Solution of Chapter 7 (Sakurai and Napolitano)
J.J. Sakurai and Jim Napolitano
Modern Quantum Mechanics, 2nd edition (Pearson, 2011).

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7.1

7.1 Liquid helium makes a transition to a macroscopic quantum fluid, called superfluid helium, when cooled below a phase-transition temperature $T = 2.17K$. Calculate the de Broglie wavelength $\lambda = h/p$ for helium atoms with average energy at this temperature, and compare it to the size of the atom itself. Use this to predict the superfluid transition temperature for other noble gases, and explain why none of them can form superfluids. (You will need to look up some empirical data for these elements.)

((**Solution**))
We assume that the kinetic energy is given by

$$E = \frac{p^2}{2M} = \frac{3}{2} k_B T$$

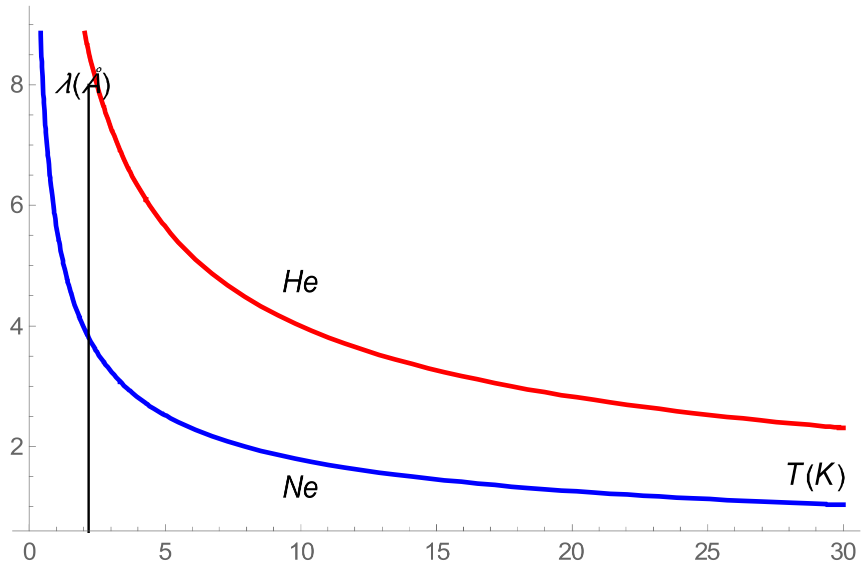
The momentum p is obtained as

$$p = \sqrt{3Mk_B T}$$

Then the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3Mk_B T}}$$

For He atom, $\lambda = 8.5729$ Å. This value of λ is much larger than the size of He atom (0.62 Å). In contrast, we make a plot of the de Broglie wavelength for He and Ne as a function of temperature. For Ne, the de Broglie wavelength is on the same order as the size of Ne atom (



From this figure, we have $T = 0.43$ K when the de Broglie length for Ne atom is equal to that that of He gas (8.5729 Å) at the critical temperature $T_l = 2.17$ K.

((**Mathematica**))

```
Clear["Global`*"];
rule1 = {kB → 1.3806504 × 10^-16, NA → 6.02214179 × 10^23,
   c → 2.99792 × 10^10, ℏ → 1.054571628 10^-27,
   me → 9.10938215 10^-28, mp → 1.672621637 × 10^-24,
   mn → 1.674927211 × 10^-24, qe → 4.8032068 × 10^-10,
   eV → 1.602176487 × 10^-12, Å → 10^-8,
   amu → 1.660538782 × 10^-24}; E1 = 3/2 kB T;
p1 = Sqrt[3 M1 kB T];
λ = (2 π ℏ)/p1;

MHe = 4.002602 amu; T1 = 2.17;
λHe = λ/Å /. {M1 → MHe} //. rule1
```

