Solution of Chapter 6 (Sakurai and Napolitano)

J.J. Sakurai and Jim Napolitano, Modern Quantum Mechanics, 2nd edition (Pearson 2011)

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6-1

6.1 The Lippmann-Schwinger formalism can also be applied to a *one*-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

(a) Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \to E + i\varepsilon$ prescription still correct? Obtain an expression for the appropriate Green's function and write an integral equation for $\langle x|\psi^{(+)}\rangle$.

(b) Consider the special case of an attractive δ-function potential

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x) \quad (\gamma > 0).$$

Solve the integral equation to obtain the transmission and reflection amplitudes. Check your results with Gottfried 1966, p. 52.

(c) The one-dimensional δ-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of γ. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when k is regarded as a complex variable.

((**Solution**))

We use the Lippmann-Schwinger equation

$$\left|\psi^{(+)}\right\rangle = |\phi\rangle + \frac{1}{E - \hat{H}_0 + i\varepsilon}\hat{V}\left|\psi^{(+)}\right\rangle$$

Green function:

$$
\begin{aligned}
G_+(x,x') &= \frac{\hbar^2}{2m}\langle x|\frac{1}{E-\hat{H}_0+i\varepsilon}|x'\rangle \\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dp'\int_{-\infty}^{\infty}dp''\langle x|p'\rangle\langle p'|\frac{1}{E-\hat{H}_0+i\varepsilon}|p''\rangle\langle p''|x'\rangle \\
&= \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp'\int_{-\infty}^{\infty}dp'' e^{\frac{ip'x}{\hbar}}\frac{1}{E-\frac{p'^2}{2m}+i\varepsilon}\langle p'|p''\rangle e^{\frac{-ip''x'}{\hbar}} \\
&= \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp' e^{\frac{ip'(x-x')}{\hbar}}\frac{1}{E-\frac{p'^2}{2m}+i\varepsilon}
\end{aligned}
$$

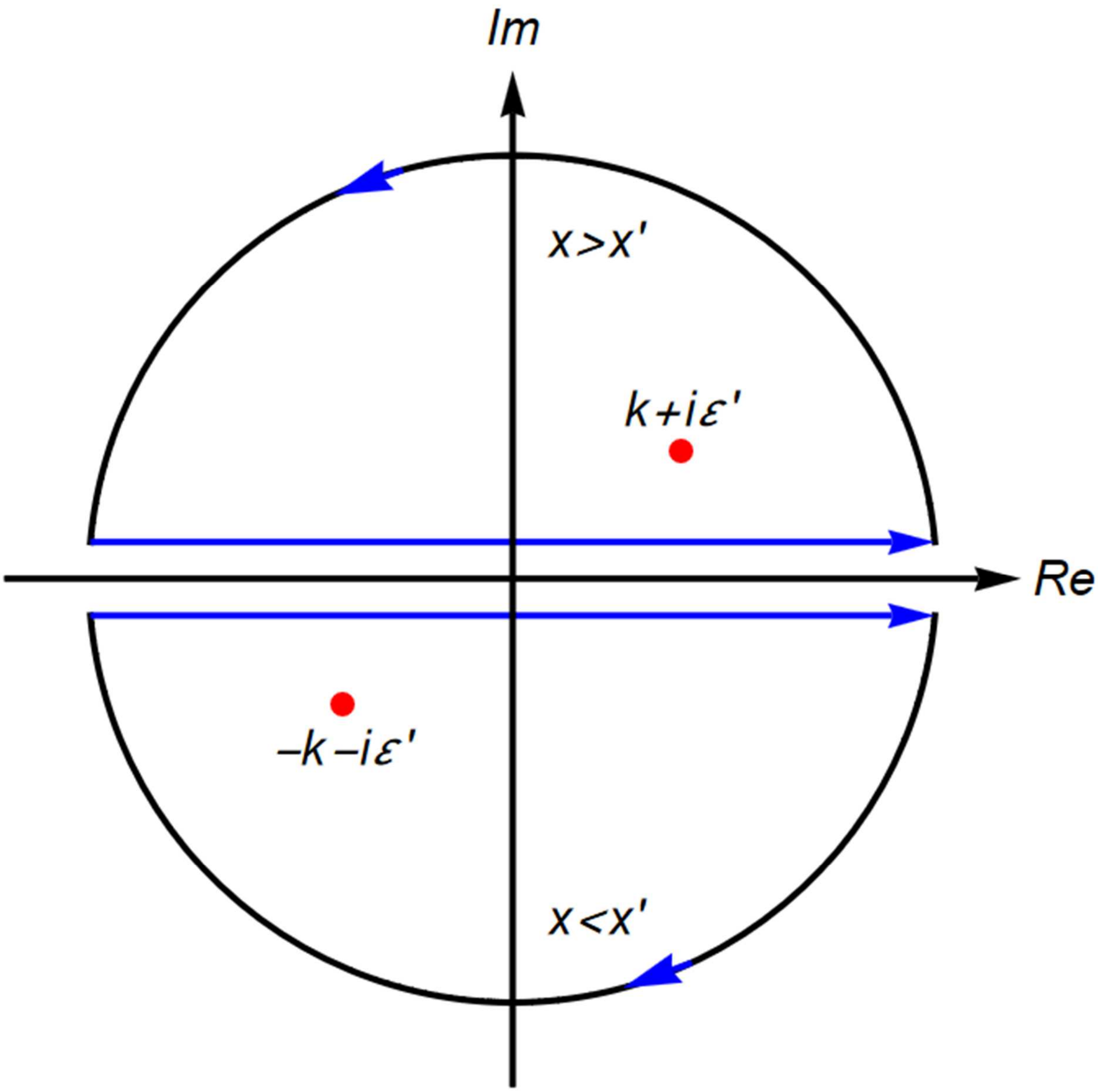
where

$$
\langle x|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ip'x}{\hbar}}, \qquad \langle p'|p''\rangle = \delta_{p',p''}
$$

Here we put

$$
E = \frac{\hbar^2k^2}{2m}, \qquad p' = \hbar k'
$$

$$
G_+(x,x') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}dk'\frac{e^{ik(x-x')}}{k'^2-k^2-i\varepsilon}
$$



$$\begin{aligned} k' &= \pm(k^2 + i\varepsilon)^{1/2} \\ &= \pm k(1+\frac{i\varepsilon}{k^2})^{1/2} \\ &= \pm k(1+\frac{i\varepsilon}{2k^2}) \\ &= \pm(k+\frac{i\varepsilon}{2k}) \\ &= \pm(k+i\varepsilon) \end{aligned}$$

((Jordan's lemma, residue theorem))

The integrand has poles in the complex k-plane at

$$k' = k + i\varepsilon, \qquad \text{and} \qquad k' = -(k + i\varepsilon)$$

When $x > x'$, we take the path in the upper plane. When $x < x'$, we take the path in the lower plane.

