-1/2}(\theta,\phi)$; by (a) applying L_- to $Y_{1/2,1/2}(\theta,\phi)$; and (b) using $L_- Y_{1/2,-1/2}(\theta,\phi) = 0$. Show that the two procedures lead to contradictory results. (This gives an argument against half-integer l-values for orbital angular momentum.)

Let us suppose $Y_l^m(\theta,\phi)$ with a half-integer l were possible. We choose the simplest case (l = 1/2).

From the definition,

$$L_+ Y_{1/2}^{1/2} = \hbar e^{i\phi}(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})e^{i\phi/2}\Theta_{1/2,1/2}(\theta) = 0$$

From this equation, we have

$$\frac{d\Theta_{1/2,1/2}(\theta)}{d\theta} = \frac{1}{2}\cot\theta\Theta_{1/2,1/2}(\theta)$$

or

$$\Theta_{1/2,1/2}(\theta) = C_1\sqrt{\sin\theta}$$

or

$$Y_{1/2}^{1/2}(\theta,\phi) = C_2 e^{i\phi/2}\sqrt{\sin\theta}$$

This expression is not permissible because it is singular at $\theta = 0$ and π.

(a)
From the property of L_-

$$L_- Y_{1/2}^{1/2} = \hbar e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})C_2 e^{i\phi/2}\sqrt{\sin\theta}$$

or

$$L_- Y_{1/2}^{1/2} \propto Y_{1/2}^{-1/2} = C_3\frac{e^{-i\phi/2}}{\sqrt{\sin\theta}}\cos\theta \qquad (1)$$

(b)

$$L_- Y_{1/2}^{-1/2} = \hbar e^{-i\phi}(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi})e^{-i\phi/2}\Theta_{1/2,-1/2}(\theta) = 0$$

From this we have

$$\frac{d\Theta_{1/2,-1/2}(\theta)}{d\theta} = \frac{1}{2}\cot\theta\Theta_{1/2,-1/2}(\theta)$$

or

$$Y_{1/2}^{-1/2}(\theta,\phi) = C_4 e^{-i\phi/2}\sqrt{\sin\theta}\,, \qquad (2)$$

Thus the two procedures lead to contradictory results.

```
(*Sakurai 3-17
      We define the ladder operators
  *)
OGD:=Exp[-ⅈ ϕ](-D[#,θ]+ⅈ Cot[θ] D[#,ϕ])&
OGU:=Exp[ⅈ ϕ](D[#,θ]+ⅈ Cot[θ] D[#,ϕ])&
```

eq1 = OGU[Exp[ⅈ ϕ/2] X[θ]]

$$e^{i\phi}\left(-\frac{1}{2}e^{\frac{i\phi}{2}}\text{Cot}[\theta]X[\theta]+e^{\frac{i\phi}{2}}X'[\theta]\right)$$

```
DSolve[eq1==0,X[θ],θ]
```

$$\{\{X[\theta]\to C[1]\sqrt{\text{Sin}[\theta]}\}\}$$

OGD[Exp[ⅈ ϕ/2] √Sin[θ]]

$$-\frac{e^{-\frac{i\phi}{2}}\text{Cos}[\theta]}{\sqrt{\text{Sin}[\theta]}}$$

eq2 = OGD[Exp[-ⅈ ϕ/2] Y[θ]] // Simplify

$$\frac{1}{2}e^{-\frac{3i\phi}{2}}(\text{Cot}[\theta]Y[\theta]-2Y'[\theta])$$

```
DSolve[eq2==0,Y[θ],θ]
```

$$\{\{Y[\theta]\to C[1]\sqrt{\text{Sin}[\theta]}\}\}$$

((**3-18**))

18. Consider an orbital angular-momentum eigenstate $|l=2, m=0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m=0, \pm 1$, and ± 2. (The spherical harmonics for $l=0$, 1, and 2 given in Appendix A may be useful.)

$$\hat{R}=Y^0_{\ell=2}(\theta=0,\phi)\hat{R}_y(\theta)$$

$$|\psi'\rangle=\hat{R}_z(\phi)\hat{R}_y(\theta)|l=2,m=0\rangle$$

$$\langle l=2,m|\psi'\rangle=\langle l=2,m|\hat{R}_z(\phi)\hat{R}_y(\theta)|l=2,m=0\rangle$$

From the Note-1 we have

$$\left[Y_{l=2}^{m}(\theta,\phi)\right]^{*} = \langle l=2,m|\hat{R}_z(\phi)\hat{R}_y(\theta)|l=2,0\rangle Y_{\ell=2}^{0}(\theta=0,\phi)$$

$$Y_{\ell=2}^{0}(\theta=0,\phi) = \sqrt{\frac{5}{4\pi}}$$ (which is independent of ϕ)

Thus we have

$$\langle l=2,m|\hat{R}_z(\phi)\hat{R}_y(\theta)|l=2,0\rangle = \frac{\left[Y_{l=2}^{m}(\theta,\phi)\right]^{*}}{Y_{\ell=2}^{0}(\theta=0,\phi)} = \sqrt{\frac{4\pi}{5}}\left[Y_{l=2}^{m}(\theta,\phi)\right]^{*}$$

We consider the case of $\phi = 0$.l

$$\langle l=2,m|\hat{R}_y(\theta)|l=2,0\rangle = \sqrt{\frac{4\pi}{5}}\left[Y_{l=2}^{m}(\theta,\phi=0)\right]^{*}$$

The probability of finding the state $|l=2,m\rangle$: $P(m)$

$$P(m=0) = \frac{4\pi}{5}\left|Y_2^0(\theta,0)\right|^2 = \frac{1}{4}(3\cos^2\theta-1)^2$$

$$P(m=\pm1) = \frac{4\pi}{5}\left|Y_2^{\pm1}(\theta,0)\right|^2 = \frac{3}{2}\sin^2\theta\cos^2\theta$$

$$P(m=\pm2) = \frac{4\pi}{5}\left|Y_2^{\pm2}(\theta,0)\right|^2 = \frac{3}{8}\sin^4\theta$$

