Comparison between theory and experiment for the scattering of identical particles

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The differential cross section for the scattering of identical two particles is different, depending on the spin. The particles with spin 0 is boson, while the particles with spin 1/2 is fermion. Although there are so many articles on the scattering of identical particles in textbooks of quantum mechanics, it is a little difficult for one to find the experimental results. The experimental results are useful for our understanding the scattering of identical particles. I tried to find such experimental results in the web sites, and find a very interesting article (Experiment No. M319, Institute für Kernphysik der Universitate zu Köű̋üln Praktikum M.) There are experimental results and experimental method for scattering of two identical particles (two isotopes, ^{12}C (no nuclear spin, boson) and ^{13}C (nuclear spin 1/2, fermion). There are three cases for the Coulomb cross section;

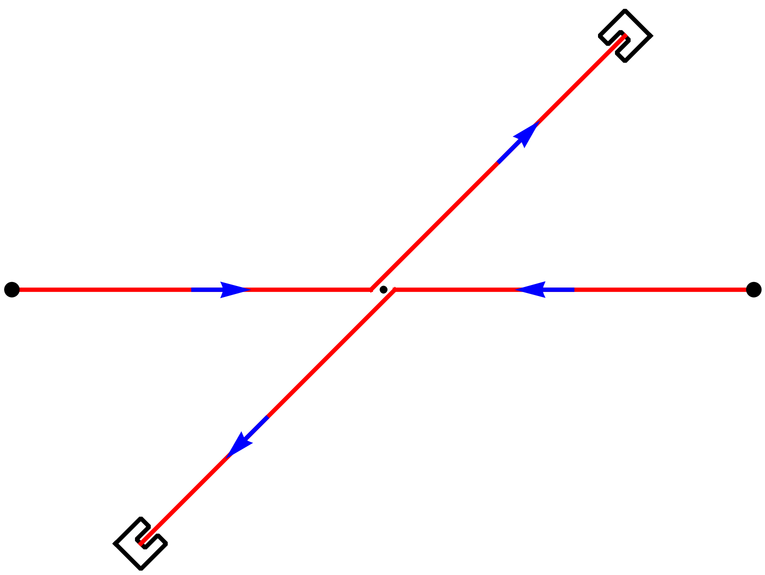
(a) ^{12}C + ^{13}C (distinct nuclei)
(b) ^{12}C + ^{12}C; (identical bosons spin 0)
(c) ^{12}C + ^{12}C; (identical fermions spin 1/2)

Here we discuss simple theory on the Coulomb scattering of two identical particles (boson, fermion) and will compare between experimental results and the corresponding theory.

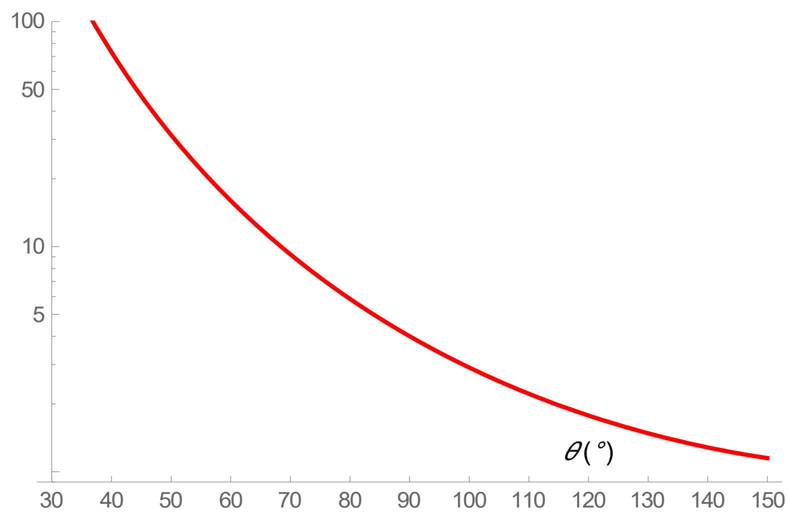
1. Overview

(a) Cross section for the Coulomb scattering; $|f(\theta)|^2$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{2k}\right)^2 \csc^4\left(\frac{\theta}{2}\right) = A\csc^4\left(\frac{\theta}{2}\right)$$