(i) For $x > x'$,

$$G_+(x,x') = -\frac{1}{2\pi} 2\pi i \operatorname{Re} s(k' = k + i\varepsilon)$$
$$= -i\frac{e^{ik(x-x')}}{2k}$$

(ii) For $x < x'$,

$$G_+(x,x') = -\frac{1}{2\pi}(-2\pi i)\operatorname{Re} s(k' = -k - i\varepsilon)$$
$$= -i\frac{e^{-ik(x-x')}}{2k}$$

Combining Eqs.(1) and (2),

$$G_+(x,x') = -\frac{i}{2k} e^{ik|x-x'|}$$

The Lippmann-Schwinger equation for $\langle x|\psi^{(+)}\rangle$;

$$\langle x|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx} - \frac{2m}{\hbar^2}\int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|} V(x') \langle x'|\psi^{(+)}\rangle$$

(b)

$$V = -\frac{\gamma\hbar^2}{2m}\delta(x) \qquad\qquad (\gamma > 0).$$

$$\langle x|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx} - \frac{2m}{\hbar^2}\int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|}[-\frac{\gamma\hbar^2}{2m}\delta(x')]\langle x'|\psi^{(+)}\rangle$$

$$= \frac{1}{\sqrt{2\pi}}e^{ikx} + \gamma\int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|}\delta(x')]\langle x'|\psi^{(+)}\rangle$$

$$= \frac{1}{\sqrt{2\pi}}e^{ikx} + \frac{i\gamma}{2k} e^{ik|x|}\langle 0|\psi^{(+)}\rangle$$

When $x = 0$,

$$\langle 0|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}} + \gamma\frac{i}{2k}\langle 0|\psi^{(+)}\rangle$$

or

$$\langle 0|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}\frac{1}{1-\frac{i\gamma}{2k}}$$

Then we get

$$\langle x|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}e^{ikx} + \frac{1}{\sqrt{2\pi}}\frac{\frac{i\gamma}{2k}}{1-\frac{i\gamma}{2k}}e^{ik|x|}$$

For $x>0$

$$\langle x|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}(\frac{1}{1-\frac{i\gamma}{2k}})e^{ikx}$$

For $x<0$

$$\langle x|\psi^{(+)}\rangle = \frac{1}{\sqrt{2\pi}}[e^{ikx} + \frac{\frac{i\gamma}{2k}}{1-\frac{i\gamma}{2k}}e^{-ikx}]$$

T: probability
R: probability of reflection

$$T = \frac{1}{1 + \left(\frac{\gamma}{2k}\right)^2}, \qquad R = 1 - T = \frac{\left(\frac{\gamma}{2k}\right)^2}{1 + \left(\frac{\gamma}{2k}\right)^2}$$

(c)
The wave function of bound state

Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

with

$$V(x) = -\frac{\gamma\hbar^2}{2m}\delta(x)$$

For $x \neq 0$, $\qquad V(x) = 0$.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

For $E = -|E| < 0$ (bound state)

$$\frac{d^2}{dx^2}\psi(x) = \frac{2m|E|}{\hbar^2}\psi(x) = \kappa^2\psi(x)$$

The solution of this equation is

$$\psi(x) = Ae^{-\kappa|x|}$$

where

$$|E| = \frac{\hbar^2\kappa^2}{2m}$$

Note that $\psi(x)$ is continuous at $x = 0$, but $d\psi(x)/dx$ is not continuous.

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dx^2}\psi(x)dx-\frac{\gamma\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi(x)dx=-\int_{-\varepsilon}^{\varepsilon}\frac{\hbar^2\kappa^2}{2m}\psi(x)dx$$

$$[\frac{d}{dx}\psi(x)]_{-\varepsilon}^{\varepsilon}=-\gamma\psi(0)$$

leading to the relation

$$\kappa=\frac{\gamma}{2}.$$

Therefore the wave function of the bound state is

$$\psi(x)=\sqrt{\frac{\gamma}{2}}e^{-\gamma|x|/2}$$

The value of k corresponding to the bound state is $k=\frac{i\gamma}{2}$.

$$T=\frac{1}{1+\left(\frac{\gamma}{2k}\right)^2}=\frac{1}{(i+\frac{\gamma}{2k})(-i+\frac{\gamma}{2k})}$$

$$R=\frac{\left(\frac{\gamma}{2k}\right)^2}{1+\left(\frac{\gamma}{2k}\right)^2}=\frac{\left(\frac{\gamma}{2k}\right)^2}{(i+\frac{\gamma}{2k})(-i+\frac{\gamma}{2k})}$$

T and R has a pole of $k=\frac{i\gamma}{2}$

6-2

6.2 Prove

$$\sigma_{\text{tot}} \simeq \frac{m^2}{\pi\hbar^4}\int d^3x \int d^3x' V(r)V(r')\frac{\sin^2 k|\mathbf{x}-\mathbf{x}'|}{k^2|\mathbf{x}-\mathbf{x}'|^2}$$

in each of the following ways.

(a) By integrating the differential cross section computed using the first-order Born approximation.

(b) By applying the optical theorem to the forward-scattering amplitude in the *second*-order Born approximation. [Note that $f(0)$ is real if the first-order Born approximation is used.]

((**Solution**))

The total cross section:

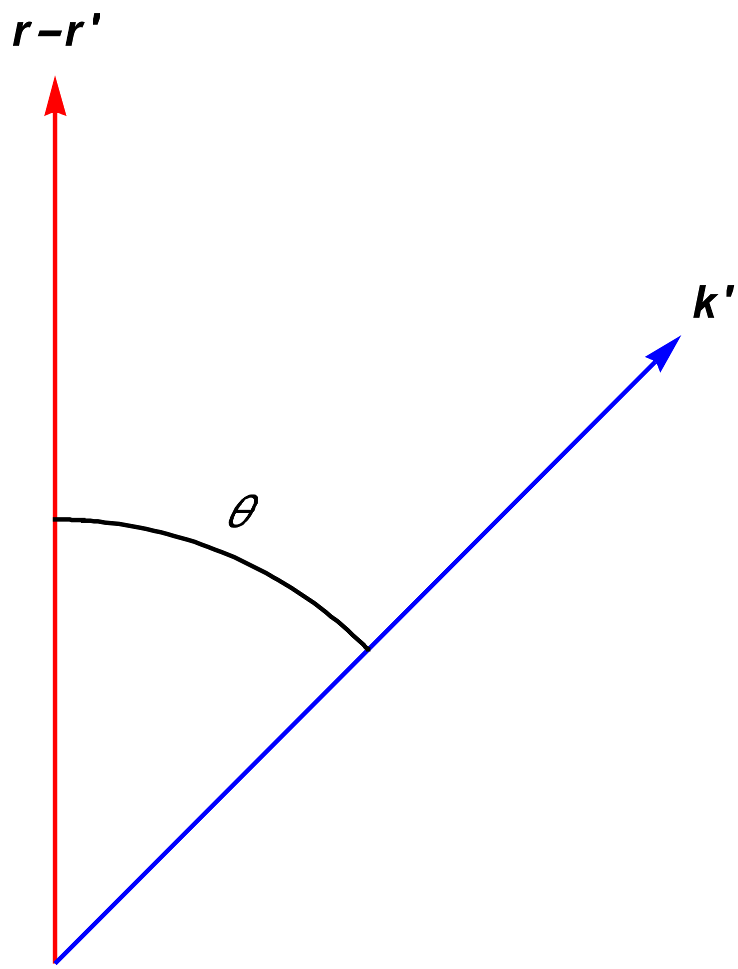
$$\sigma_{tot} = \int \left| f^{(1)}(\boldsymbol{k}',\boldsymbol{k})\right|^2 d\Omega_{k'}$$

with

$$f^{(1)}(\boldsymbol{k}',\boldsymbol{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3\boldsymbol{r} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} V(\boldsymbol{r})$$