$$\frac{12.6287}{\sqrt{T}}$$

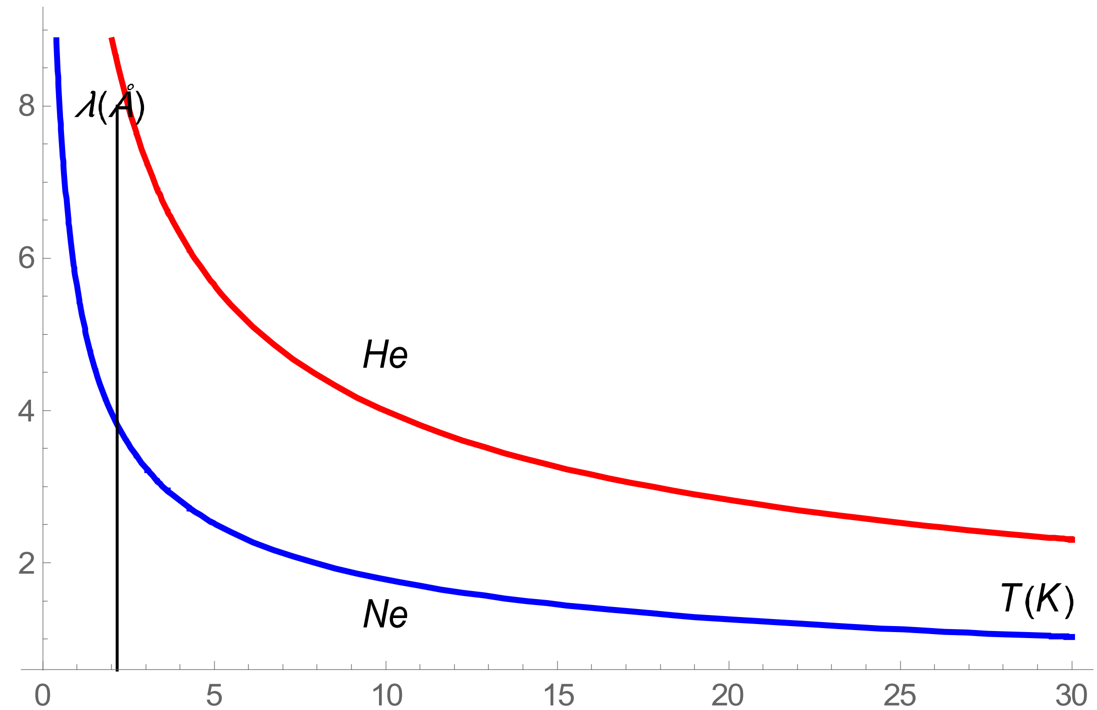
```
λHe /. T → T1
```

8.5729

```
MNe = 20.1797 amu;

λNe = λ/Å /. {M1 → MNe} //. rule1;

f1 = Plot[{λHe, λNe}, {T, 0, 30},
  PlotStyle → {{Red, Thick}, {Blue, Thick}}];
f2 =
 Graphics[
  {Text[Style["T(K)", Black, 12, Italic],
    {29, 1.5}], Text[Style["He", Black, 12, Italic],
    {10, 4.7}], Text[Style["Ne", Black, 12, Italic],
    {10, 1.3}],
   Text[Style["λ(Å)", Black, 12, Italic], {2, 8}],
   Line[{{2.17, 0}, {2.17, 8}}]}];
Show[f1, f2]
```



7.2

7.2 (a) N identical spin $\frac{1}{2}$ particles are subjected to a one-dimensional simple harmonic-oscillator potential. Ignore any mutual interactions between the particles. What is the ground-state energy? What is the Fermi energy?

(b) What are the ground-state and Fermi energies if we ignore the mutual interactions and assume N to be very large?

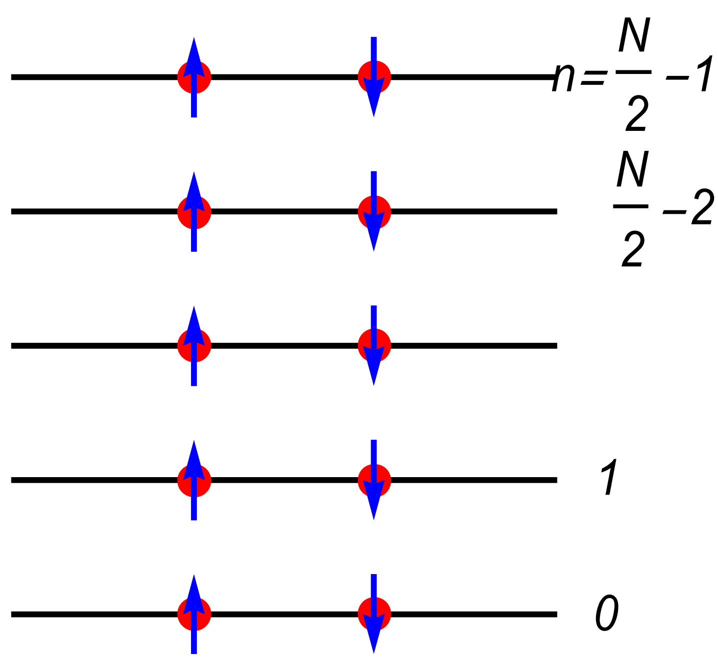
((**Solution**))

(a)

E_0: Ground state energy

E_f: Fermi energy

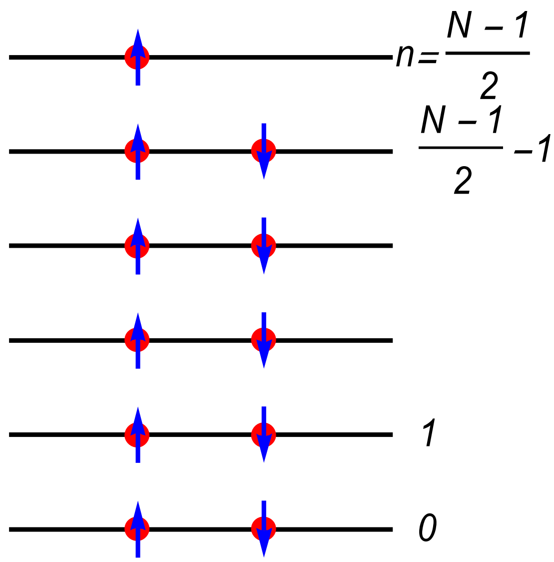
(i) For even N,



$$E_f = \left(\frac{1}{2} + \frac{N}{2} - 1\right)\hbar\omega = \frac{N-1}{2}\hbar\omega$$

$$E_0 = 2\times\sum_{n=0}^{N/2-1}(n+\frac{1}{2})\hbar\omega$$
$$= \frac{N}{2}(\frac{N}{2}-1)\hbar\omega + \frac{N}{2}\hbar\omega$$
$$= \frac{1}{4}N^2\omega$$

(ii) For odd N



$$E_f = (\frac{1}{2}+\frac{N-1}{2})\hbar\omega = \frac{N}{2}\hbar\omega$$

$$E_0 = 2\times\sum_{n=0}^{(N-3)/2}(n+\frac{1}{2})\hbar\omega + \hbar\omega(1+\frac{N-1}{2})$$
$$= \frac{1}{4}(N-1)^2\hbar\omega + \frac{N}{2}\hbar\omega$$
$$= \frac{1}{4}(N^2+1)\hbar\omega$$

(b) In the limit of $N\to\infty$

$$E_F = \frac{1}{2}N\hbar\omega, \qquad E_0 = \frac{1}{4}N^2\hbar\omega$$

7.3

7.3 It is obvious that two nonidentical spin 1 particles with no orbital angular momenta (that is, s-states for both) can form $j = 0$, $j = 1$, and $j = 2$. Suppose, however, that the two particles are *identical*. What restrictions do we get?