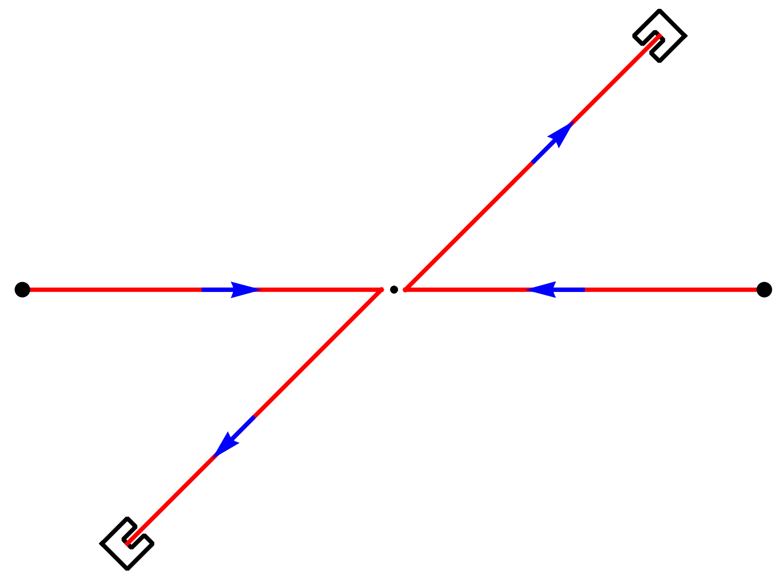


We make a plot of $\frac{d\sigma}{d\Omega}/A$ as a function of the scattering angle θ.



(b) Cross section for the Coulomb scattering; $|f(\pi-\theta)|^2$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{2k}\right)^2 \sec^4\left(\frac{\theta}{2}\right) = A\sec^4\left(\frac{\theta}{2}\right)$$



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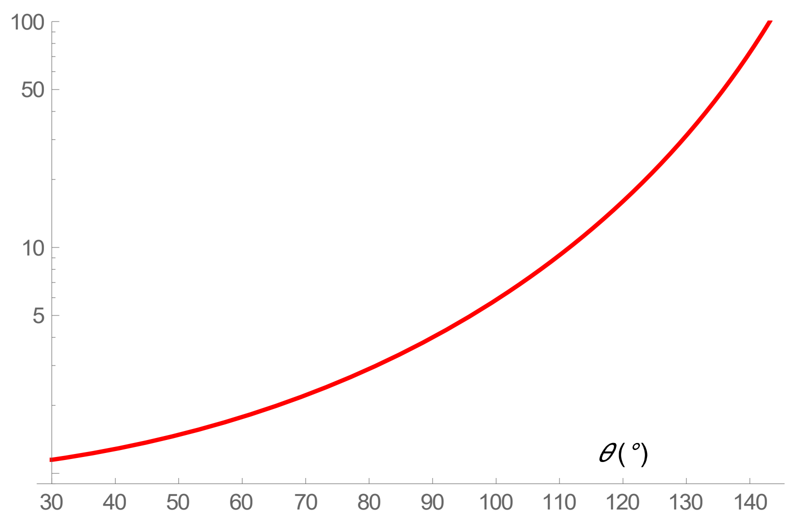
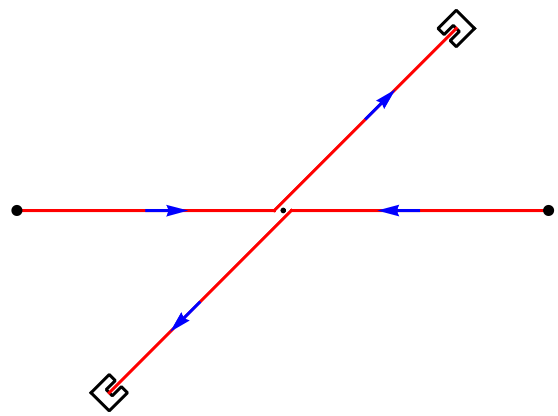
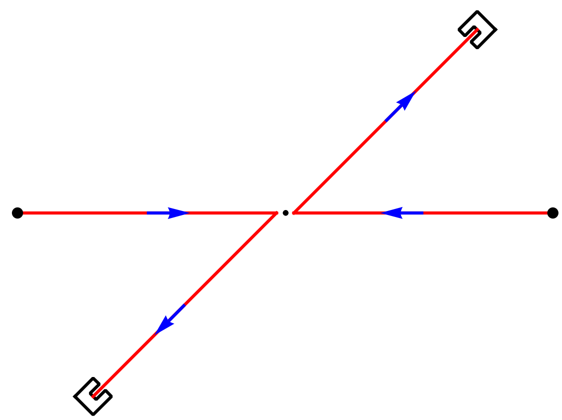


(c) Cross section for the Coulomb scattering; $|f(\theta)|^2 + |f(\pi-\theta)|^2$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{2k}\right)^2 [\csc^4\left(\frac{\theta}{2}\right) + \sec^4\left(\frac{\theta}{2}\right)]$$