So we get

$$\sigma_{tot} = \frac{\mu^2}{4\pi^2\hbar^4}\int d\Omega_{k'}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}' \exp[i(\boldsymbol{k}-\boldsymbol{k}')\cdot(\boldsymbol{r}-\boldsymbol{r}')]V(\boldsymbol{r})V(\boldsymbol{r}')$$



We now calculate

$$I = \int d\Omega_{k'} \exp[i(\boldsymbol{k} - \boldsymbol{k}') \cdot (\boldsymbol{r} - \boldsymbol{r}')]$$
$$= \exp[i\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')] \int d\Omega_{k'} \exp[-i\boldsymbol{k}' \cdot (\boldsymbol{r} - \boldsymbol{r}')]$$

Here

$$d\Omega_{k'} = 2\pi \sin\theta d\theta$$

$$\exp[-i\boldsymbol{k}' \cdot (\boldsymbol{r} - \boldsymbol{r}')] = \exp[-ik' |\boldsymbol{r} - \boldsymbol{r}'| \cos\theta] = \exp[-ik |\boldsymbol{r} - \boldsymbol{r}'| \cos\theta]$$

since $k' = k$. Then we get

$$I = \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\int 2\pi \sin\theta d\theta \exp[-ik|\boldsymbol{r}-\boldsymbol{r}'|\cos\theta]$$

$$= 2\pi \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\int_{-1}^{1} d(\cos\theta)\exp[-ik|\boldsymbol{r}-\boldsymbol{r}'|\cos\theta]$$

$$= \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\frac{4\pi\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k|\boldsymbol{r}-\boldsymbol{r}'|}$$

$$\sigma_{tot} = \frac{\mu^2}{\pi\hbar^4}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\frac{\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r})V(\boldsymbol{r}')$$

Since the potential has a spherical symmetry, σ_{tot} does not depend on the direction of $\boldsymbol{k}$,

$$\sigma_{tot} = \frac{1}{4\pi}\int \sigma_{tot} d\Omega_{\boldsymbol{k}}$$

Note that

$$\int d\Omega_{\boldsymbol{k}} \exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')] = \frac{4\pi\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k|\boldsymbol{r}-\boldsymbol{r}'|}.$$

$$\sigma_{tot} = \frac{\mu^2}{\pi\hbar^4}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\frac{\sin^2(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2|\boldsymbol{r}-\boldsymbol{r}'|^2}V(\boldsymbol{r})V(\boldsymbol{r}')$$

(b)

$$\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im}[f^{(2)}(\theta=0)]$$

where

$\boldsymbol{k}'=\boldsymbol{k}$ (forward scattering)

Note that

$$f^{(2)}(\boldsymbol{k}'=\boldsymbol{k},\boldsymbol{k}) = -\frac{2\mu}{\hbar^2}\frac{1}{4\pi}(2\pi)^3\langle\boldsymbol{k}|\hat{V}(E_k-\hat{H}_0+i\varepsilon)^{-1}\hat{V}|\boldsymbol{k}\rangle$$
$$= -\frac{2\mu}{\hbar^2}\frac{1}{4\pi}(2\pi)^3\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{V}(E_k-\hat{H}_0+i\varepsilon)^{-1}\hat{V}|\boldsymbol{r}'\rangle\langle\boldsymbol{r}'|\boldsymbol{k}\rangle$$
$$= -\frac{2\mu}{\hbar^2}\frac{1}{4\pi}(2\pi)^3\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}'|\boldsymbol{k}\rangle V(\boldsymbol{r})V(\boldsymbol{r}')\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle$$
$$= -\frac{2\mu}{\hbar^2}\frac{1}{4\pi}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}V(\boldsymbol{r})V(\boldsymbol{r}')\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle$$
$$= \frac{\mu^2}{4\pi^2\hbar^4}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}V(\boldsymbol{r})\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')$$

where

$$\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle = -\frac{2\mu}{\hbar^2}\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}$$

Thus we have

$$\sigma_{tot} = \frac{4\pi}{k}\mathrm{Im}[\frac{\mu^2}{4\pi^2\hbar^4}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}V(\boldsymbol{r})\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')]$$

Since the potential has a spherical symmetry, σ_{tot} does not depend on the direction of $\boldsymbol{k}$,