((**Solution**))
We suppose that two particles (S = 1) are identical. The particle with S = 1 is boson

$$|\psi\rangle = |\psi_{orbital}\rangle|\chi_{spin}\rangle$$

$|\psi\rangle$ is symmetric with respect to interchange of two particles.
$|\psi_{orbital}\rangle$ is symmetric because these two particles have no orbital angular momentum.
Therefore $|\chi_{spin}\rangle$ should be symmetric.

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

$J = 2$ (symmetric)

$$|j = 2, m = 2\rangle = |1,1\rangle$$
$$|2,1\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle]$$
$$|2,0\rangle = \frac{1}{\sqrt{6}}(|1,-1\rangle + |-1,1\rangle + +2|0,0\rangle)$$
$$|2,-1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle + |-1,0\rangle]$$
$$|2,-2\rangle = |-1,-1\rangle$$

J=0 (antisymmetric)

$$|j = 1, m = 1\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle]$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle)$$
$$|1,-1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle - |-1,0\rangle]$$

J=0 (symmetric)

$$\left| j=0, m=0 \right\rangle = \frac{1}{\sqrt{3}}(\left|1,-1\right\rangle + \left|-1,1\right\rangle - \left|0,0\right\rangle)$$

Therefore, J =2 and J = 0, which is symmetric with respect to the interchange of two particles, are allowed.

7.4

7.4 Discuss what would happen to the energy levels of a helium atom if the electron were a spinless boson. Be as quantitative as you can.

((**Solution**))

If the electron is a boson, not having spin, the total wave function should be symmetric for the interchange of two electrons. There are two cases.

(1) Triplet spin states (symmetric, Ortho) and symmetric spatial wave function.

Ground state: $\left|1s\right\rangle_1 \left|1s\right\rangle_2$,

Excited state: $\left|1s\right\rangle_1 \left|2s\right\rangle_2 + \left|2s\right\rangle_1 \left|1s\right\rangle_2$

(2) Singlet spin state (antisymmetric, Para) and antisymmetric spatial wave functions.

Excited state: $\left|1s\right\rangle_1 \left|2s\right\rangle_2 - \left|2s\right\rangle_1 \left|1s\right\rangle_2$

Then we have the matrix of $\hat{V}$ under the basis of $\{\left|1s, 2s\right\rangle, \left|2s, 1s\right\rangle\}$ as

$$\hat{V} = \begin{pmatrix} J & K \\ K & J \end{pmatrix} = J\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + K\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = J\hat{1} + K\hat{\sigma}_x$$

where

$$J = \left\langle 1s, 2s \right| \hat{V} \left| 1s, 2s \right\rangle$$

$$K = \langle 1s, 2s|\hat{V}|2s, 1s\rangle$$

The eigenvalue problem:

$$\hat{V}|+x\rangle = (J\hat{1} + K\hat{\sigma}_x)|+x\rangle = (J+K)|+x\rangle$$

where

$$|+x\rangle = \frac{1}{\sqrt{2}}[|1s, 2s\rangle + |2s, 1s\rangle]$$ (symmetric orbital state;)

with the eigenvalue $J + K$

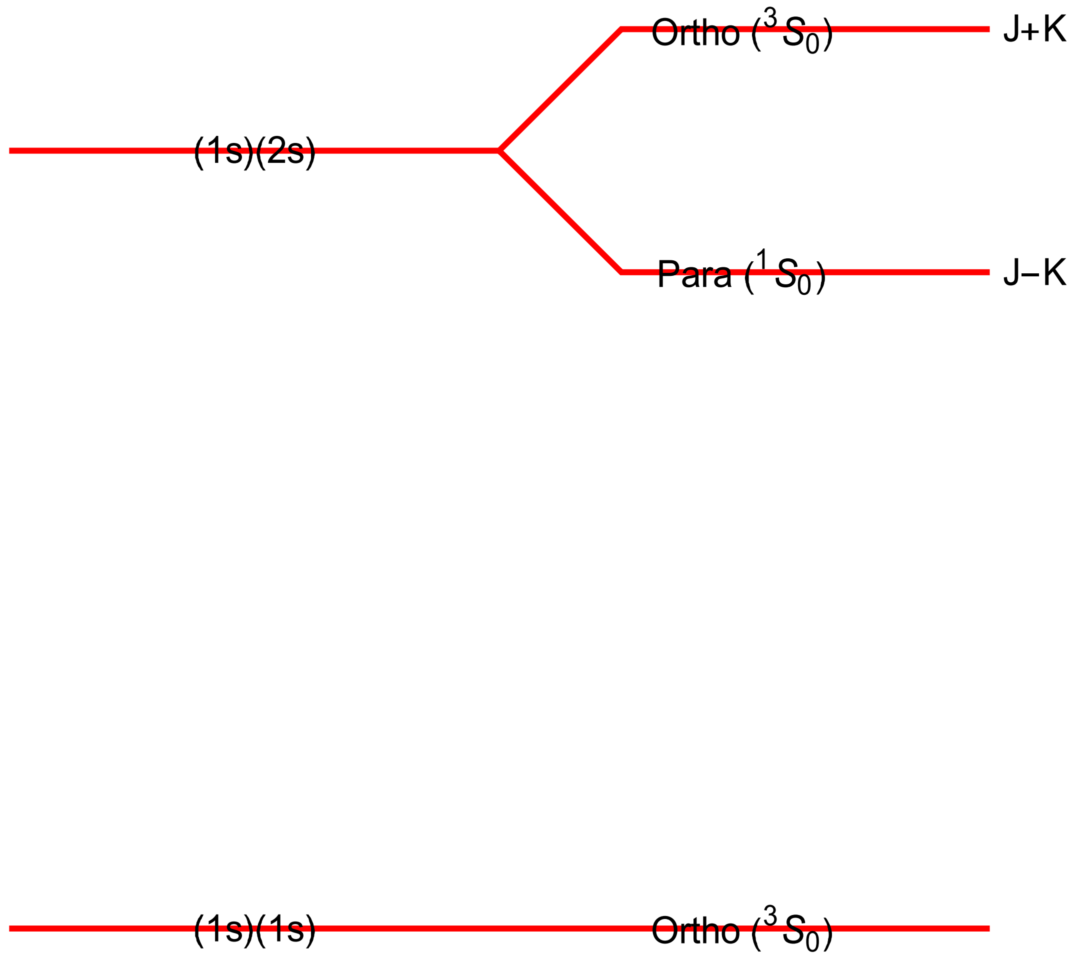
$$\hat{V}|-x\rangle = (J\hat{1} + K\hat{\sigma}_x)|-x\rangle = (J-K)|+x\rangle$$