Either or



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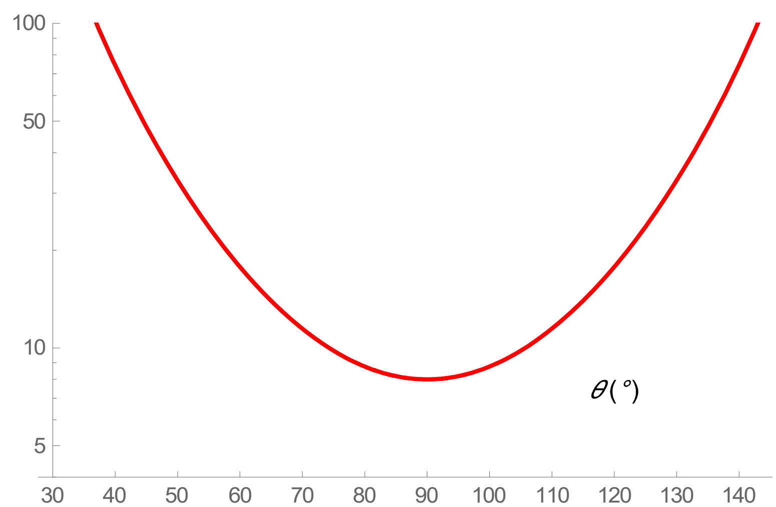


2. Scattering of identical particles (boson and fermion)

In this experiment scattering of ^{12}C (nuclear spin 0; boson) and ^{13}C (nuclear spin 1/2, fermion) particles at a ^{12}C and ^{13}C target is investigated. The aim is to determine scattering cross sections for the reactions of identical particles and non-identical particles in dependence of the angle. The ^{12}C and ^{13}C ions are accelerated to 10MeV and hit the target in the target chamber. The scattered particles are detected at angles between 10 and 80 degrees. Using identical bosons and fermions as target and projectile quantum mechanical effects are observed.

Due to their indistinguishability both contributions interfere. The differential cross section in dependence of the angle shows an interference pattern (oscillations). The interference patterns derive from the fact that the cross section has to be calculated quantum mechanically. Depending on the symmetry of the wave function, the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) + f(\pi-\theta) \right|^2$$
$$= \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 + 2\,\mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

for identical bosons,

$$\frac{d\sigma}{d\Omega} = \left| f(\theta) - f(\pi-\theta) \right|^2$$
$$= \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 - 2\,\mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

for identical fermions,

For spin 1/2 (fermion);

$$D_{1/2} \times D_{1/2} = D_1 + D_0$$

The total spin of two identical particles with spin 1/2 is $S = 1$ (triplet, symmetric) and S = 0 (singlet, antisymmetric). Because of the antisymmetric character of total wave function, the orbital wave function is antisymmetric for $S = 1$, while it is symmetric for $S = 0$.

$$\left(\frac{d\sigma}{d\Omega}\right)^{unpol} = \frac{3}{4}\left| f(\theta) - f(\pi-\theta) \right|^2 + \frac{1}{4}\left| f(\theta) + f(\pi-\theta) \right|^2$$
$$= \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 - \mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

For the spin state ($\left| +z,+z \right\rangle$(one of the triplet state), the orbital wave function is antisymmetric, the differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)^{\uparrow\uparrow} = \left| f(\theta) - f(\pi-\theta) \right|^2$$
$$= \left| f(\theta) \right|^2 + \left| f(\pi-\theta) \right|^2 - 2\,\mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

3. Mott formula for the Coulomb scattering of two identical

(a) Two identical spinless particles (bosons)

The differential cross section for the Coulomb scattering of two identical bosons is derived by Mott as

$$\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{2k}\right)^2 \{\csc^4\left(\frac{\theta}{2}\right) + \sec^4\left(\frac{\theta}{2}\right) + 2\csc^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right)\cos[2\gamma \ln(\tan\frac{\theta}{2})]\}$$

