Second quantization in Relativistic Quantum Mechanics

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We use the Dirac spinors such that

$$u^{(R)} = \sqrt{\frac{R + mc^2}{2R}} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{R + mc^2} \\ \dfrac{c(p_x + ip_y)}{R + mc^2} \end{pmatrix}$$

$$u^{(L)} = \sqrt{\frac{R + mc^2}{2R}} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{R + mc^2} \\ -\dfrac{cp_z}{R + mc^2} \end{pmatrix}$$

$$v^{(R)} = \sqrt{\frac{R + mc^2}{2R}} \begin{pmatrix} -\dfrac{cp_z}{R + mc^2} \\ \dfrac{c(p_x + ip_y)}{R + mc^2} \\ 1 \\ 0 \end{pmatrix}$$

$$v^{(L)} = \sqrt{\frac{R + mc^2}{2R}} \begin{pmatrix} -\dfrac{c(p_x - ip_y)}{R + mc^2} \\ \dfrac{cp_z}{R + mc^2} \\ 0 \\ 1 \end{pmatrix}$$

where

$$u^{(R)^+}u^{(L)} = u^{(R)^+}v^{(R)} = u^{(R)^+}v^{(L)} = u^{(L)^+}v^{(R)} = u^{(L)^+}v^{(L)} = v^{(R)^+}v^{(L)} = 0$$

such that

$$u^{(R)^+}u^{(L)} = \left(\sqrt{\frac{R+mc^2}{2R}}\right)^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{R+mc^2} & \frac{c(p_x - ip_y)}{R+mc^2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{R+mc^2} \\ -\frac{cp_z}{R+mc^2} \end{pmatrix} = 0$$

$$u^{(R)^+}v^{(R)} = \left(\sqrt{\frac{R+mc^2}{2R}}\right)^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{R+mc^2} & \frac{c(p_x - ip_y)}{R+mc^2} \end{pmatrix} \begin{pmatrix} -\frac{cp_z}{R+mc^2} \\ \frac{c(p_x + ip_y)}{R+mc^2} \\ 1 \\ 0 \end{pmatrix} = 0$$

$$u^{(R)^+}v^{(L)} = \left(\sqrt{\frac{R+mc^2}{2R}}\right)^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{R+mc^2} & \frac{c(p_x - ip_y)}{R+mc^2} \end{pmatrix} \begin{pmatrix} -\frac{c(p_x - ip_y)}{R+mc^2} \\ \frac{cp_z}{R+mc^2} \\ 0 \\ 1 \end{pmatrix} = 0$$

and

$$u^{(R)^+}u^{(R)} = u^{(L)^+}u^{(L)} = v^{(R)^+}v^{(R)} = v^{(L)^+}v^{(L)} = 1,$$

such that

$$u^{(R)^+}u^{(R)} = \left(\sqrt{\frac{R+mc^2}{2R}}\right)^2 \begin{pmatrix} 1 & 0 & \frac{cp_z}{R+mc^2} & \frac{c(p_x - ip_y)}{R+mc^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{R+mc^2} \\ \frac{c(p_x + ip_y)}{R+mc^2} \end{pmatrix} = 1$$

We define a quantum field operator by using the four spinors,

$$\psi(\boldsymbol{r}) = \psi^{(+)}(\boldsymbol{r}) + \psi^{(-)}(\boldsymbol{r})$$

with

$$\psi^{(+)}(\boldsymbol{r}) = \frac{1}{(2\pi\hbar)^{3/2}}\int d\boldsymbol{p}[u^{R}(\boldsymbol{p})\hat{a}_{R}(\boldsymbol{p}) + u^{:L}(\boldsymbol{p})\hat{a}_{L}(\boldsymbol{p})]\exp(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r})$$

and

$$\begin{aligned}\psi^{(-)}(\boldsymbol{r}) &= \frac{1}{(2\pi\hbar)^{3/2}}\int d\boldsymbol{p}[v^{R}(-\boldsymbol{p})\hat{b}_{R}{}^{+}(\boldsymbol{p}) + v^{:L}(-\boldsymbol{p})\hat{b}_{L}{}^{+}(\boldsymbol{p})]\exp(-\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}) \\ &= \frac{1}{(2\pi\hbar)^{3/2}}\int d\boldsymbol{p}[v^{R}(\boldsymbol{p})\hat{b}_{R}{}^{+}(-\boldsymbol{p}) + v^{:L}(\boldsymbol{p})\hat{b}_{L}{}^{+}(-\boldsymbol{p})]\exp(\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r})\end{aligned}$$

Creation operator and annihilation operator for electrons;

$$\hat{a}, \qquad \hat{a}^{+}$$

Creation operator and annihilation operator for positrons

$$\hat{b}, \qquad \hat{b}^{+}$$

The anti-commutation relations:

$$\{\hat{a}_{R}(\boldsymbol{p}), \hat{a}_{R}{}^{+}(\boldsymbol{p}')\} = \{\hat{a}_{L}(\boldsymbol{p}), \hat{a}_{L}{}^{+}(\boldsymbol{p}')\} = \delta(\boldsymbol{p} - \boldsymbol{p}')$$

$$\{\hat{b}_{R}(\boldsymbol{p}), \hat{b}_{R}{}^{+}(\boldsymbol{p}')\} = \{\hat{b}_{L}(\boldsymbol{p}), \hat{b}_{L}{}^{+}(\boldsymbol{p}')\} = \delta(\boldsymbol{p} - \boldsymbol{p}')$$