where

$$|-x\rangle = \frac{1}{\sqrt{2}}[|1s, 2s\rangle - |2s, 1s\rangle]$$ (antisymmetric orbital state)

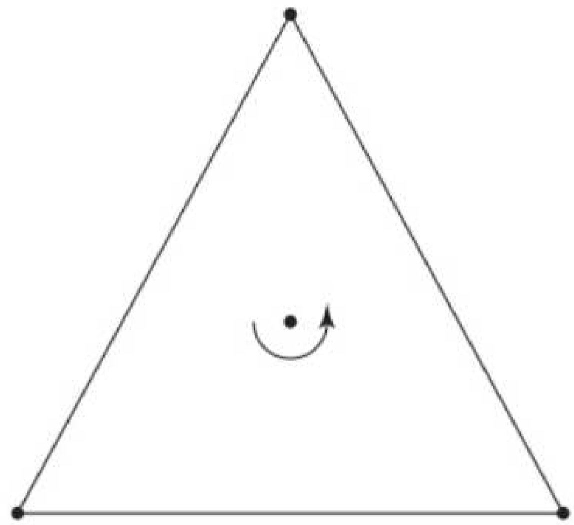
with the eigenvalue $J - K$

((Energy diagram))



7.5

7.5 Three spin 0 particles are situated at the corners of an equilateral triangle (see the accompanying figure). Let us define the z-axis to go through the center and in the direction normal to the plane of the triangle. The whole system is free to rotate about the z-axis. Using statistics considerations, obtain restrictions on the magnetic quantum numbers corresponding to J_z.



((**Solution**))

$$|\psi\rangle = |j,m\rangle$$

$$\begin{aligned}|\psi'\rangle &= \exp(-\frac{i}{\hbar}\hat{J}_z\theta)|\psi\rangle \\ &= \exp(-\frac{i}{\hbar}m\hbar\theta)|\psi\rangle \\ &= \exp(-im\theta)|\psi\rangle\end{aligned}$$

When $\theta = \dfrac{2\pi}{3}$

$$|\psi'\rangle = |j,m\rangle$$

because of the symmetric wave function (boson).

$$e^{-im\frac{2\pi}{3}} = 1$$

So we have

$$m = 3n \qquad (n\text{: integer})$$

7.6

7.6 Consider three weakly interacting, identical spin 1 particles.

(a) Suppose the space part of the state vector is known to be symmetrical under interchange of *any* pair. Using notation $|+\rangle|0\rangle|+\rangle$ for particle 1 in $m_s = +1$, particle 2 in $m_s = 0$, particle 3 in $m_s = +1$, and so on, construct the normalized spin states in the following three cases:

(i) All three of them in $|+\rangle$.

(ii) Two of them in $|+\rangle$, one in $|0\rangle$.

(iii) All three in different spin states.

What is the total spin in each case?

(b) Attempt to do the same problem when the space part is antisymmetrical under interchange of any pair.

((**Solution**))

Since the particle has a spin 1, $|\psi\rangle$ is symmetric (boson)

$$|\psi\rangle = |\psi_{space}\rangle|\chi_{spin}\rangle$$

(a)

Suppose that $|\psi_{space}\rangle$ is assumed to be symmetric. Then $|\chi_{spin}\rangle$ should be symmetric.

$$\begin{aligned}(D_1 \times D_1) \times D_1 &= (D_2 + D_1 + D_0) \times D_1 \\ &= (D_2 \times D_1) + (D_1 \times D_1) + (D_0 \times D_1)\end{aligned}$$

(i) All three of them in $|+\rangle = |m_s = 1\rangle$

$$|\alpha\rangle = |+\rangle|+\rangle|+\rangle$$

(ii) Two of them in $|+\rangle = |m_s = 1\rangle$. and one in $|0\rangle = |m_s = 0\rangle$

$$|\beta\rangle = \frac{1}{\sqrt{3}}[|0\rangle|+\rangle|+\rangle + |+\rangle|0\rangle|+\rangle + |+\rangle|+\rangle|0\rangle]$$