with

$$\gamma = \frac{(Ze)^2}{\hbar v}$$

(b) Two identical spin 1/2 fermions,

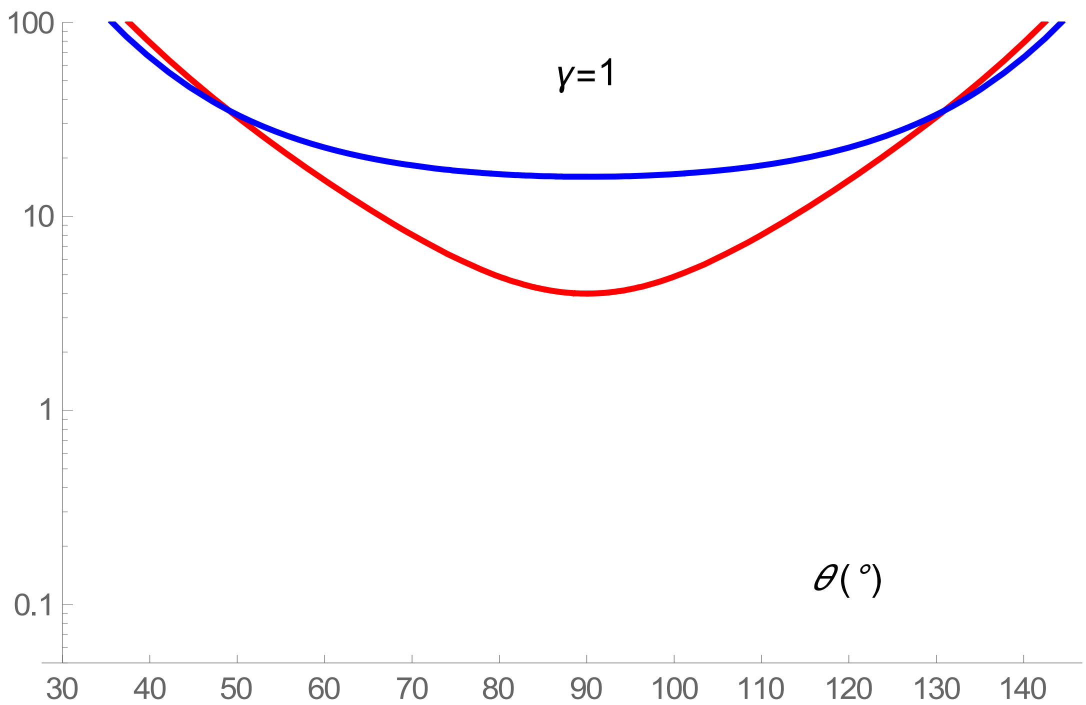
The differential cross section for the Coulomb scattering of two identical fermions with spin 1/2 is derived by Mott as

$$\frac{d\sigma}{d\Omega} = \left(\frac{\gamma}{2k}\right)^2 \{\csc^4\left(\frac{\theta}{2}\right) + \sec^4\left(\frac{\theta}{2}\right) - \csc^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right)\cos[2\gamma \ln(\tan\frac{\theta}{2})]\}$$

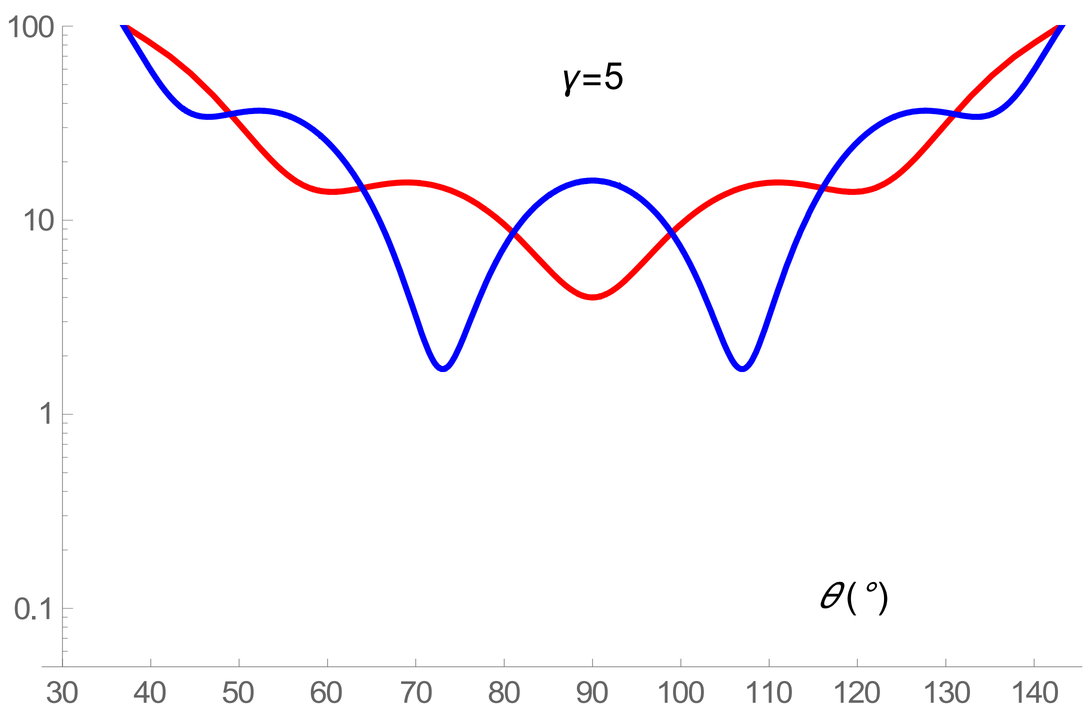
We make a plot of $\frac{d\sigma}{d\Omega}/A$ as a function of the scattering angle θ for identical two bosons, and identical two fermions, where the parameter γ is changed as a parameter.