((Note-1))

$$|\mathbf{n}\rangle = |\Re\mathbf{e}_z\rangle = \hat{R}_z(\phi)\hat{R}_y(\theta)|\mathbf{e}_z\rangle$$

$$= \sum_{m'}\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\langle lm'|\mathbf{e}_z\rangle$$

$$\langle lm|\mathbf{n}\rangle = \sum_{m'}\langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\langle lm'|\mathbf{e}_z\rangle = \sum_{m'}\langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|lm'\rangle\delta_{m',0}Y_\ell^0(\theta=0,\phi)$$

$$\left[Y_\ell^m(\theta,\phi)\right]^{*} = \langle lm|\hat{R}_z(\phi)\hat{R}_y(\theta)|l0\rangle Y_\ell^0(\theta=0,\phi)$$

where

$$\langle \mathbf{n} | lm \rangle = Y_\ell^m(\mathbf{n}) = Y_\ell^m(\theta, \phi)$$

and

$$\langle \mathbf{e}_z | lm \rangle = Y_\ell^m(\mathbf{e}_z) = \delta_{m,0} Y_\ell^0(\theta = 0, \phi) \text{ with } \phi \text{ undetermined}$$

((Note-2)) Mathematica 5.2

```
Table[{m,SphericalHarmonicY[2,m,θ,ϕ]},{m,-2,2,1}]//TableForm
```

−2	$\frac{1}{4} e^{-2 i \phi} \sqrt{\frac{15}{2\pi}} \text{Sin}[\theta]^2$
−1	$\frac{1}{2} e^{-i \phi} \sqrt{\frac{15}{2\pi}} \text{Cos}[\theta] \text{Sin}[\theta]$
0	$\frac{1}{4} \sqrt{\frac{5}{\pi}} (-1 + 3 \text{Cos}[\theta]^2)$
1	$-\frac{1}{2} e^{i \phi} \sqrt{\frac{15}{2\pi}} \text{Cos}[\theta] \text{Sin}[\theta]$
2	$\frac{1}{4} e^{2 i \phi} \sqrt{\frac{15}{2\pi}} \text{Sin}[\theta]^2$

$P[0] = \frac{1}{4} (-1 + 3 \text{Cos}[\theta]^2)^2$

$\frac{1}{4} (-1 + 3 \text{Cos}[\theta]^2)^2$

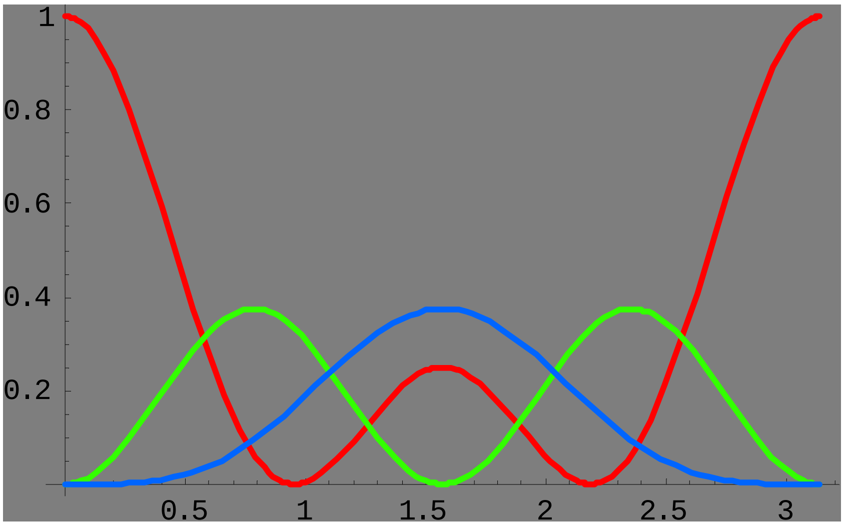
$P[1] = \frac{3}{2} \text{Sin}[\theta]^2 \text{Cos}[\theta]^2$

$\frac{3}{2} \text{Cos}[\theta]^2 \text{Sin}[\theta]^2$

$P[2] = \frac{3}{8} \text{Sin}[\theta]^4$

$\frac{3 \text{Sin}[\theta]^4}{8}$

```
Plot[Evaluate[{P[0],P[1],P[2]}],{θ,0,
π},PlotStyle→Table[Hue[0.3
i],{i,0,3}],Prolog→AbsoluteThickness[2],Background→GrayLeve
l[0.5]]
```



-Graphics-

((**3-20**))

20. We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2$, 1, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j, m\}$ eigenkets in terms of $|j_1 j_2; m_1 m_2\rangle$. Write your answer as

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}|+,0\rangle - \frac{1}{\sqrt{2}}|0,+\rangle, \ldots,$$

where + and 0 stand for $m_{1,2} = 1, 0$, respectively.

We use the relation by Clebsch-Gordan.

$j_1 = 1, j_2 = 1$ ($|m_1| \leq 1, |m_2| \leq 1$)

$D_1 \times D_1 = D_2 + D_1 + D_0$

(i) $j = 2$ ($|m| \leq 2$)

$m = m_1 + m_2$

$|1,1\rangle|1,1\rangle$ ($m = 2$)

$$\frac{|1,1\rangle|1,0\rangle + |1,0\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 1)$$

$$\frac{|1,1\rangle|1,-1\rangle + 2|1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{6}} \quad (m = 0)$$

$$\frac{|1,1\rangle|1,-1\rangle + |1,-1\rangle|1,0\rangle}{\sqrt{2}} \quad (m = -1)$$

$$|1,-1\rangle|1,-1\rangle \quad (m = -2)$$

(ii) $j = 1$ ($|m| \leq 1$)

$$\frac{|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 1)$$

$$\frac{|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 0)$$

$$\frac{|1,0\rangle|1,-1\rangle - |1,-1\rangle|1,0\rangle}{\sqrt{2}} \quad (m = -1)$$

(iii) $j = 0$ ($m = 0$)

$$\frac{|1,1\rangle|1,-1\rangle - |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{3}}$$

((Mathematica 5.2))

```
(*Determination of CG co-efficient
  *)
 (* j1=1/2,j2=1/2*)

CG[j1_,j2_,j_]:=Table[Sum[ClebschGordan[{j1,k1},{j2,k2},{j,k
1+k2}]  a[j1,k1]  b[j2,k2]    KroneckerDelta[k1+k2,m],{k1,-
j1,j1},{k2,-j2,j2}],{m,-j,j}]
 CG[1/2,1/2,1]//TableForm
```

$$a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

$$\frac{a\left[\frac{1}{2},\frac{1}{2}\right] b\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{1}{2},-\frac{1}{2}\right] b\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$$

$$a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
 CG[1/2,1/2,0]//TableForm
```

$$\frac{a\left[\frac{1}{2},\frac{1}{2}\right] b\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}} - \frac{a\left[\frac{1}{2},-\frac{1}{2}\right] b\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$$