$$\bar{\sigma}_{tot} = \frac{1}{4\pi}\int\sigma_{tot}d\Omega_{\boldsymbol{k}}$$

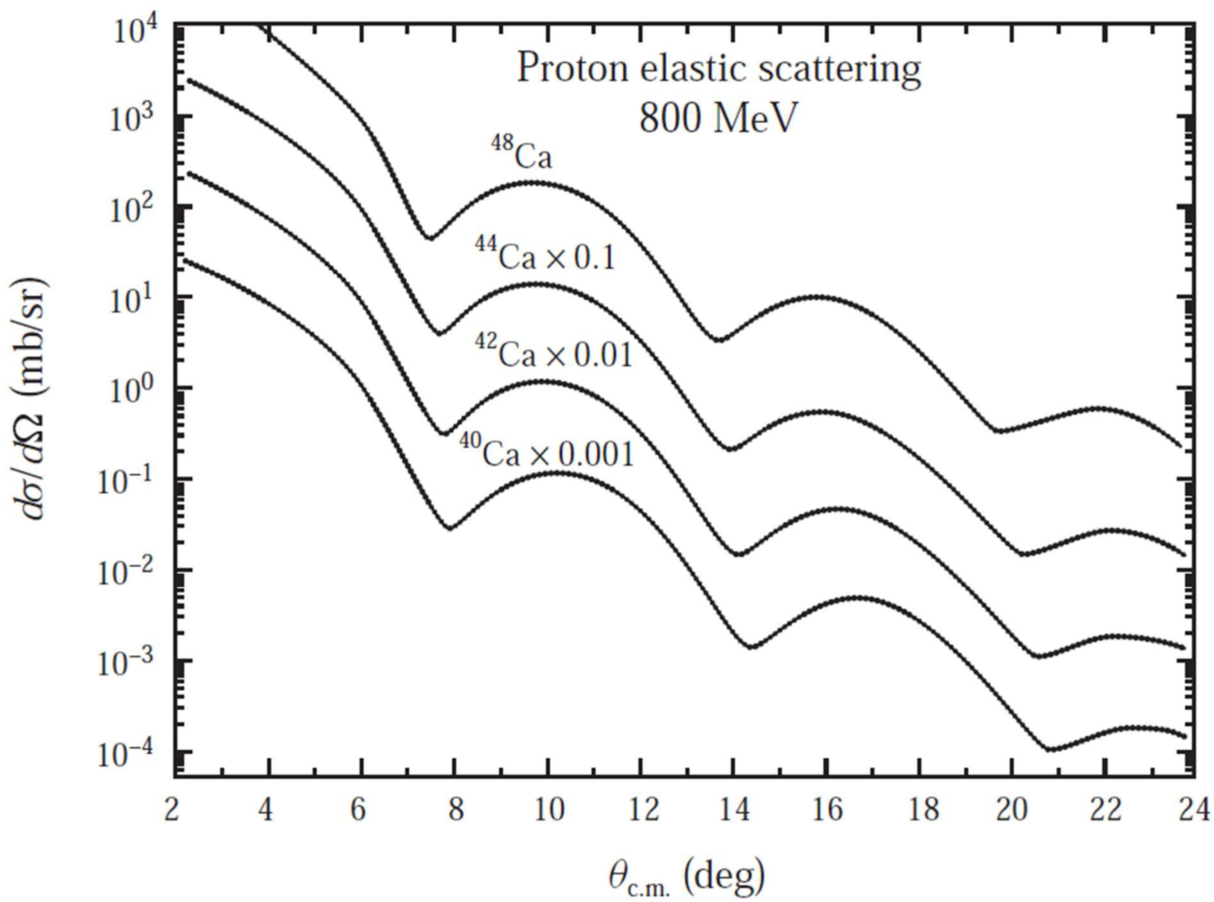
Note that

$$\int d\Omega_{\boldsymbol{k}}\exp[i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')] = \frac{4\pi\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k|\boldsymbol{r}-\boldsymbol{r}'|}.$$

$$\bar{\sigma}_{tot} = \frac{\mu^2}{\pi\hbar^4}\mathrm{Im}[\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\frac{\sin(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r})\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(\boldsymbol{r}')]$$
$$= \frac{\mu^2}{\pi\hbar^4}\int d^3\boldsymbol{r}\int d^3\boldsymbol{r}'\frac{\sin^2(k|\boldsymbol{r}-\boldsymbol{r}'|)}{k^2|\boldsymbol{r}-\boldsymbol{r}'|^2}V(\boldsymbol{r})V(\boldsymbol{r}')]$$

6-3

6.3 Estimate the radius of the ^{40}Ca nucleus from the data in Figure 6.6 and compare to that expected from the empirical value $\approx 1.4A^{1/3}$ fm, where A is the nuclear mass number. Check the validity of using the first-order Born approximation for these data.



((**Solution**))

Low energy soft-sphere scattering

Suppose that the potential has a spherical symmetry such that

$$V(r) = \begin{cases} V_0 & r \le R \\ 0 & r > R \end{cases}$$

The scattering amplitude is obtained as

$$f^{(1)}(\theta) = -\frac{1}{Q}\frac{2\mu}{\hbar^2}V_0\int_0^R dr' r' \sin(Qr')$$
$$= -\frac{1}{Q}\frac{2\mu}{\hbar^2}V_0\frac{\sin(QR) - QR\cos(QR)}{Q^2}$$
$$= -\frac{2\mu}{\hbar^2}R^3 V_0\frac{\sin(QR) - QR\cos(QR)}{Q^3R^3}$$

where Q is the magnitude of the scattering vector,

$$Q = 2k\sin(\frac{\theta}{2}).$$

We make a plot of $\left|f^{(1)}(Q)\right|^2 / \left|f^{(1)}(Q=0)\right|^2$ as a function QR.

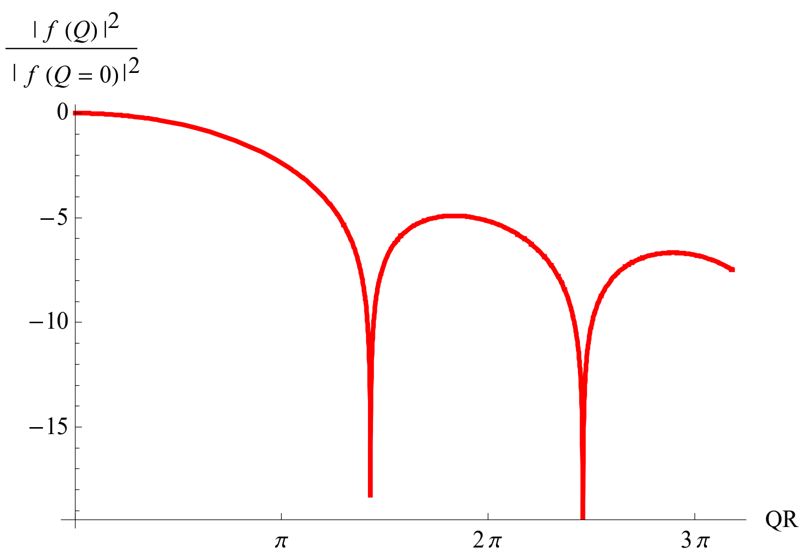


Fig. Plot of $\left|f^{(1)}(Q)\right|^2 / \left|f^{(1)}(Q=0)\right|^2$ as a function QR. The value of $\left|f^{(1)}(Q)\right|^2$ becomes zero when QR = 4.49341, 7.72525, 10.9041, 14.0662, …..

$E = 800$ MeV.

$$k = \sqrt{\frac{2m_p E}{\hbar^2}} = 6.20923 \text{ fm}^{-1}.$$

where m_p is the mass of proton. A is the atomic mass of Ca. $Q = 2k\sin\frac{\theta}{2}$ (fm^{-1})

A	$1.4A^{1/3}$	$\theta(°)$	$Q = 2k\sin\dfrac{\theta}{2}$ (fm^{-1})	a (fm)
^{40}Ca				
40	4.78793	7.95	0.860863	5.21966
40	4.78793	14.2	1.53494	5.03293
40	4.78793	20.9	2.25243	4.84104
^{42}Ca				
42	4.86644	7.85	0.850051	5.28604
42	4.86644	14.	1.51343	5.10446
42	4.86644	20.5	2.20979	4.93446
^{44}Ca				
44	4.94249	7.6	0.82302	5.45966
44	4.94249	13.8	1.49192	5.17808
44	4.94249	20.1	2.16712	5.03162
^{48}Ca				
48	5.08794	7.3	0.790577	5.68371
48	5.08794	13.4	1.44887	5.33191
48	5.08794	19.5	2.10306	5.18487

((**Mathematica**))

```
Clear["Global`*"];
rule1 = {c → 2.99792 × 10^10, ℏ → 1.054571628 10^-27, mp → 1.672621637 × 10^-24,
  eV → 1.602176487 × 10^-12, meV → 1.602176487 × 10^-15, keV → 1.602176487 × 10^-9,
  MeV → 1.602176487 × 10^-6, E1 → 800 MeV, fm → 10^-13};

k = Sqrt[(2 mp E1)/ℏ^2] /. rule1; k fm //. rule1
```