(iii) All three in different spin states

$$|\gamma\rangle = \frac{1}{\sqrt{6}}[|0\rangle|+\rangle|-\rangle + |+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle$$
$$+|0\rangle|-\rangle|+\rangle + |-\rangle|0\rangle|+\rangle + |-\rangle|+\rangle|0\rangle]$$

$$\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} + \hat{J}_{3z}$$

$$\hat{\boldsymbol{J}}^2 = (\hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_1 + \hat{\boldsymbol{J}}_3)^2 = \hat{\boldsymbol{J}}_1{}^2 + \hat{\boldsymbol{J}}^2{}_1 + \hat{\boldsymbol{J}}_3{}^3$$
$$+(\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+} + 2\hat{J}_{1z}\hat{J}_{2z})$$
$$+(\hat{J}_{2+}\hat{J}_{3-} + \hat{J}_{2-}\hat{J}_{3+} + 2\hat{J}_{2z}\hat{J}_{3z})$$
$$+(\hat{J}_{3+}\hat{J}_{1-} + \hat{J}_{3-}\hat{J}_{1+} + 2\hat{J}_{3z}\hat{J}_{1z})$$

(i) $\hat{J}_z|\alpha\rangle = 3\hbar|\alpha\rangle$, $\hat{\boldsymbol{J}}^2|\alpha\rangle = 12\hbar^2|\alpha\rangle$
leading to $j = 3$ and $m = 3$.

(ii) $\hat{J}_z|\beta\rangle = 2\hbar|\beta\rangle$, $\hat{\boldsymbol{J}}^2|\beta\rangle = 12\hbar^2|\beta\rangle$
leading to $j = 3$ and $m = 2$.

(iii) $\hat{J}_z|\gamma\rangle = 0\hbar|\gamma\rangle$, $\hat{\boldsymbol{J}}^2|\gamma\rangle = 8\hbar^2|\gamma\rangle + \sqrt{6}\hbar^2|0\rangle|0\rangle|0\rangle$
$m = 0$ but $|\gamma\rangle$ is not an eigenstate of $\hat{\boldsymbol{J}}^2$

(b) Suppose that $|\psi_{space}\rangle$ is assumed to be antisymmetric. Then $|\chi_{spin}\rangle$ should be antisymmetric. All states should be different.

For (i) and (ii), it is impossible to construct the spin states.

For (iii)

$$|\delta\rangle = \frac{1}{\sqrt{6}}\begin{vmatrix} |+\rangle_1 & |0\rangle_1 & |-\rangle_1 \\ |+\rangle_2 & |0\rangle_2 & |-\rangle_2 \\ |+\rangle_3 & |0\rangle_3 & |-\rangle_3 \end{vmatrix}$$

$\hat{J}_z|\delta\rangle = 0$, $\hat{\boldsymbol{J}}^2|\delta\rangle = 0$

leading to $j = 0$ and $m = 0$.

7.7

7.7 Show that, for an operator a that, with its adjoint, obeys the anticommutation relation $\{a, a^\dagger\} = aa^\dagger + a^\dagger a = 1$, the operator $N = a^\dagger a$ has eigenstates with the eigenvalues 0 and 1.

((**Solution**))

$$\hat{N} = \hat{a}^+ \hat{a}$$

The commutation relation: $\{\hat{a}, \hat{a}^+\} = \hat{1}$

$$\hat{a}\hat{a}^+ = \hat{1} - \hat{N}$$

We assume that

$$\hat{N}|\eta\rangle = \eta|\eta\rangle$$

$|\eta\rangle$ is the eigenket of $\hat{N}$ with the eigenvalue η. Since

$$\langle\eta|\hat{N}|\eta\rangle = \langle\eta|\hat{a}^+\hat{a}|\eta\rangle = \eta\langle\eta|\eta\rangle = \langle\alpha|\alpha\rangle \geq 0$$

where

$$\hat{a}|\eta\rangle = |\alpha\rangle$$

Thus we have $\eta \geq 0$. We note that

$$\hat{N}\hat{a}^+|\eta\rangle = \hat{a}^+\hat{a}\hat{a}^+|\eta\rangle = \hat{a}^+(\hat{1} - \hat{N}|\eta\rangle = (1-\eta)\hat{a}^+|\eta\rangle$$

which means that $\hat{a}^+|\eta\rangle$ is the eigenket of $\hat{N}$ with the eigenvalue $(1-\eta)$. Thus we have

$1-\eta \geq 0$. The eigenvalue η is between 0 and 1.

7.8

7.8 Suppose the electron were a spin-$\frac{3}{2}$ particle obeying Fermi-Dirac statistics. Write the configuration of a hypothetical Ne ($Z = 10$) atom made up of such "electrons" [that is, the analog of $(1s)^2(2s)^2(2p)^6$]. Show that the configuration is highly degenerate. What is the ground state (the lowest term) of the hypothetical Ne atom in spectroscopic notation ($^{2S+1}L_J$, where S, L, and J stand for the total spin, the total orbital angular momentum, and the total angular momentum, respectively) when exchange splitting and spin-orbit splitting are taken into account?

((**Solution**))
Ne atom Z = 10

(i)
First we consider a spin 1/2 particle obeying Fermi-Dirac statistics.
s = 1/2 (↑,↓)
multiplicity due to spin: 2s+1=2

(1s)

$l = 0$	($2l$+1 = 1)		
s = 1/2	(2s+1 = 2)	1x2=2	$(1s)^2$

(2s)

$l = 0$	($2l$+1 = 1)		
s = 1/2	(2s+1 = 2)	1x2=2	$(1s)^2$

(2p)

$l = 1$	($2l$+1 = 3)		
s = 1/2	(2s+1 = 2)	3x2=6	$(2p)^6$

Therefore, the ground state configuration is $(1s)^2(2s)^2(2p)^6$.

(ii)
Next we assume that an electron is a spin 3/2 particle, obeying the Fermi-Dirac ststistics.