```
 (*j1=1,j2=1*)
 CG[1,1,2]//TableForm
```

$a[1,-1]\,b[1,-1]$

$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}} + \frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$

$\frac{a[1,1]\,b[1,-1]}{\sqrt{6}} + \sqrt{\frac{2}{3}}\,a[1,0]\,b[1,0] + \frac{a[1,-1]\,b[1,1]}{\sqrt{6}}$

$\frac{a[1,1]\,b[1,0]}{\sqrt{2}} + \frac{a[1,0]\,b[1,1]}{\sqrt{2}}$

$a[1,1]\,b[1,1]$

```
CG[1,1,1]//TableForm
```

$\frac{a[1,0]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,0]}{\sqrt{2}}$

$\frac{a[1,1]\,b[1,-1]}{\sqrt{2}} - \frac{a[1,-1]\,b[1,1]}{\sqrt{2}}$

$\frac{a[1,1]\,b[1,0]}{\sqrt{2}} - \frac{a[1,0]\,b[1,1]}{\sqrt{2}}$

```
CG[1,1,0]//TableForm
```

$\frac{a[1,1]\,b[1,-1]}{\sqrt{3}} - \frac{a[1,0]\,b[1,0]}{\sqrt{3}} + \frac{a[1,-1]\,b[1,1]}{\sqrt{3}}$

((3-21))

21. a. Evaluate

$$\sum_{m=-j}^{j} |d_{mm'}^{(j)}(\beta)|^2 m$$

for *any j* (integer or half-integer); then check your answer for $j = \frac{1}{2}$.

b. Prove, for any j,

$$\sum_{m=-j}^{j} m^2 |d_{m'm}^{(j)}(\beta)|^2 = \frac{1}{2} j(j+1)\sin^2\beta + m'^2 \frac{1}{2}(3\cos^2\beta - 1).$$

[*Hint*: This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

(a) For any j,

$$\sum_{m=-j}^{j} m\left|d_{m.m'}^{(j)}(\beta)\right|^2 = \sum_{m=-j}^{j} m d_{m.m'}^{(j)}{}^{*}(\beta) d_{m.m'}^{(j)}(\beta)$$

$$= \sum_{m=-j}^{j} \langle j,m'|\exp(\frac{i\hat{J}_y\beta}{\hbar})|j,m\rangle m\langle j,m|\exp(-\frac{i\hat{J}_y\beta}{\hbar})|j,m'\rangle$$

$$= \frac{1}{\hbar}\sum_{m=-j}^{j} \langle j,m'|\exp(\frac{i\hat{J}_y\beta}{\hbar})\hat{J}_z|j,m\rangle\langle j,m|\exp(-\frac{i\hat{J}_y\beta}{\hbar})|j,m'\rangle$$

$$= \frac{1}{\hbar}\langle j,m'|\exp(\frac{i\hat{J}_y\beta}{\hbar})\hat{J}_z\exp(-\frac{i\hat{J}_y\beta}{\hbar})|j,m'\rangle$$

(closure relation)

$$\sum_{m=-j}^{j} m\left|d_{m.m'}^{(j)}(\beta)\right|^2 = \frac{1}{\hbar}\langle j,m'|\hat{J}_z\cos\beta - \hat{J}_x\sin\beta)|j,m'\rangle = m'\cos\beta$$

Note that

$\langle j,m'|\hat{J}_z|j,m'\rangle = m'$, $\langle j,m'|\hat{J}_x|j,m'\rangle = 0$

since $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$ and $\langle j,m'|\hat{J}_\pm|j,m'\rangle = 0$

((Note))

The proof of the formula $\hat{I} = \exp(\frac{i\hat{J}_y\beta}{\hbar})\hat{J}_z\exp(-\frac{i\hat{J}_y\beta}{\hbar}) = \hat{J}_z\cos\beta - \hat{J}_x\sin\beta$

We use the relation

$$\exp(x\hat{A})\hat{B}\exp(-x\hat{A}) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

where $x = \frac{i\beta}{\hbar}$, $\hat{A} = \hat{J}_y$, and $\hat{B} = \hat{J}_z$.

Then we have

$$\hat{I} = \exp(\frac{i\hat{J}_y\beta}{\hbar})\hat{J}_z\exp(-\frac{i\hat{J}_y\beta}{\hbar}) = \hat{J}_z + \frac{1}{1!}\frac{i\beta}{\hbar}[\hat{J}_y,\hat{J}_z] + \frac{1}{2!}\left(\frac{i\beta}{\hbar}\right)^2[\hat{J}_y,[\hat{J}_y,\hat{J}_z] + ...$$

$$= \hat{J}_z + \frac{1}{1!}\frac{i\beta}{\hbar}[\hat{J}_y, \hat{J}_z] + \frac{1}{2!}\left(\frac{i\beta}{\hbar}\right)^2 [\hat{J}_y, [\hat{J}_y, \hat{J}_z] + .\frac{1}{3!}\left(\frac{i\beta}{\hbar}\right)^3 [\hat{J}_y, [\hat{J}_y, [\hat{J}_y, \hat{J}_z]] +$$

$$= \hat{J}_z[1 - -\frac{1}{2}(\beta)^2 + ...] - \hat{J}_x[\beta - .\frac{1}{3!}\left(\frac{\beta}{\hbar}\right)^3 + ...] = \hat{J}_z \cos\phi - \hat{J}_x \sin\beta$$

(b)

$$\sum_{m=-j}^{j} m^2 \left| d_{m.m'}^{(j)}(\beta) \right|^2 = \sum_{m=-j}^{j} m^2 d_{m.m'}^{(j)}{}^*(\beta) d_{m.m'}^{(j)}(\beta)$$

$$= \sum_{m=-j}^{j} \langle j, m' | \exp(\frac{i\hat{J}_y\beta}{\hbar}) | j, m \rangle m^2 \langle j, m | \exp(-\frac{i\hat{J}_y\beta}{\hbar}) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \sum_{m=-j}^{j} \langle j, m' | \exp(\frac{i\hat{J}_y\beta}{\hbar}) \hat{J}_z{}^2 | j, m \rangle \langle j, m | \exp(-\frac{i\hat{J}_y\beta}{\hbar}) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \langle j, m' | \exp(\frac{i\hat{J}_y\beta}{\hbar}) \hat{J}_z{}^2 \exp(-\frac{i\hat{J}_y\beta}{\hbar}) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \langle j, m' | \exp(\frac{i\hat{J}_y\beta}{\hbar}) \hat{J}_z \exp(-\frac{i\hat{J}_y\beta}{\hbar}) \exp(\frac{i\hat{J}_y\beta}{\hbar}) \hat{J}_z \exp(-\frac{i\hat{J}_y\beta}{\hbar}) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \langle j, m' | [\exp(\frac{i\hat{J}_y\beta}{\hbar}) \hat{J}_z \exp(-\frac{i\hat{J}_y\beta}{\hbar})]^2) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \langle j, m' | [\hat{J}_z \cos\beta - \hat{J}_x \sin\beta]^2) | j, m' \rangle$$

$$= \frac{1}{\hbar^2} \langle j, m' | [\hat{J}_z{}^2 \cos^2\beta + \hat{J}_x{}^2 \sin^2\beta - \sin\beta\cos\beta(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)] | j, m' \rangle$$

$$= m'^2 \cos^2\beta + \frac{1}{2}[j(j+1) - m'^2]\sin^2\beta$$

Since

$$\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)\hat{J}_z + \hat{J}_z(\hat{J}_+ + \hat{J}_-)$$

$$\langle j,m'|\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x|j,m'\rangle = 0 .$$

$$\langle j,m'|\hat{J}_x{}^2|j,m'\rangle = \langle j,m'|\hat{J}_x{}^2|j,m'\rangle = \frac{1}{4}\langle j,m'|\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+|j,m'\rangle = \frac{1}{2}\hbar^2[j(j+1) - m'^2]$$

((3-22))

22. a. Consider a system with $j=1$. Explicitly write

$$\langle j=1, m'|J_y|j=1, m\rangle$$

in 3×3 matrix form.

b. Show that for $j=1$ only, it is legitimate to replace $e^{-iJ_y\beta/\hbar}$ by

$$1 - i\left(\frac{J_y}{\hbar}\right)\sin\beta - \left(\frac{J_y}{\hbar}\right)^2(1-\cos\beta).$$

–

c. Using (b), prove

$d^{(j=1)}(\beta) =$

$$\begin{pmatrix} \left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}.$$

(a)

$$\hat{J}_x = \hbar\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \hat{J}_y = \hbar\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad \hat{J}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) and (c)

Taylor expansion:

$$\exp(-\frac{i}{\hbar}\theta\hat{J}_y) = 1 + \frac{1}{1!}(-\frac{i}{\hbar}\theta\hat{J}_y) + \frac{1}{2!}(-\frac{i}{\hbar}\theta\hat{J}_y)^2 + \frac{1}{3!}(-\frac{i}{\hbar}\theta\hat{J}_y)^3 + \frac{1}{4!}(-\frac{i}{\hbar}\theta\hat{J}_y)^4 + ..$$

where

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

Note that

$$\hat{J}_+|1\rangle = 0, \qquad \hat{J}_+|0\rangle = \sqrt{2}\hbar|1\rangle, \qquad \hat{J}_+|-1\rangle = \sqrt{2}\hbar|0\rangle$$

$$\hat{J}_-|1\rangle = \sqrt{2}\hbar|0\rangle, \qquad \hat{J}_-|0\rangle = \sqrt{2}\hbar|-1\rangle, \qquad \hat{J}_-|-1\rangle = 0$$

$$\hat{J}_y|1\rangle = \frac{i\hbar}{\sqrt{2}}|0\rangle, \qquad \hat{J}_y|0\rangle = \frac{-i\hbar}{\sqrt{2}}(|1\rangle - |-1\rangle), \qquad \hat{J}_y|-1\rangle = -\frac{i\hbar}{\sqrt{2}}|0\rangle$$

$$\hat{J}_y = \hbar\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \qquad \hat{J}_y{}^2 = -\hbar^2\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\hat{J}_y{}^3 = \hbar^2\hat{J}_y, \qquad \hat{J}_y{}^4 = \hat{J}_y{}^3\hat{J}_y = \hbar^2\hat{J}_y\hat{J}_y = \hbar^2\hat{J}_y{}^2,$$

$$\hat{J}_y{}^5 = \hat{J}_y{}^4\hat{J}_y = \hbar^2\hat{J}_y{}^2\hat{J}_y = \hbar^2\hat{J}_y{}^3 = \hbar^4\hat{J}_y$$

Therefore

$$\exp(-\frac{i}{\hbar}\theta\hat{J}_y) = 1 + \frac{\hat{J}_y}{\hbar}[(-\theta) + \frac{1}{3!}(-i\theta)^3 + \frac{1}{5!}(-i\theta)^5 + ..]$$
$$+ \frac{\hat{J}_y{}^2}{\hbar^2}[\frac{1}{2!}(-i\theta)^2 + \frac{1}{4!}(-i\theta)^4 + ..]$$

$$= 1 - \frac{\hat{J}_y}{\hbar}(i\sin\theta) + \frac{\hat{J}_y{}^2}{\hbar^2}(\cos\theta - 1)$$

$$= \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

We also get

$$\exp(-\frac{i}{\hbar}\phi\hat{J}_z) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}$$

which is a diagonal matrix.

(a) We use Mathematica to derive the matrix expression for the angular m omentum with j = 1.