6.20923

```
a1[θ_] := (4.49341 /. rule1)/(2 k Sin[θ/2] fm) /. rule1; a2[θ_] := 7.72525/(2 k Sin[θ/2] fm) /. rule1;
a3[θ_] := (10.9041 /. rule1)/(2 k Sin[θ/2] fm) /. rule1;
Q[θ_] = 2 k Sin[θ/2] fm /. rule1;

f1 = List[{40, 1.4 × 40^(1/3), 7.95, Q[7.95 °], a1[7.95 °]},
  {42, 1.4 × 42^(1/3), 7.85, Q[7.85 °], a1[7.85 °]},
  {44, 1.4 × 44^(1/3), 7.6, Q[7.6 °], a1[7.6 °]},
  {48, 1.4 × 48^(1/3), 7.3, Q[7.3 °], a1[7.3 °]}];
f2 = List[{40, 1.4 × 40^(1/3), 14.2, Q[14.2 °], a2[14.2 °]},
  {42, 1.4 × 42^(1/3), 14.0, Q[14.0 °], a2[14.0 °]},
  {44, 1.4 × 44^(1/3), 13.8, Q[13.8 °], a2[13.8 °]},
  {48, 1.4 × 48^(1/3), 13.4, Q[13.4 °], a2[13.4 °]}];
f3 = List[{40, 1.4 × 40^(1/3), 20.9, Q[20.9 °], a3[20.9 °]},
  {42, 1.4 × 42^(1/3), 20.5, Q[20.5 °], a3[20.5 °]},
  {44, 1.4 × 44^(1/3), 20.1, Q[20.1 °], a3[20.1 °]},
  {48, 1.4 × 48^(1/3), 19.5, Q[19.5 °], a3[19.5 °]}];
```

f3 // TableForm

40	4.78793	20.9	2.25243	4.84104
42	4.86644	20.5	2.20979	4.93446
44	4.94249	20.1	2.16712	5.03162
48	5.08794	19.5	2.10306	5.18487

s1 = List[f1[[1]], f2[[1]], f3[[1]]]; s1 // TableForm

40	4.78793	7.95	0.860863	5.21966
40	4.78793	14.2	1.53494	5.03293
40	4.78793	20.9	2.25243	4.84104

s2 = List[f1[[2]], f2[[2]], f3[[2]]]; s2 // TableForm

42	4.86644	7.85	0.850051	5.28604
42	4.86644	14.	1.51343	5.10446
42	4.86644	20.5	2.20979	4.93446

s3 = List[f1[[3]], f2[[3]], f3[[3]]]; s3 // TableForm

44	4.94249	7.6	0.82302	5.45966
44	4.94249	13.8	1.49192	5.17808
44	4.94249	20.1	2.16712	5.03162

s4 = List[f1[[4]], f2[[4]], f3[[4]]]; s4 // TableForm

48	5.08794	7.3	0.790577	5.68371
48	5.08794	13.4	1.44887	5.33191
48	5.08794	19.5	2.10306	5.18487

6-4

6.4 Consider a potential

$$V = 0 \quad \text{for } r > R, \qquad V = V_0 = \text{constant} \quad \text{for } r < R,$$

where V_0 may be positive or negative. Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2 k^2/2m$ and $kR \ll 1$, the differential cross section is isotropic and that the total cross section is given by

$$\sigma_{tot} = \left(\frac{16\pi}{9}\right)\frac{m^2 V_0^2 R^6}{\hbar^4}.$$

Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta.$$

Obtain an approximate expression for B/A.

((**Solution**))

$$V = 0 \quad \text{for } r > R. \qquad\qquad V = V_0 \quad \text{for } r < R.$$

where V_0 may be either positive or negative.

(a)

$$|V_0| << E = \frac{\hbar^2 k^2}{2m} \qquad \text{and } kR << 1$$

$$\tan\delta_l = \frac{kR\, j_l'(kR) - \beta_l\, j_l(kR)}{kR\, n_l'(kR) - \beta_l\, n_l(kR)} \qquad (1)$$

For $r < R$

$$[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k_{in}{}^2 - \frac{l(l+1)}{r^2}]R_l(r) = 0$$

with

$$k_{in}{}^2 = k^2 - \frac{2m}{\hbar^2} V_0$$

We put

$$\rho = k_{jn} r$$

$$[\frac{d^2}{d\rho^2} + \frac{2}{\rho}\frac{d}{d\rho} + 1 - \frac{l(l+1)}{\rho^2}]R_l(\rho) = 0$$

We have a solution given by

$$R_l(\rho) = C_1 j_l(\rho)$$

because of the regularity at the origin.

$$R_l(r) = C_1 j_l(k_{in} r)$$

$$\begin{aligned}\beta_l &= [\frac{k_{in} r}{C_1 j_l(k_{in} r)} C_1 \frac{d}{d(k_{in} r)} j_l(k_{in} r)]_{r=R} \\ &= \frac{k_{in} R\ j_l'(k_{in} R)}{j_l(k_{in} R)}\end{aligned} \tag{2}$$

$$\begin{aligned}\tan\delta_l &= \frac{kR\ j_l'(kR) - \dfrac{k_{in} R\ j_l'(k_{in} R)}{j_l(k_{in} R)}\ j_l(kR)}{kR\ n_l'(kR) - \dfrac{k_{in} R\ j_l'(k_{in} R)}{j_l(k_{in} R)}\ n_l(kR)} \\ &= \frac{kR\ j_l'(kR) j_l(k_{in} R) - k_{in} R\ j_l'(k_{in} R)\ j_l(kR)}{kR\ n_l'(kR) j_l(k_{in} R) - k_{in} R\ j_l'(k_{in} R) n_l(kR)}\end{aligned}$$

We consider the case of $l = 0$.

$$\tan\delta_0 = \frac{kR\ j_0'(kR) j_0(k_{in} R) - k_{in} R\ j_0'(k_{in} R)\ j_0(kR)}{kR\ n_0'(kR) j_0(k_{in} R) - k_{in} R\ j_0'(k_{in} R) n_0(kR)}$$

When $kR << 1$

$$j_0(kR) = \frac{\sin(kR)}{kR} = 1 - \frac{1}{6}(kR)^2$$

$$j_0'(kR) = -\frac{1}{3}(kR)$$

$$n_0(kR) = -\frac{\cos(kR)}{kR} = -\frac{1}{kR} + \frac{1}{2}kR$$

$$n_0'(kR) = \frac{1}{2} + \frac{1}{(kR)^2}$$

Then we have

$$\begin{aligned}
\delta_0 &\approx \tan\delta_0 \\
&= \frac{1}{3}kR^3({k_{in}}^2 - k^2) \\
&= \frac{1}{3}kR^3(k^2 - \frac{2m}{\hbar^2}V_0 - k^2) \\
&= -\frac{2mV_0}{3\hbar^2}kR^3
\end{aligned}$$

$$\sigma_{tot} \approx \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{16\pi}{9}\frac{m^2{V_0}^2R^6}{\hbar^4}$$

(b)

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= |f(\theta)|^2 \\
&= \frac{1}{k^2}\sum_{l=0}\sum_{l'=0}(2l+1)(2l'+1)e^{i\delta_l}e^{-i\delta_{l'}}\sin\delta_l\sin\delta_{l'}P_l(\cos\theta)P_{l'}(\cos\theta)
\end{aligned}$$

We take three terms with $(l,l') = (0,0)$, $(l,l') = (1,0)$, and $(l,l') = (0,1)$.