Blue line (bosons)
Red line (fermion, unpolarized)

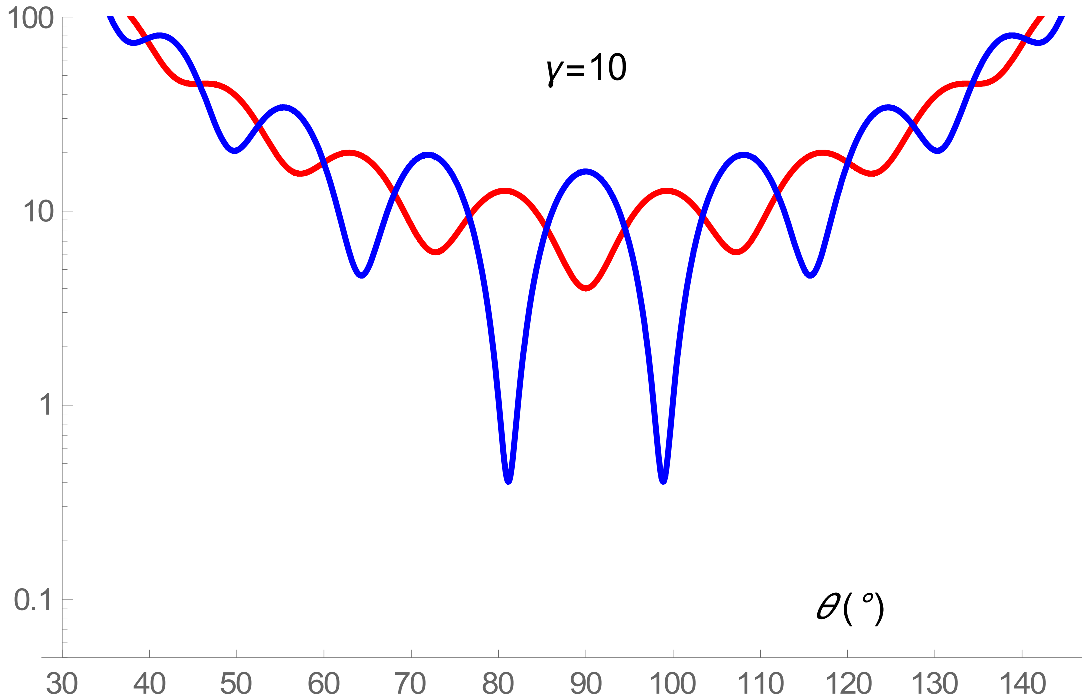
The scattering cross section shows a local maximum at $\theta = \frac{\pi}{2}$ for the two identical bosons, while it shows a local minimum at $\theta = \frac{\pi}{2}$ for the two identical fermions (unpolarized spins).



100
10
1
0.1
γ=1
θ(°)
30
40
50
60
70
80
90
100
110
120
130
140



100
10
1
0.1
γ=5
θ(°)
30
40
50
60
70
80
90
100
110
120
130
140



100
10
1
0.1
γ=10
θ(°)
30
40
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70
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100
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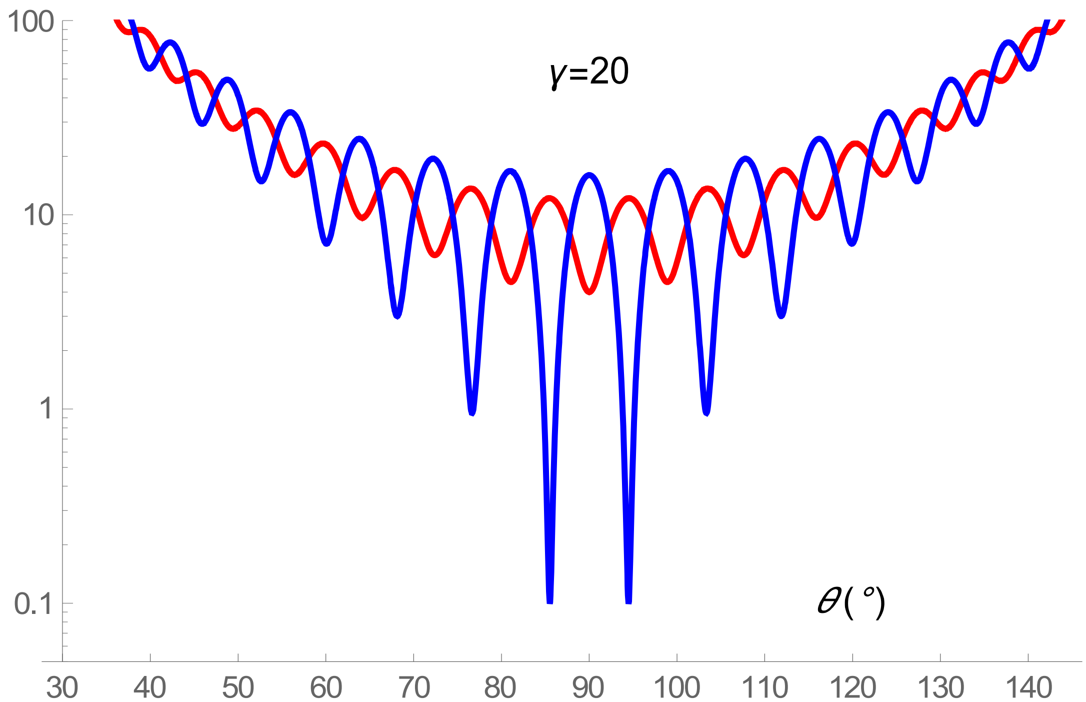