```
(*Sakurai 3-22*)
(*a*)
Jx[ℓ_, n_, m_] :=
  1/2 ℏ Sqrt[(ℓ - m) (ℓ + m + 1)] KroneckerDelta[n, m + 1] +
  1/2 ℏ Sqrt[(ℓ + m) (ℓ - m + 1)] KroneckerDelta[n, m - 1]
Jy[ℓ_, n_, m_] :=
  -1/2 ⅈ ℏ Sqrt[(ℓ - m) (ℓ + m + 1)] KroneckerDelta[n, m + 1] +
  1/2 ⅈ ℏ Sqrt[(ℓ + m) (ℓ - m + 1)] KroneckerDelta[n, m - 1]
Jz[ℓ_,n_,m_]:=m ℏ KroneckerDelta[n,m]
(*Matrices j = 1*)
J1=Table[Jx[1,n,m],{n,1,-1,-1},{m,1,-1,-1}]
```

$$\left\{\left\{0, \frac{\hbar}{\sqrt{2}}, 0\right\}, \left\{\frac{\hbar}{\sqrt{2}}, 0, \frac{\hbar}{\sqrt{2}}\right\}, \left\{0, \frac{\hbar}{\sqrt{2}}, 0\right\}\right\}$$

```
J2=Table[Jy[1,n,m],{n,1,-1,-1},{m,1,-1,-1}]
```

$$\left\{\left\{0, -\frac{i\hbar}{\sqrt{2}}, 0\right\}, \left\{\frac{i\hbar}{\sqrt{2}}, 0, -\frac{i\hbar}{\sqrt{2}}\right\}, \left\{0, \frac{i\hbar}{\sqrt{2}}, 0\right\}\right\}$$