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &\approx \frac{1}{k^2}[\sin^2\delta_0 + 6\cos(\delta_0 - \delta_1)\sin\delta_0\sin\delta_1\cos\theta] \\
&= A + B\cos\theta
\end{aligned}$$

where

$$\frac{B}{A} = \frac{6\cos(\delta_0 - \delta_1)\sin\delta_0 \sin\delta_1}{\sin^2\delta_0}$$
$$= \frac{6\cos(\delta_0 - \delta_1)\sin\delta_1}{\sin\delta_0}$$
$$\approx \frac{6\delta_1}{\delta_0}$$

Here we note that

$$\tan\delta_1 = \frac{kR\ j_1'(kR) j_1(k_{in}R) - k_{in}R\ j_1'(k_{in}R)\ j_1(kR)}{kR\ n_1'(kR) j_1(k_{in}R) - k_{in}R\ j_1'(k_{in}R) n_1(kR)}$$

$$j_1(kR) = \frac{1}{3}(kR) - \frac{1}{30}(kR)^3$$

$$j_1'(kR) = \frac{1}{3} - \frac{1}{10}(kR)^2$$

$$n_1(kR) = -\frac{1}{(kR)^2} - \frac{1}{2} + \frac{1}{8}x^2$$

$$n_1'(kR) = \frac{2}{(kR)^3} + \frac{x}{4}$$

$$\delta_1 \approx \tan\delta_1$$
$$= \frac{1}{45}k^3R^5(k_{in}{}^2 - k^2)$$
$$= -\frac{2mV_0}{45\hbar^2}k^3R^5$$

$$\frac{B}{A} \approx \frac{2}{5}(kR)^2$$

6-5

6.5 A spinless particle is scattered by a weak Yukawa potential

$$V = \frac{V_0 e^{-\mu r}}{\mu r},$$

where $\mu > 0$ but V_0 can be positive or negative. It was shown in the text that the first-order Born amplitude is given by

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{[2k^2(1-\cos\theta)+\mu^2]}.$$

(a) Using $f^{(1)}(\theta)$ and assuming $|\delta_l| \ll 1$, obtain an expression for δ_l in terms of a Legendre function of the second kind,

$$Q_l(\zeta) = \frac{1}{2}\int_{-1}^{1} \frac{P_l(\zeta')}{\zeta - \zeta'} d\zeta'.$$

(b) Use the expansion formula

$$Q_l(\zeta) = \frac{l!}{1\cdot 3\cdot 5 \cdots (2l+1)} \times \left\{ \frac{1}{\zeta^{l+1}} + \frac{(l+1)(l+2)}{2(2l+3)} \frac{1}{\zeta^{l+3}} + \frac{(l+1)(l+2)(l+3)(l+4)}{2\cdot 4\cdot (2l+3)(2l+5)} \frac{1}{\zeta^{l+5}} + \cdots \right\} \quad (|\zeta| > 1)$$

to prove each assertion.

(i) δ_l is negative (positive) when the potential is repulsive (attractive).

(ii) When the de Broglie wavelength is much longer than the range of the potential, δ_l is proportional to k^{2l+1}. Find the proportionality constant.

((**Solution**))

Yukawa potential

$$V(r) = \frac{V_0}{\mu r} e^{-\mu r}$$

The first Born approximation

$$f^{(1)}(\theta) = -\frac{2mV_0}{\hbar^2 \mu} \frac{1}{q^2 + \mu^2}$$

where

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$$

$$\begin{aligned} q^2 &= k^2 + k'^2 - 2kk' \cos\theta \\ &= 2k^2 (1 - \cos\theta) \end{aligned}$$

where the angle between the vctors ***k*** and ***k'*** is θ. $k' = k$

(b)
The scattering amplitude $f(\theta)$ can be expanded in terms of the phase shift δ_l as

$$f(\theta) = \frac{1}{k} \sum_{l'=0}^{\infty} (2l'+1) e^{i\delta_{l'}} \sin\delta_{l'} P_{l'}(\cos\theta)$$

$$\begin{aligned} I &= \int_{-1}^{1} d(\cos\theta) P_l(\cos\theta) f(\theta) \\ &= \int_{-1}^{1} d(\cos\theta) \frac{1}{k} \sum_{l'=0}^{\infty} (2l'+1) e^{i\delta_{l'}} \sin\delta_{l'} P_l(\cos\theta) P_{l'}(\cos\theta) \end{aligned}$$

Note that

$$\int_{-1}^{1} d(\cos\theta) P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1} \delta_{l,l'}$$

Then we get

$$
\begin{aligned}
I &= \frac{1}{k}\sum_{l'=0}^{\infty}(2l'+1)e^{i\delta_{l'}}\sin\delta_{l'}\frac{2}{2l+1}\delta_{l,l'} \\
&= \frac{1}{k}(2l+1)e^{i\delta_l}\sin\delta_l\frac{2}{2l+1} \\
&= \frac{2}{k}e^{i\delta_l}\sin\delta_l \\
&\approx \frac{2}{k}\delta_l
\end{aligned}
$$

for $|\delta_l| << 1$. Thus δ_l can be estimated as

$$
\begin{aligned}
\delta_l &= \frac{k}{2}I \\
&= \frac{k}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)f(\theta) \\
&= \frac{k}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)\left(-\frac{2mV_0}{\hbar^2\mu}\right)\frac{1}{2k^2(1-\cos\theta)+\mu^2} \\
&= -\frac{mV_0}{2\hbar^2\mu k}\int_{-1}^{1} d\varsigma'\frac{P_l(\varsigma')}{1+\frac{\mu^2}{2k^2}-\varsigma'} \qquad (1) \\
&= -\frac{mV_0}{\hbar^2\mu k}\frac{1}{2}\int_{-1}^{1} d\varsigma'\frac{P_l(\varsigma')}{1+\frac{\mu^2}{2k^2}-\varsigma'} \\
&= -\frac{mV_0}{\hbar^2\mu k}f_l(1+\frac{\mu^2}{2k^2})
\end{aligned}
$$