(1s)
$l = 0$ ($2l$+1 = 1)

$s = 3/2$ $(2s+1 = 4)$ 1x4=4 $(1s)^4$

(2s)

$l = 0$ $(2l+1 = 1)$

$s = 1/2$ $(2s+1 = 4)$ 1x4=4 $(1s)^4$

(2p)

$l = 1$ $(2l+1 = 3)$

$s = 3/2$ $(2s+1 = 4)$ 3x4=12 $(2p)^{12}$

Therefore, the ground state configuration is

$$(1s)^4(2s)^4(2p)^2$$

This configuration is highly degenerate because only two electrons are occupied in the twelve states.

$${}_{12}C_2 = 66 \qquad \text{degeneracy}$$

This electron (s = 3/2) obey a FD statistics.

$$\left|\psi\right\rangle = \left|\psi_{orbital}\right\rangle\left|\psi_{spin}\right\rangle$$

is antisymmetric under the exchange of particles.

$\left|\psi_{orbital}\right\rangle$: symmetric, $\left|\psi_{spin}\right\rangle$: antisymmetric

$\left|\psi_{orbital}\right\rangle$: antisymmetric, $\left|\psi_{spin}\right\rangle$: symmetric

Here we consider the two particles with $l = 1$ and $s = 3/2$.

((Orbital degeneracy))

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

with

D_2: $(L = 2)$; symmetric

D_1: (L = 1);anti symmetric

D_0: (L = 0); symmetric

((Spin degeneracy))

$$D_{3/2} \times D_{3/2} = D_3 + D_2 + D_1 + D_0$$

D_3: (S = 3); symmetric

D_2: (S = 2); anti symmetric

D_1: (S = 1); symmetric

D_0: (S = 0); symmetric

(iii) Total angular momentum

We now consider the combination of L and S.

For L = 2 (symmetric) and S = 2 (antisymmetric)

$D_2 \times D_2 = D_4 + D_3 + D_2 + D_1 + D_0$, leading to

J = 4, 3, 2, 1, and 0

$^{2S+1=5}D_{J=4}$, $^5D_3$, $^5D_2$, $^5D_1$, $^5D_0$,

For L = 2 (symmetric) and S = 0 (antisymmetric)

$D_2 \times D_0 = D_2$, leading to

J = 2 ($^1D_2$)

For L = 1 (antisymmetric) and S = 3 (symmetric)

J = 4, 3, 2 ($^7P_4$, $^7P_3$, $^7P_2$)

For L = 1 (symmetric), S = 1 (symmetric)

J = 2, 1, 0 ($^3P_2$, $^3P_1$, $^3P_0$)

For $L = 0$ (symmetric), $S = 2$ (antisymmetric

$$J = 2 \qquad ({}^5S_2).$$

For $L = 0$ (symmetric), $S = 0$

$$J = 0 \qquad ({}^1S_0).$$

Next we consider the lowest state for the exchange splitting and orbit-spin interaction.

For the exchange splitting, the energy is low for the wave functions having the antisymmetric: antisymmetric spatial part, leading to $L = 1$.
For the spin-orbit coupling,

$$2\boldsymbol{L}\cdot\boldsymbol{S} = \boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2 = \hbar^2[J(J+1) - L(L+1) - S(S+1)]$$

The energy is lower for small J and large S.
A possibility is that $L = 1$, $S = 3$, and $J = 2$.

Then we have ${}^7P_2$ (minimum energy)

((**Note**))

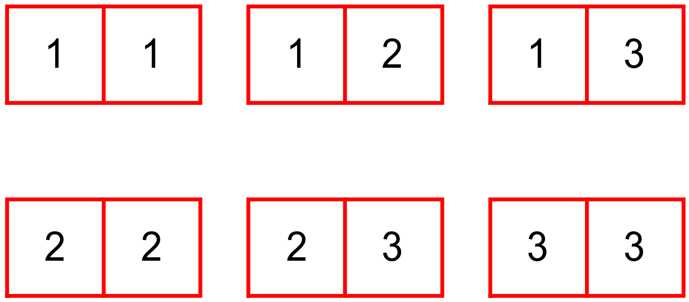
$(2p)^2$

$L = 1$. s = 3/2.

((**Space symmetry**))