100
10
1
0.1
γ=20
θ(°)
30
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50
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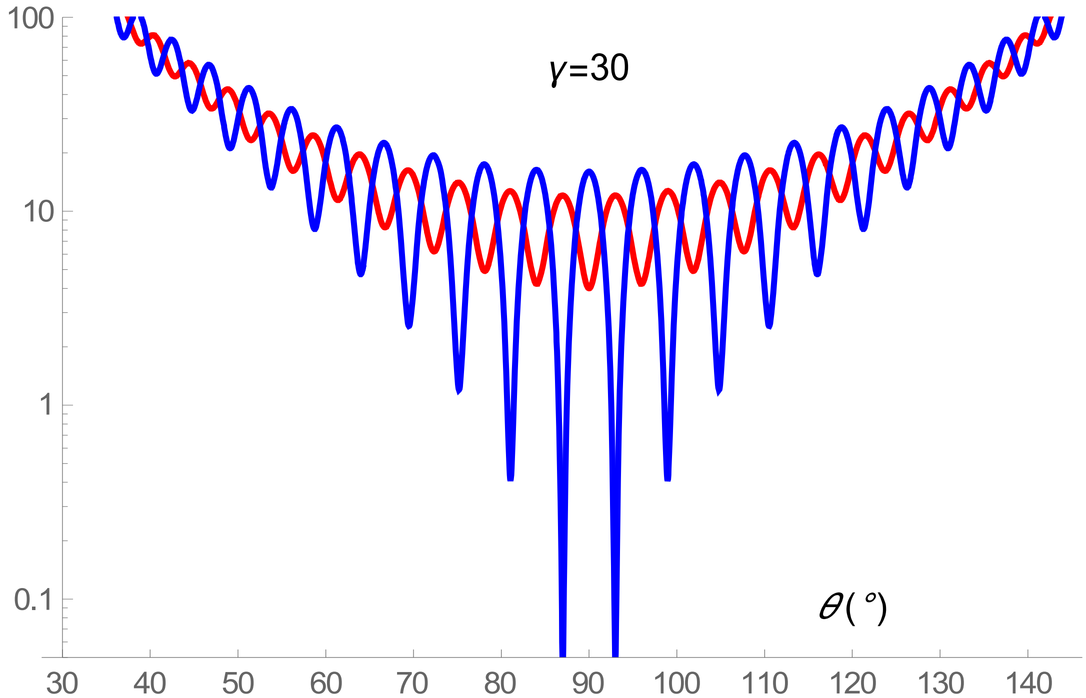
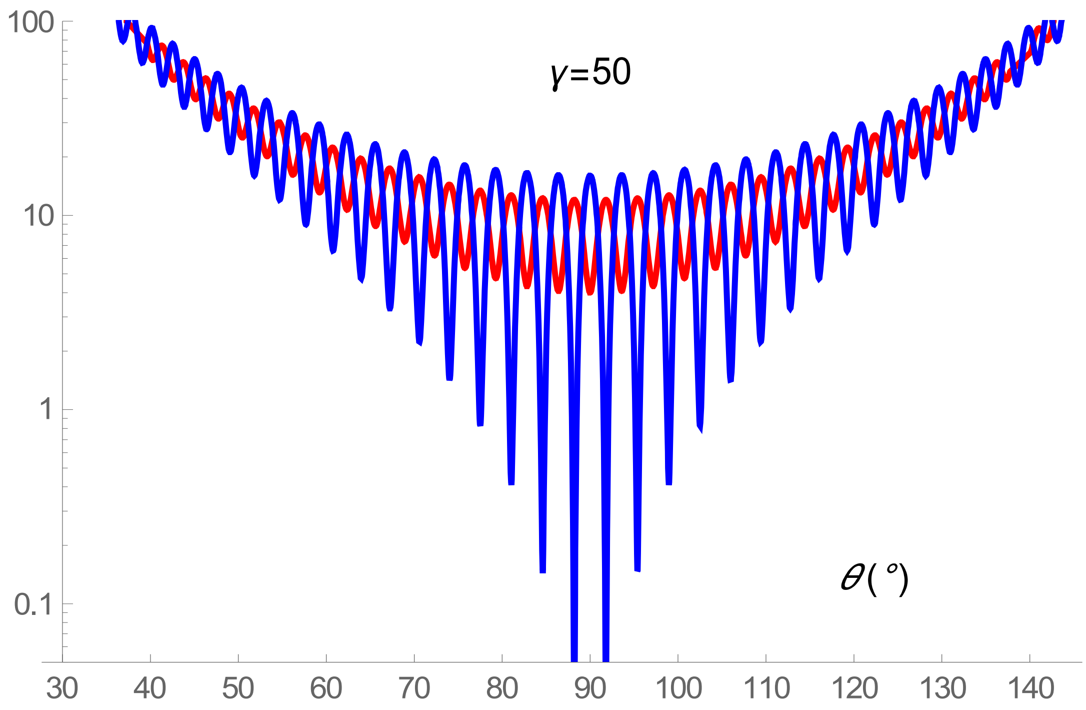
70
80
90
100
110
120
130
140