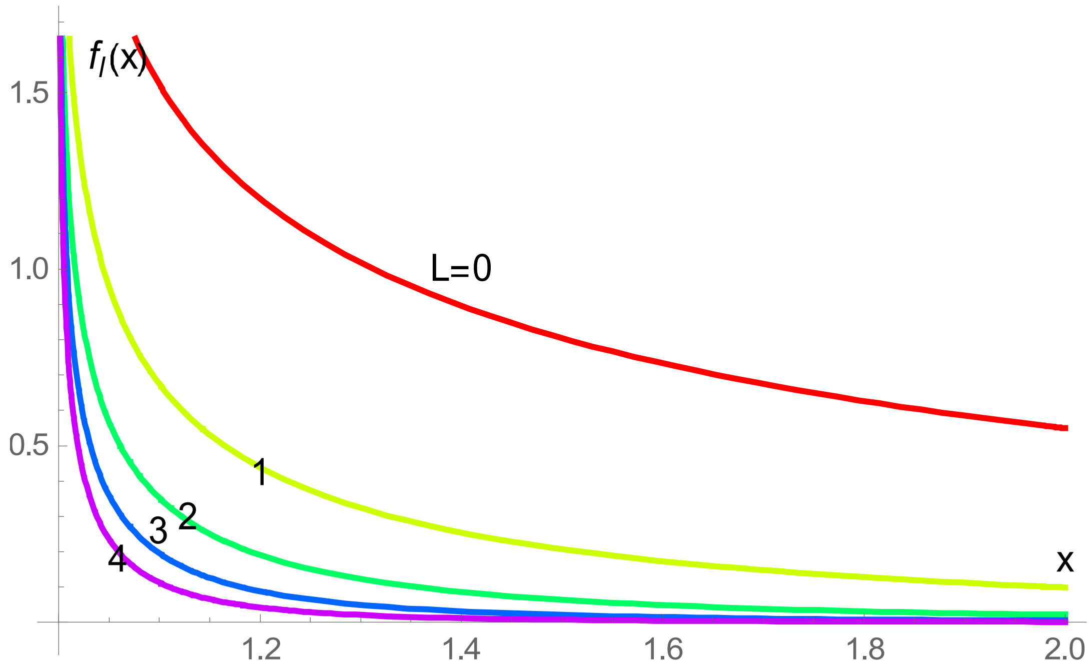
where $f_l(\varsigma)$ is defined by

$$
f_l(x) = \frac{1}{2}\int_{-1}^{1} dy\frac{P_l(y)}{x-y}
$$

with

$$
x = 1+\frac{\mu^2}{2k^2}
$$

We make a plot of $f_l(x)$ as a function of x for x>1, where l is changed as a parameter. (numerical integration using the Mathematicva).



(b)
(i)
As is shown in the above figure, we have

$$f_l(x) > 0 \text{ for } x > 1.$$

$$x = 1 + \frac{\mu^2}{2k^2}$$

From Eq.(1)

$\delta_l > 0$ when $V_0 > 0$ (repulsive)

$\delta_l < 0$ when $V_0 < 0$ (attractive)

(ii) de Broglie wave length $\lambda = \frac{2\pi}{k} >> \frac{1}{\mu}$ (the range of potential) or $\frac{\mu}{k} >> 1$

$$\delta_l = -\frac{mV_0}{\hbar^2 \mu k} Q_l (1 + \frac{\mu^2}{2k^2})$$
$$\approx -\frac{mV_0}{\hbar^2 \mu k} Q_l (\frac{\mu^2}{2k^2})$$
$$= -\frac{mV_0}{\hbar^2 \mu k} \frac{l!}{(2l+1)!!} \left(\frac{2k^2}{\mu^2} \right)^{l+1}$$
$$= -\frac{mV_0}{\hbar^2 \mu^{2l+3}} \frac{2^{l+1} l!}{(2l+1)!!} k^{2l+1}$$

6-6

6.6 Check explicitly the $x - p_x$ uncertainty relation for the ground state of a particle confined inside a hard sphere: $V = \infty$ for $r > a$, $V = 0$ for $r < a$. (*Hint:* Take advantage of spherical symmetry.)

((**Solution**))

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rR_{n\ell}(r)] + [k^2 - \frac{1}{r^2}\ell(\ell+1)]R_{n\ell}(r) = 0$$

When $l = 0$,

$$\frac{\partial^2}{\partial r^2}[rR_{n\ell}(r)] + k^2 r R_{n\ell}(r) = 0$$

When

$$u_n(r) = rR_{n\ell}(r)$$

with the boundary condition $u_n(a) = u_n(0)$
The solution is obtained as

$$u_n(r) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi r}{a}) \qquad \text{with } n = 1, 2, 3, 4, \ldots$$

From the parity

$$\langle n, l=0, m=0|\hat{x}|n, l=0, m=0\rangle = 0 \qquad \text{(odd parity)}$$

$$\langle n, l=0, m=0|\hat{p}_x|n, l=0, m=0\rangle = 0$$

((**Proof**))

The Hamiltonian of the hydrogen atom is given by

$$\hat{H} = \frac{1}{2\mu}\hat{\boldsymbol{p}}^2 + V(|\hat{\boldsymbol{r}}|)$$

The commutation relation:

$$\begin{aligned}
[\hat{H}, \hat{x}] &= \frac{1}{2\mu}[\hat{p}_x{}^2, \hat{x}] \\
&= \frac{1}{2\mu}\{\hat{p}_x[\hat{p}_x, \hat{x}] + [\hat{p}_x, \hat{x}]\hat{p}_x\} \\
&= \frac{\hbar}{\mu i}\hat{p}_x
\end{aligned}$$

or

$$\hat{p}_x = \frac{i\mu}{\hbar}[\hat{H}, \hat{x}]$$

$$\begin{aligned}
\langle n', l', m'|\hat{p}_x|n, l, m\rangle &= \frac{i\mu}{\hbar}\langle n', l', m'|[\hat{H}, \hat{x}]|n, l, m\rangle \\
&= \frac{i\mu}{\hbar}(E_{n'} - E_n)\langle n', l', m'|\hat{x}|n, l, m\rangle
\end{aligned}$$