```
J3=Table[Jz[1,n,m],{n,1,-1,-1},{m,1,-1,-1}]
```

{{ℏ,0,0},{0,0,0},{0,0,-ℏ}}

```
J1//MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
J2//MatrixForm
```

$$\begin{pmatrix} 0 & -\frac{i\hbar}{\sqrt{2}} & 0 \\ \frac{i\hbar}{\sqrt{2}} & 0 & -\frac{i\hbar}{\sqrt{2}} \\ 0 & \frac{i\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

```
J3//MatrixForm
```

$$\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

```
(*b*)
I1={{1,0,0},{0,1,0},{0,0,1}}
```

{{1,0,0},{0,1,0},{0,0,1}}

```
I1//MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{P1 = I1 - i\,\frac{J2}{\hbar}\,Sin[\beta] - \frac{1}{\hbar^2}\,J2.J2\,(1 - Cos[\beta])\ //\ Simplify}$$

$$\left\{\left\{\text{Cos}\left[\frac{\beta}{2}\right]^2, -\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Sin}\left[\frac{\beta}{2}\right]^2\right\}, \left\{\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}[\beta], -\frac{\text{Sin}[\beta]}{\sqrt{2}}\right\}, \left\{\text{Sin}\left[\frac{\beta}{2}\right]^2, \frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}\left[\frac{\beta}{2}\right]^2\right\}\right\}$$

```
P1//MatrixForm
```

$$\begin{pmatrix} \text{Cos}\left[\frac{\beta}{2}\right]^2 & -\frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Sin}\left[\frac{\beta}{2}\right]^2 \\ \frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Cos}[\beta] & -\frac{\text{Sin}[\beta]}{\sqrt{2}} \\ \text{Sin}\left[\frac{\beta}{2}\right]^2 & \frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Cos}\left[\frac{\beta}{2}\right]^2 \end{pmatrix}$$

```
(*c*)
```

$$\mathbf{MJ2 = MatrixExp\left[-\frac{i}{\hbar}\,J2\,\beta\right]\ //\ Simplify}$$