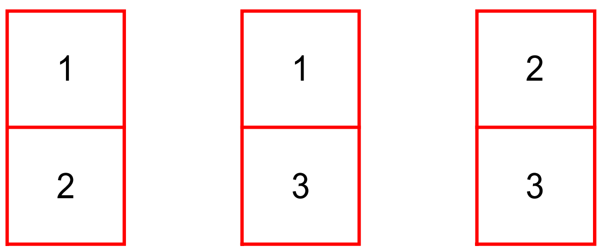
$$D_1 \times D_1 = D_2 + D_1 + D_0$$

(a) D_2 ($L = 2$; symmetric), D_0 ($L = 0$; symmetric),



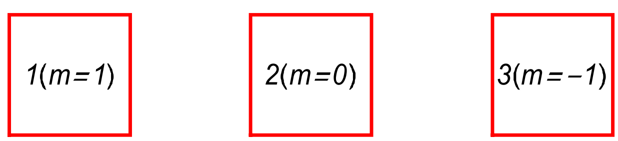
m = 2, 1, 0, 0, -1, and -2

(b) D_1 (anti-symmetric); $L = 1$



m =1 m = 0 m = -1

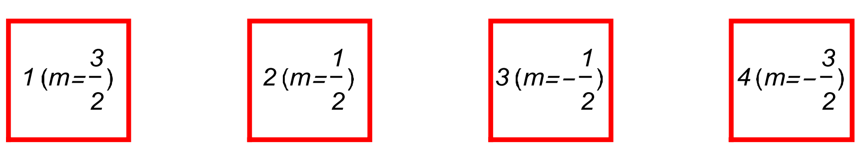
where



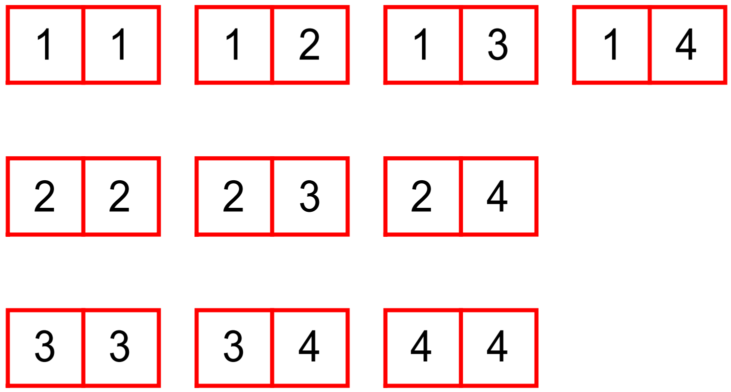
((**Spin-symmetry**))

$$D_{3/2} \times D_{3/2} = D_3 + D_2 + D_1 + D_0$$

with

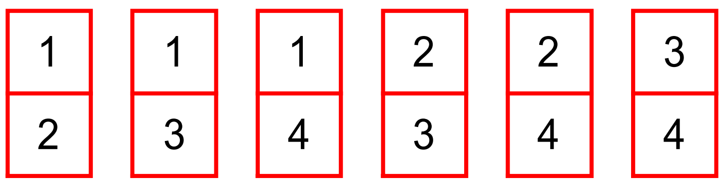


((**Symmetric state**))



m = 3, 2, 1, 0, 1, 0, -1, -1, -2, -3 (S = 3 and 1)

((**Anti-symmetric state**))



m = 2, 1, 0, 0, -1, -2 (S = 2 and 0)

7.9

7.9 Two identical spin $\frac{1}{2}$ fermions move in one dimension under the influence of the infinite-wall potential $V = \infty$ for $x < 0$, $x > L$, and $V = 0$ for $0 \le x \le L$.

(a) Write the ground-state wave function and the ground-state energy when the two particles are constrained to a triplet spin state (ortho state).

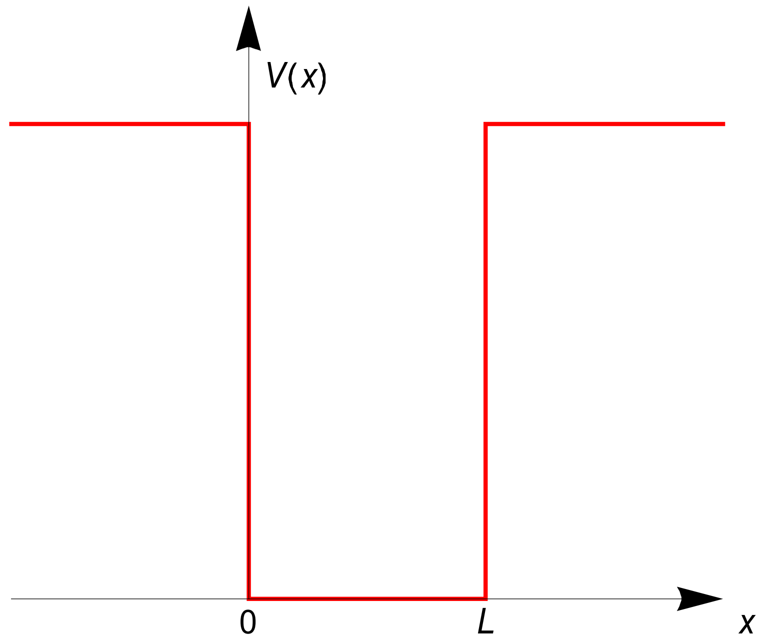
(b) Repeat (a) when they are in a singlet spin state (para state).

(c) Let us now suppose that the two particles interact mutually via a very short-range attractive potential that can be approximated by

$$V = -\lambda\delta(x_1 - x_2) \quad (\lambda > 0).$$

Assuming that perturbation theory is valid even with such a singular potential, discuss semiquantitatively what happens to the energy levels obtained in (a) and (b).

((**Solution**))



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

with

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\frac{d^2}{dx^2}\psi(x) + k^2\psi(x) = 0$$

with

$$\psi(0) = \psi(L) = 0$$

Then we have

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$$

with

$$E_n(k) = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$n = 1, 2, 3, \ldots$

Since S = 1/2 (fermion),

$|\psi\rangle = |\psi_{space}\rangle|\chi_{spin}\rangle$ is antisymmetric under the exchange of pairs.

(a)

$|\chi_{spin}\rangle$ is symmetric (spin triplet). $|\psi_{space}\rangle$ is antisymmetric. The ground state:

$$|\psi_{space}^{anti}\rangle = \frac{1}{\sqrt{2}}[|1\rangle|2\rangle - |2\rangle|1\rangle]$$

with

$$\langle x_1|1\rangle = \sqrt{\frac{2}{L}}\sin(\frac{\pi x_1}{L}), \qquad \langle x_1|2\rangle = \sqrt{\frac{2}{L}}\sin(\frac{2\pi x_2}{L})$$

$$E = \frac{\hbar^2\pi^2}{2mL^2}(1^2 + 2^2) = \frac{5\hbar^2\pi^2}{2mL^2}$$

$$\begin{aligned}\langle x_1, x_2|\psi_{space}^{anti}\rangle &= \frac{1}{\sqrt{2}}[\langle x_1|1\rangle\langle x_2|2\rangle - \langle x_1|2\rangle\langle x_2|1\rangle \\ &= \frac{1}{L}[\sin(\frac{\pi x_1}{L})\sin(\frac{2\pi x_2}{L}) - \sin(\frac{2\pi x_1}{L})\sin(\frac{\pi x_2}{L})]\end{aligned}$$