4. Experimental results

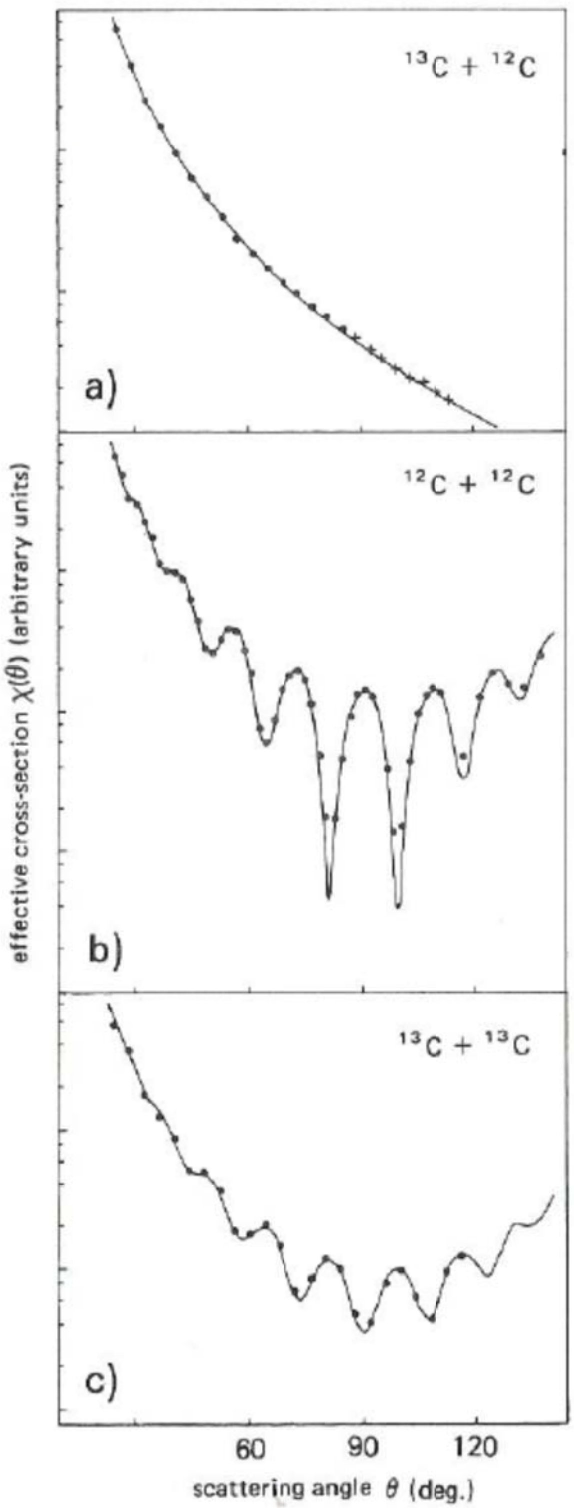


Fig. Coulomb differential cross sections of ^{12}C and ^{13}C carbon isotopes_ hence of nearly equal masses- each at nergy 2 MeV (J.-M. Levy-Leblond and F. Balibar).