$$\{\{\text{Cos}[\frac{\beta}{2}]^2, -\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Sin}[\frac{\beta}{2}]^2\},$$
$$\{\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}[\beta], -\frac{\text{Sin}[\beta]}{\sqrt{2}}\}, \{\text{Sin}[\frac{\beta}{2}]^2, \frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}[\frac{\beta}{2}]^2\}\}$$

```
MJ2//MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\frac{\beta}{2}]^2 & -\frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Sin}[\frac{\beta}{2}]^2 \\ \frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Cos}[\beta] & -\frac{\text{Sin}[\beta]}{\sqrt{2}} \\ \text{Sin}[\frac{\beta}{2}]^2 & \frac{\text{Sin}[\beta]}{\sqrt{2}} & \text{Cos}[\frac{\beta}{2}]^2 \end{pmatrix}$$

((**3-24**))

24. Consider a system made up of two spin $\frac{1}{2}$ particles. Observer A specializes in measuring the spin components of one of the particles (s_{1z}, s_{1x} and so on), while observer B measures the spin components of the other particle. Suppose the system is known to be in a spin-singlet state, that is, $S_{\text{total}} = 0$.
   a. What is the probability for observer A to obtain $s_{1z} = \hbar/2$ when observer B makes no measurement? Same problem for $s_{1x} = \hbar/2$.
   b. Observer B determines the spin of particle 2 to be in the $s_{2z} = \hbar/2$ state with certainty. What can we then conclude about the outcome of observer A's measurement if (i) A measures s_{1z} and (ii) A measures s_{1x}? Justify your answer.

The singlet state is given by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|+-\rangle - |-+\rangle \qquad (1)$$

$|+\rangle_x = \frac{1}{\sqrt{2}}[|+\rangle + |-\rangle]$ and $|-\rangle_x = \frac{1}{\sqrt{2}}[|+\rangle - |-\rangle]$

or

$$|+\rangle = \frac{1}{\sqrt{2}}[|+\rangle_x + |-\rangle_x]$$

$$|-\rangle = \frac{1}{\sqrt{2}}[|+\rangle_x - |-\rangle_x]$$

Therefore $|\psi\rangle$ can be rewritten as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}[|+\rangle_x + |-\rangle_x])(\frac{1}{\sqrt{2}}[|+\rangle_x - |-\rangle_x]) - \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}[|+\rangle_x - |-\rangle_x])(\frac{1}{\sqrt{2}}[|+\rangle_x + |-\rangle_x])$$

$$= -\frac{1}{\sqrt{2}}(|+\rangle_x|-\rangle_x + |-\rangle_x|+\rangle_x)$$

Apart from the overall sign, which in any case is a matter of convention, we could have guessed this from Eq.(1), because spin-singlet states have no preferred direction in space.

(a) Observer A specializes in measuring the spin components of one of the particles (1), while the observer M measures the spin components of the other particle.

What is the probability for the observer A to obtain $S_{1z} = \hbar/2$ when the observer B makes no measurement? Same problem for $S_{1x} = \hbar/2$.

$$P(S_{1z} = \hbar/2) = |\langle ++|\psi\rangle|^2 + |\langle +-|\psi\rangle|^2 = \frac{1}{2}$$

$$P(S_{1x} = \hbar/2) = |{}_x\langle +|{}_x\langle +|\psi\rangle|^2 + |{}_x\langle +|{}_x\langle -|\psi\rangle|^2 = \frac{1}{2}$$

(b) Observer B determines the spin of particles 2 to be in the $S_{2z} = \hbar/2$ state with certainty. What can we then conclude about the outcome of the observer A's measurement if (i) A measures S_{1z} and (ii) A measures S_{1x}? Justify your answer.

After the measurement of $S_{2z} = \hbar/2$ by observer B, the state becomes

$$|\psi'\rangle = -|-+\rangle$$

(i)

$$P(S_{1z} = \frac{\hbar}{2}) = |\langle ++|\psi'\rangle|^2 + |\langle +-|\psi'\rangle|^2 = 0$$

$$P(S_{1z} = -\frac{\hbar}{2}) = |\langle -+|\psi'\rangle|^2 + |\langle --|\psi'\rangle|^2 = 1$$

(ii)

$$P(S_{1x} = \frac{\hbar}{2}) = \left|{}_x\langle + |{}_x\langle + | \psi' \rangle\right|^2 + \left|{}_x\langle + |{}_x\langle - | \psi' \rangle\right|^2 = \frac{1}{2}$$

$$P(S_{1x} = -\frac{\hbar}{2}) = \left|{}_x\langle - |{}_x\langle + | \psi' \rangle\right|^2 + \left|{}_x\langle - |{}_x\langle - | \psi' \rangle\right|^2 = \frac{1}{2}$$

((3-25))

25. Consider a spherical tensor of rank 1 (that is, a vector)

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}, \qquad V_0^{(1)} = V_z.$$

Using the expression for $d^{(j=1)}$ given in Problem 22, evaluate

$$\sum_{q'} d_{qq'}^{(1)}(\beta) V_{q'}^{(1)}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

$$\begin{pmatrix} V'^{(1)}_1 \\ V'^{(1)}_0 \\ V'^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \begin{pmatrix} V^{(1)}_1 \\ V^{(1)}_0 \\ V^{(1)}_{-1} \end{pmatrix}$$

On the other hand,

$$\Re = \begin{pmatrix} \Re_{11} & \Re_{12} & \Re_{13} \\ \Re_{21} & \Re_{22} & \Re_{23} \\ \Re_{31} & \Re_{32} & \Re_{33} \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\begin{pmatrix} V_1^{(1)} \\ V_0^{(1)} \\ V_{-1}^{(1)} \end{pmatrix} = \begin{pmatrix} -\frac{V_x + iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x - iV_y}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\begin{pmatrix} V'^{(1)}_1 \\ V'^{(1)}_0 \\ V'^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} V_x' \\ V_y' \\ V_z' \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \text{ or}$$

or

$$\begin{pmatrix} V'^{(1)}_1 \\ V'^{(1)}_0 \\ V'^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}^{-1} \begin{pmatrix} V_1^{(1)} \\ V_0^{(1)} \\ V_{-1}^{(1)} \end{pmatrix}$$

Using Mathematica 5.2, we have

$$\begin{pmatrix} V'^{(1)}_1 \\ V'^{(1)}_0 \\ V'^{(1)}_{-1} \end{pmatrix} = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \begin{pmatrix} V_1^{(1)} \\ V_0^{(1)} \\ V_{-1}^{(1)} \end{pmatrix}$$

((Mathematica 5.2))

$$\mathbf{A} = \left\{\left\{-\frac{1}{\sqrt{2}},\ -\frac{i}{\sqrt{2}},\ 0\right\},\ \{0,\ 0,\ 1\},\ \left\{\frac{1}{\sqrt{2}},\ -\frac{i}{\sqrt{2}},\ 0\right\}\right\}$$

$$\left\{\left\{-\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right\}, \{0, 0, 1\}, \left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0\right\}\right\}$$

A//MatrixForm

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Inverse[A]//MatrixForm

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

M={{Cos[β],0,Sin[β]},{0,1,0},{-Sin[β],0,Cos[β]}}

{{Cos[β],0,Sin[β]},{0,1,0},{-Sin[β],0,Cos[β]}}

M//MatrixForm

$$\begin{pmatrix} \text{Cos}[\beta] & 0 & \text{Sin}[\beta] \\ 0 & 1 & 0 \\ -\text{Sin}[\beta] & 0 & \text{Cos}[\beta] \end{pmatrix}$$

A.M.Inverse[A]//Simplify

$$\left\{\left\{\text{Cos}\left[\frac{\beta}{2}\right]^2, -\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Sin}\left[\frac{\beta}{2}\right]^2\right\}, \left\{\frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}[\beta], -\frac{\text{Sin}[\beta]}{\sqrt{2}}\right\}, \left\{\text{Sin}\left[\frac{\beta}{2}\right]^2, \frac{\text{Sin}[\beta]}{\sqrt{2}}, \text{Cos}\left[\frac{\beta}{2}\right]^2\right\}\right\}$$

((**3-26**))

26. a. Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicitly write $T^{(1)}_{\pm 1, 0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

b. Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T^{(2)}_{\pm 2, \pm 1, 0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

(a)

The spherical tensor of rank-1:

The quantity

$$P_{l,m}(x,y,z) = r^l Y_l^m(\theta,\phi)$$

is a homogeneous polynomial of order *l*.

The quantity $P_{1,q}(x,y,z) = \sqrt{\frac{4\pi}{3}} r Y_1^q(\theta,\phi)$ is a first order homogeneous polynomial in *x*, *y*, and *z*.

$$P_{1,1}(x,y,z) = \sqrt{\frac{4\pi}{3}} r Y_1^1(\theta,\phi) = -\frac{x+iy}{\sqrt{2}}$$

$$T_1^{(1)} = -\frac{\hat{V}_x + i\hat{V}_y}{\sqrt{2}}, \qquad T_1^{(1)} = -\frac{\hat{U}_x + i\hat{U}_y}{\sqrt{2}}$$

$$P_{1,0}(x,y,z) = \sqrt{\frac{4\pi}{3}} r Y_1^0(\theta,\phi) = z$$

$$T_0^{(1)} = \hat{V}_z, \qquad T_0^{(1)} = \hat{U}_z$$

$$P_{1,-1}(x,y,z) = \sqrt{\frac{4\pi}{3}} r Y_1^{-1}(\theta,\phi) = \frac{x-iy}{\sqrt{2}}$$

$$T_{-1}^{(1)} = \frac{\hat{V}_x - i\hat{V}_y}{\sqrt{2}}, \qquad T_{-1}^{(1)} = \frac{\hat{U}_x - i\hat{U}_y}{\sqrt{2}}$$

(b)
The spherical tensor of rank-2:

We use the following theorem.

When $\hat{U}_{q_1}^{(k_1)}$ and $\hat{V}_{q_2}^{(k_2)}$ be irreducible spherical tensors of rank k_1 and k_2, respectively. Then

$$\hat{T}_q^{(k)} = \sum_{q_1,q_2} \langle k_1, k_2; q_1, q_2 | k_1, k_2; k, q \rangle \hat{U}_{q_1}^{(k_1)} V_{q_2}^{(k_2)}$$

is a spherical (irreducible) tensor of rank *k*.

$\langle k_1, k_2; q_1, q_2 | k_1, k_2; k, q \rangle$ is the Clebsch-Gordan coefficient.

Using the Mathematica 5.2 we get

$$T_2^{(2)} = X_1^{(1)} Z_1^{(1)} = U_1 V_1$$

$$T_1^{(2)} = \frac{X_1^{(1)} Z_0^{(1)} + X_0^{(1)} Z_1^{(1)}}{\sqrt{2}} = \frac{U_1 V_0 + U_0 V_1}{\sqrt{2}}$$

$$T_0^{(2)} = \frac{X_1^{(1)} Z_{-1}^{(1)} + 2 X_0^{(1)} Z_0^{(1)} + X_{-1}^{(1)} Z_1^{(1)}}{\sqrt{6}} = \frac{U_1 V_{-1} + 2 U_0 V_0 + U_{-1} V_1}{\sqrt{6}}$$

$$T_{-1}^{(2)} = \frac{X_0^{(1)} Z_{-1}^{(1)} + X_{-1}^{(1)} Z_0^{(1)}}{\sqrt{2}} = \frac{U_0 V_{-1} + U_{-1} V_1}{\sqrt{2}}$$

$$T_2^{(2)} = X_{-1}^{(1)} Z_{-1}^{(1)} = U_{-1} V_{-1}$$

((Mathematica 5.2))