(b)

$|\chi_{spin}\rangle$ is antisymmetric (spin singlet). $|\psi_{space}\rangle$ is symmetric. The ground state:
Ground state:

$$|\psi_{space}^{symm}\rangle = |1\rangle|1\rangle$$

with

$$E = \frac{\hbar^2\pi^2}{2mL^2}(1^2 + 1^2) = \frac{\hbar^2\pi^2}{mL^2}$$

$$\left\langle x_1, x_2 \middle| \psi_{space}^{symm} \right\rangle = \langle x_1 | 1 \rangle \langle x_2 | 1 \rangle$$
$$= \frac{2}{L}\sin(\frac{\pi x_1}{L})\sin(\frac{\pi x_2}{L})$$

(c)

$$\Delta E = \left\langle \text{ground state} \middle| \hat{V} \middle| \text{ground state} \right\rangle$$

For the case (a), the wave function of ground state is antisymmetric. This means that two particles are not located in the same site. Therefore,

$$\Delta E = 0,$$

because of $V = -\lambda\delta(x_1 - x_2)$.

For the case (b), the wave function of the ground state is symmetric, indicating that two particles are located on the same place. Therefore the energy is lowered due to an attractive potential.

$$\Delta E = -\lambda \int dx_1 \int dx_2 \left(\frac{2}{L}\right)^2 \sin^2(\frac{\pi x_1}{L})\sin^2(\frac{\pi x_2}{L})\delta(x_1 - x_2)$$
$$= -\frac{4\lambda}{L^2}\int_0^L dx_1 \sin^4(\frac{\pi x_1}{L})$$
$$= -\frac{4\lambda}{L^2}\frac{3L}{8}$$
$$= -\frac{3\lambda}{2L}$$

((Note))

$$\int_0^L \text{Sin}\left[\frac{\pi x}{L}\right]^4 dx$$

$$\frac{3L}{8}$$

7.10

7.10 Prove the relations (7.6.11), and then carry through the calculation to derive (7.6.17).

((From Pearson)) by J.J. Sakurai and Jim Napolitano

10. To prove the orthogonality relations (7.6.11), start with the definitions

$$
\begin{aligned}
\hat{\mathbf{e}}_{\mathbf{k}\pm} &= \mp\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \pm i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right) \qquad \text{so we have} \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right] \cdot \left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \cdot \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[\pm 1 + 0 - 0 \pm \lambda\lambda'\right] \\
&= \pm 1 \qquad \text{if}\lambda = \lambda' \\
&= 0 \qquad \text{if}\lambda \neq \lambda' \\
\hat{\mathbf{e}}_{\mathbf{k}\lambda}^{*} \times \hat{\mathbf{e}}_{\pm\mathbf{k}\lambda'} &= \left[-\lambda\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} - \lambda i\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}\right)\right] \times \left[-\lambda'\frac{1}{\sqrt{2}}\left(\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda' i\hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right)\right] \\
&= \frac{\lambda\lambda'}{2}\left[\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + i\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)} - i\lambda\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(1)} + \lambda\lambda'\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \times \hat{\mathbf{e}}_{\pm\mathbf{k}}^{(2)}\right] \\
&= \frac{\lambda\lambda'}{2}\left[0 \pm i\lambda'\hat{\mathbf{k}} \pm i\lambda\hat{\mathbf{k}} + 0\right] \\
&= \pm i\hat{\mathbf{k}} \qquad \text{if}\lambda = \lambda' \\
&= 0 \qquad \text{if}\lambda \neq \lambda'
\end{aligned}
$$

The first result (7.6.11a) serves to collapse the two sums over λ and λ' into one, when calculating $|\mathbf{E}|^2 = \mathbf{E}^* \cdot \mathbf{E}$ from (7.6.14), and the integral (7.6.15) collapses the two sums over $\mathbf{k}$ and $\mathbf{k}'$ into one, leading to (7.6.16). The expression for the magnetic field is

$$\mathbf{B}(\mathbf{x},t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{x},t) = \frac{i}{c} \sum_{\mathbf{k},\lambda} \omega_k \left[\mathbf{A}_{\mathbf{k},\lambda} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - \mathbf{A}^*_{\mathbf{k},\lambda} e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \right] \hat{\mathbf{k}} \times \hat{\mathbf{e}}^{(\lambda)}_{\mathbf{k}}$$

which is very similar to (7.6.14), differing by the presence of $\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k},\lambda}$ instead of $\hat{\mathbf{e}}_{\mathbf{k},\lambda}$. But

$$\hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k},\pm} = -\mp \frac{1}{\sqrt{2}} \left[\hat{\mathbf{k}} \times \hat{\mathbf{e}}^{(1)}_{\mathbf{k}} \pm i \hat{\mathbf{k}} \times \hat{\mathbf{e}}^{(2)}_{\mathbf{k}} \right] = -\mp \frac{1}{\sqrt{2}} \left[\hat{\mathbf{e}}^{(2)}_{\mathbf{k}} \mp i \hat{\mathbf{e}}^{(1)}_{\mathbf{k}} \right] = i \hat{\mathbf{e}}_{\mathbf{k},\pm}$$

so that the calculation of $|\mathbf{B}|^2 = \mathbf{B}^* \cdot \mathbf{B}$ carries through directly as for the electric field. The cross terms, however, have opposite sign, and therefore cancel when adding the contributions to the energy from electric and magnetic fields, leading to (7.6.17).