(a) $^{12}C + {}^{13}C$: distinct nuclei.
(b) $^{12}C + {}^{12}C$: identical bosons.
(b) $^{13}C + {}^{13}C$: identical fermions.
The dots correspond to experimental measurements.

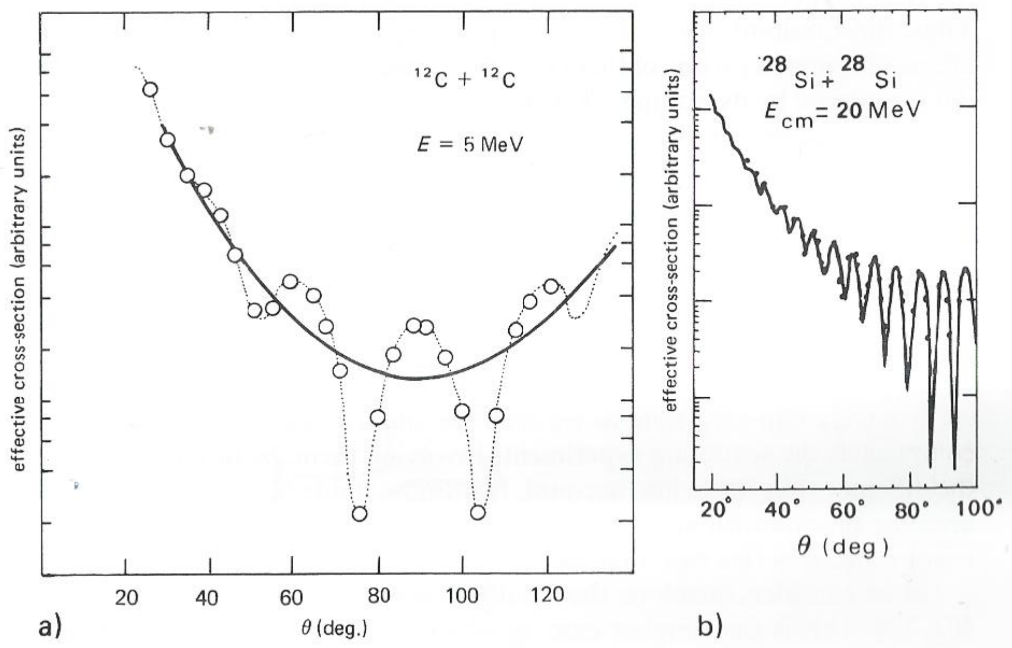


Fig. Coulomb scattering of two identical bosons. (a) Scattering of two ^{12}C nuclei of kinetic energy 5 MeV. (b) Scattering of two ^{28}Si nuclei having kinetic energy 20 MeV. The nuclear spin is zero for ^{28}Si (J.-M. Levy-Leblond and F. Balibar).

5. Summary

The generalization of the formula is

$$\frac{d\sigma}{d\Omega} = \left| f(\theta)^2 \right| + \left| f(\pi-\theta) \right|^2 + \frac{(-1)^{2s}}{2s+1} 2\,\mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

where s is a spin such that $s = 0$ (boson) and s = 1/2 (fermion).

In other words,

$$\frac{d\sigma}{d\Omega} = \left| f(\theta)^2 \right| + \left| f(\pi-\theta) \right|^2 + 2\,\mathrm{Re}[f(\theta) f^*(\pi-\theta)] \qquad \text{for s = 0 (boson)}$$

$$\frac{d\sigma}{d\Omega} = \left| f(\theta)^2 \right| + \left| f(\pi-\theta) \right|^2 - \mathrm{Re}[f(\theta) f^*(\pi-\theta)]$$

for s = 1/2 (fermion, unpolarized)

$$\left| f(\theta) \right|^2 = \left(\frac{\gamma}{2k} \right)^2 \csc^4\left(\frac{\theta}{2} \right)$$

$$\left| f(\pi-\theta) \right|^2 = \left(\frac{\gamma}{2k} \right)^2 \sec^4\left(\frac{\theta}{2} \right)$$

$$\mathrm{Re}[f(\theta) f^*(\pi-\theta)] = \left(\frac{\gamma}{2k} \right)^2 \csc^2\left(\frac{\theta}{2} \right) \sec^2\left(\frac{\theta}{2} \right) \cos[2\gamma \ln(\tan\frac{\theta}{2})]\}$$

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Institut fur Kernphysik der Universitat at zu Koln Praktikum M (Experiment No. M319) Coulomb scattering of identical particles, spin, and statistics *https://www.ikp.uni-koeln.de/students/fp/download/AnleitungVers19eng.pdf*

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