```
(*Determination of CG co-efficient
  *)

CG[j1_,j2_,j_]:=Table[Sum[ClebschGordan[{j1,k1},{j2,k2},{j,k1+k2}]  X[j1,k1]  Z[j2,k2]    KroneckerDelta[k1+k2,m],{k1,-j1,j1},{k2,-j2,j2}],{m,-j,j}]
 CG[1,1,2]//TableForm
```

$X[1,-1]\, Z[1,-1]$

$\frac{X[1,0]\, Z[1,-1]}{\sqrt{2}} + \frac{X[1,-1]\, Z[1,0]}{\sqrt{2}}$

$\frac{X[1,1]\, Z[1,-1]}{\sqrt{6}} + \sqrt{\frac{2}{3}}\, X[1,0]\, Z[1,0] + \frac{X[1,-1]\, Z[1,1]}{\sqrt{6}}$

$\frac{X[1,1]\, Z[1,0]}{\sqrt{2}} + \frac{X[1,0]\, Z[1,1]}{\sqrt{2}}$

$X[1,1]\, Z[1,1]$

((3-27))

27. Consider a spinless particle bound to a fixed center by a central force potential.

a. Relate, as much as possible, the matrix elements

$$\langle n', l', m'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|n, l, m\rangle \quad \text{and} \quad \langle n', l', m'|z|n, l, m\rangle$$

using *only* the Wigner-Eckart theorem. Make sure to state under what conditions the matrix elements are nonvanishing.

b. Do the same problem using wave functions $\psi(\mathbf{x}) = R_{nl}(r)Y_l^m(\theta, \phi)$.

$$\langle n', l'm'| \mp \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})|n, l, m\rangle = \langle n', l'm'|T_{\pm 1}^{(1)}|n, l, m\rangle$$

$$\langle n', l'm'|\hat{z}|n, l, m\rangle = \langle n', l'm'|T_0^{(1)}|n, l, m\rangle$$

We now consider the matrix element

$$\langle n', l'm'|T_q^{(1)}|n, l, m\rangle$$

with q = 0 and 1. According to the Wigner-Eckart theorem, this matrix element is equal to zero unless

$$m' = m \pm q$$
$$l' = l + 1, l, l - 1$$

However, we know that

$$\hat{\pi}T_{\pm 1}^{(1)}\hat{\pi} = -T_{\pm 1}^{(1)} \text{ and } \hat{\pi}|n, l, m\rangle = (-1)^l|n, l, m\rangle$$

where $\hat{\pi}$ is the parity operator.

$$\langle n', l', m'|T_{\pm 1}^{(1)}|n, l, m\rangle = 0$$

when $l' = l$.

Thus the matrix element is equal to zero unless

$$m' = m \pm q$$
$$l' = l + 1, l - 1$$

((Note))

$\hat{T}_1^{(1)} = -\frac{\hat{x}+i\hat{y}}{\sqrt{2}}$, $\hat{T}_0^{(1)} = \hat{z}$, $\hat{T}_{-1}^{(1)} = \frac{\hat{x}-i\hat{y}}{\sqrt{2}}$

((**3-28**))

28. a. Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.
    b. The expectation value

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

is known as the *quadrupole moment*. Evaluate

$$e\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle,$$

(where $m' = j, j-1, j-2, \ldots$) in terms of Q and appropriate Clebsch-Gordan coefficients.

(a)

$$\hat{x}\hat{y} = \frac{\hat{T}_2^{(2)} - \hat{T}_{-2}^{(2)}}{2i}$$

$$\hat{x}\hat{z} = \frac{-\hat{T}_1^{(2)} + \hat{T}_{-1}^{(2)}}{2}$$

$$\hat{x}^2 - \hat{y}^2 = \hat{T}_2^{(2)} + \hat{T}_{-2}^{(2)}$$

(b)

$$Q = e\langle \alpha; j, m' = j|2\hat{z}^2 - \hat{x}^2 - \hat{y}^2|\alpha; j, m = j\rangle = e\sqrt{6}\langle \alpha; j, m' = j|\hat{T}_0^{(2)}|\alpha; j, m = j\rangle$$

$$e\langle \alpha; j, m'|\hat{x}^2 - \hat{y}^2|\alpha; j, m = j\rangle = e\langle \alpha; j, m'|\hat{T}_2^{(2)} + \hat{T}_{-2}^{(2)}|\alpha; j, m = j\rangle$$

$$= e\langle \alpha; j, j-2|\hat{T}_{-2}^{(2)}|\alpha; j, m = j\rangle$$

from the Wigner-Eckart theorem

$$Q = e\sqrt{6}\langle \alpha; j, m' = j|\hat{T}_0^{(2)}|\alpha; j, m = j\rangle = e\sqrt{6}\langle j,2; j,0| j,2; j, j\rangle \frac{\langle \alpha' j\|\hat{T}^{(k)}\|\alpha j\rangle}{\sqrt{2j+1}}$$

$$e\langle \alpha; j, j-2|\hat{T}_{-2}^{(2)}|\alpha; j, m=j\rangle = e\langle j,2; j,-2| j,2; j, j-2\rangle \frac{\langle \alpha j \| \hat{T}^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}}$$

Then we have

$$\frac{e\langle \alpha; j, j-2|\hat{T}_{-2}^{(2)}|\alpha; j, m=j\rangle}{Q} = \frac{e\langle j,2; j,-2| j,2; j, j-2\rangle \frac{\langle \alpha j \| \hat{T}^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}}}{e\sqrt{6}\langle j,2; j,0| j,2; j, j\rangle \frac{\langle \alpha' j \| \hat{T}^{(k)} \| \alpha j\rangle}{\sqrt{2j+1}}}$$

or

$$\frac{e\langle \alpha; j, j-2|\hat{T}_{-2}^{(2)}|\alpha; j, m=j\rangle}{Q} = \frac{\langle j,2; j,-2| j,2; j, j-2\rangle}{\sqrt{6}\langle j,2; j,0| j,2; j, j\rangle} = \frac{1}{\sqrt{j(2j-1)}}$$

((Note-1))

$$\hat{x}^2 - \hat{y}^2 = \hat{T}_2^{(2)} + \hat{T}_{-2}^{(2)}$$

$$\hat{x}\hat{y} = \frac{\hat{T}_2^{(2)} - \hat{T}_{-2}^{(2)}}{2i}$$

$$\hat{y}\hat{z} = \frac{\hat{T}_1^{(2)} + \hat{T}_{-1}^{(2)}}{-2i}$$

$$\hat{x}\hat{z} = \frac{\hat{T}_1^{(2)} - \hat{T}_{-1}^{(2)}}{-2}$$

$$\left(\frac{\hat{x}^2 + \hat{y}^2 - 2\hat{z}^2}{\sqrt{6}}\right) = -\hat{T}_0^{(2)}$$

((Note-2))

$$\langle \alpha'; j', m' | \hat{T}_q^{(k)} | \alpha; j, m \rangle = \langle j, k; m, q | j, k; j', m' \rangle \frac{\langle \alpha' j' \| \hat{T}^{(k)} \| \alpha j \rangle}{\sqrt{2j+1}}$$

((3-29))

29. A spin $\frac{3}{2}$ nucleus situated at the origin is subjected to an external inhomogeneous electric field. The basic electric quadrupole interaction may by taken to be

$$H_{\text{int}} = \frac{eQ}{2s(s-1)\hbar^2}\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 S_x^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)_0 S_y^2 + \left(\frac{\partial^2\phi}{\partial z^2}\right)_0 S_z^2\right],$$

where ϕ is the electrostatic potential satisfying Laplace's equation and the coordinate axes are so chosen that

$$\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_0 = \left(\frac{\partial^2\phi}{\partial y\partial z}\right)_0 = \left(\frac{\partial^2\phi}{\partial x\partial z}\right)_0 = 0.$$

Show that the interaction energy can be written as

$$A(3S_z^2 - \mathbf{S}^2) + B(S_+^2 + S_-^2),$$

and express A and B in terms of $(\partial^2\phi/\partial x^2)_0$ and so on. Determine the energy eigenkets (in terms of $|m\rangle$, where $m = \pm\frac{3}{2}, \pm\frac{1}{2}$) and the corresponding energy eigenvalues. Is there any degeneracy?

We solve the last part of the problem using Mathematica 5.2.

```
(* Sakurai 3-29 Spin 3/2 matrix elements*)
Sx = 1/2 {{0, √3, 0, 0}, {√3, 0, 2, 0},
    {0, 2, 0, √3}, {0, 0, √3, 0}}
{{0, √3/2, 0, 0}, {√3/2, 0, 1, 0},
 {0, 1, 0, √3/2}, {0, 0, √3/2, 0}}
Sx//MatrixForm
```

$$\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Sy =

$$\frac{1}{2}\left\{\left\{0, -\sqrt{3}\,i, 0, 0\right\}, \left\{\sqrt{3}\,i, 0, -2\,i, 0\right\}, \left\{0, 2\,i, 0, -\sqrt{3}\,i\right\}, \left\{0, 0, \sqrt{3}\,i, 0\right\}\right\}$$

$$\left\{\left\{0, -\frac{i\sqrt{3}}{2}, 0, 0\right\}, \left\{\frac{i\sqrt{3}}{2}, 0, -i, 0\right\}, \left\{0, i, 0, -\frac{i\sqrt{3}}{2}\right\}, \left\{0, 0, \frac{i\sqrt{3}}{2}, 0\right\}\right\}$$

Sy//MatrixForm

$$\begin{pmatrix} 0 & -\frac{i\sqrt{3}}{2} & 0 & 0 \\ \frac{i\sqrt{3}}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{i\sqrt{3}}{2} \\ 0 & 0 & \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

$$\mathbf{Sz} = \frac{1}{2}\{\{3, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, -1, 0\}, \{0, 0, 0, -3\}\}$$

$$\left\{\left\{\frac{3}{2}, 0, 0, 0\right\}, \left\{0, \frac{1}{2}, 0, 0\right\}, \left\{0, 0, -\frac{1}{2}, 0\right\}, \left\{0, 0, 0, -\frac{3}{2}\right\}\right\}$$

Sz//MatrixForm

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

```
Sp=Sx + ⅈ Sy
```

$$\{\{0, \sqrt{3}, 0, 0\}, \{0, 0, 2, 0\}, \{0, 0, 0, \sqrt{3}\}, \{0, 0, 0, 0\}\}$$

```
Sm=Sx-ⅈ Sy
```

$$\{\{0, 0, 0, 0\}, \{\sqrt{3}, 0, 0, 0\}, \{0, 2, 0, 0\}, \{0, 0, \sqrt{3}, 0\}\}$$

```
H=A (3 Sz.Sz- Sx.Sx-Sy.Sy-Sz.Sz)+B(Sp.Sp + Sm.Sm)
```

$$\{\{3 A, 0, 2\sqrt{3} B, 0\}, \{0, -3 A, 0, 2\sqrt{3} B\}, \{2\sqrt{3} B, 0, -3 A, 0\}, \{0, 2\sqrt{3} B, 0, 3 A\}\}$$

```
H//MatrixForm
```

$$\begin{pmatrix} 3 A & 0 & 2\sqrt{3} B & 0 \\ 0 & -3 A & 0 & 2\sqrt{3} B \\ 2\sqrt{3} B & 0 & -3 A & 0 \\ 0 & 2\sqrt{3} B & 0 & 3 A \end{pmatrix}$$

```
Eigensystem[H]//Simplify
```

$$\Big\{\Big\{-\sqrt{9 A^2 + 12 B^2}, -\sqrt{9 A^2 + 12 B^2}, \sqrt{9 A^2 + 12 B^2}, \sqrt{9 A^2 + 12 B^2}\Big\},$$

$$\Big\{\Big\{0, -\frac{\sqrt{3} A + \sqrt{3 A^2 + 4 B^2}}{2 B}, 0, 1\Big\}, \Big\{\frac{\sqrt{3} A - \sqrt{3 A^2 + 4 B^2}}{2 B}, 0, 1, 0\Big\},$$

$$\Big\{0, \frac{-\sqrt{3} A + \sqrt{3 A^2 + 4 B^2}}{2 B}, 0, 1\Big\}, \Big\{\frac{\sqrt{3} A + \sqrt{3 A^2 + 4 B^2}}{2 B}, 0, 1, 0\Big\}\Big\}\Big\}$$