When $n' = n$

$$\langle n', l', m'|\hat{p}_x|n, l, m\rangle = 0$$

Thus we have

$$\langle n, l=0, m=0|\hat{p}_x|n, l=0, m=0\rangle = 0$$

From the spherical symmetry, we get

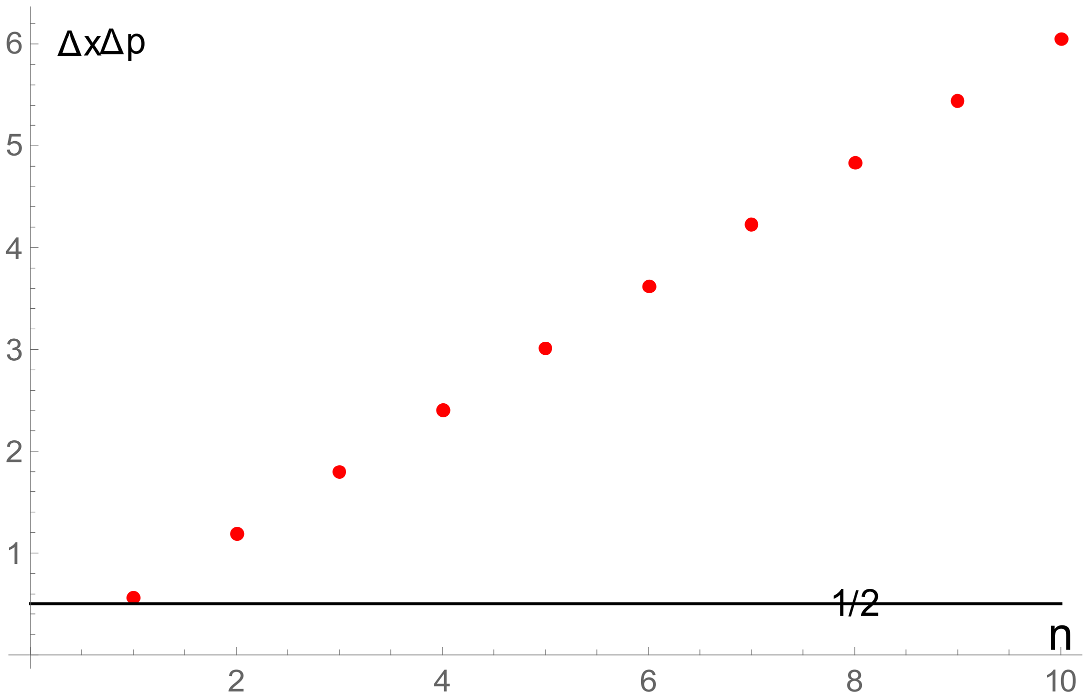
$$\begin{aligned}
\langle n, l=0, m=0|\hat{x}^2|n, l=0, m=0\rangle &= \frac{1}{3}\langle n, l=0, m=0|\hat{\boldsymbol{r}}^2|n, l=0, m=0\rangle \\
&= \frac{1}{3}\int_0^a r^2 dr R_n^*(r) r^2 R_n(r) \\
&= \frac{1}{3}\int_0^a r^2 dr \frac{u_n^*(r)}{r} r^2 \frac{u_n(r)}{r} \\
&= \frac{1}{3}\int_0^a r^2 dr u_n^*(r) u_n(r) \\
&= \frac{1}{9}a^2 - \frac{1}{6n^2\pi^2}
\end{aligned}$$

$$\begin{aligned}
\langle n, l=0, m=0|\hat{p}_x{}^2|n, l=0, m=0\rangle &= \frac{1}{3}\langle n, l=0, m=0|\hat{\boldsymbol{p}}^2|n, l=0, m=0\rangle \\
&= \frac{1}{3}\int_0^a r^2 dr R_n^*(r) p_r{}^2 R_n(r) \\
&= \frac{1}{3}\int_0^a r^2 dr \frac{u_n^*(r)}{r} p_r{}^2 \frac{u_n(r)}{r} \\
&= \frac{1}{3}\int_0^a r dr u_n^*(r) p_r{}^2 \frac{u_n(r)}{r} \\
&= \frac{n^2\pi^2\hbar^2}{3a^2}
\end{aligned}$$

Then we have

$$(\Delta x)^2 = \langle x^2\rangle = a^2\left(\frac{1}{9} - \frac{1}{6n^2\pi^2}\right). \qquad (\Delta p)^2 = \langle p^2\rangle = \frac{n^2\pi^2\hbar^2}{3a^2}$$

$$(\Delta x)(\Delta p) = \frac{\hbar}{3}\sqrt{\frac{1}{3}n^2\pi^2 - \frac{1}{2}} > \frac{\hbar}{2}$$




```
Clear["Global`*"]; pr := (ħ/i)/r D[r #, r] &;

ur = Sqrt[2/a] Sin[n π r/a]; f1 = 1/3 r^2 ur^2 // Simplify;

g1 = 1/3 r ur pr[pr[ur/r]];

Integrate[f1, {r, 0, a}] //
 Simplify[#, n ∈ Integers] &
```

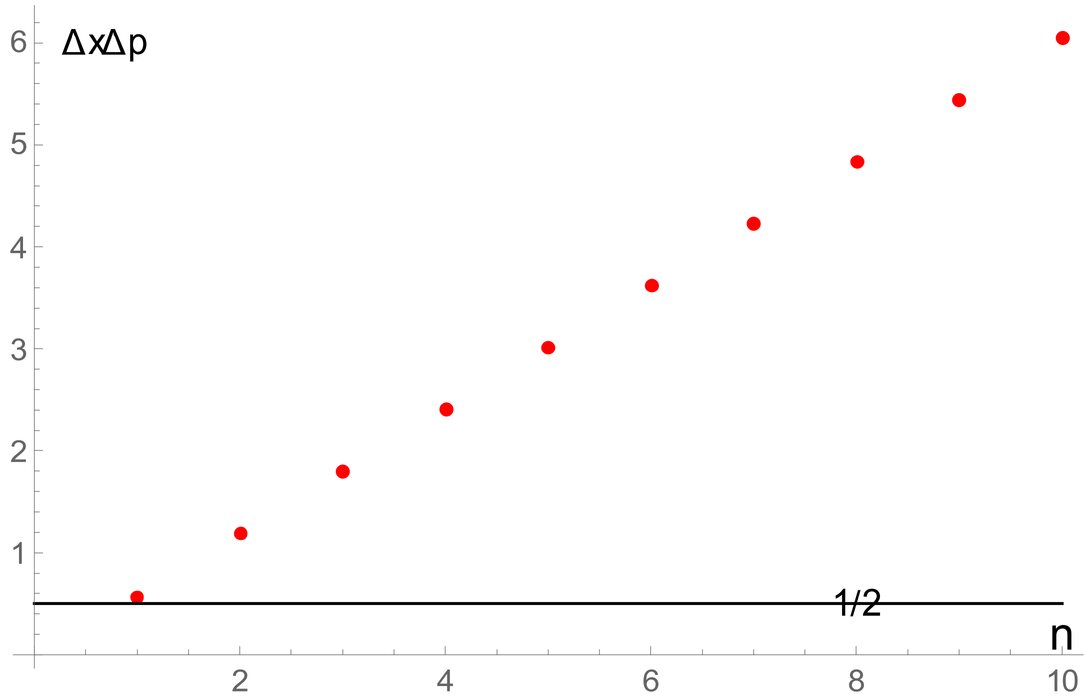
$$\frac{1}{18} a^2 \left(2 - \frac{3}{n^2 \pi^2}\right)$$

```
Integrate[g1, {r, 0, a}] //
 Simplify[#, n ∈ Integers] &
```

$$\frac{n^2 \pi^2 \hbar^2}{3 a^2}$$

$$p1 = \frac{1}{3}\sqrt{\frac{n^2\pi^2}{3} - \frac{1}{2}};$$

```
k1 = ListPlot[Table[{n, p1}, {n, 1, 10}],
  PlotStyle → {Red, Thick}];
k2 =
 Graphics[{Text[Style["ΔxΔp", Black, 12], {0.7, 6}],
   Text[Style["n", Black, 15], {10, 0.2}],
   Text[Style["1/2", Black, 12], {8, 0.5}] ,
   Line[{{0, 0.5}, {10, 0.5}}]}];
Show[k1, k2]
```



6-7

6.7 Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} 0 & \text{for } r > a \\ \infty & \text{for } r < a. \end{cases}$$

(a) Derive an expression for the s-wave ($l = 0$) phase shift. (You need not know the detailed properties of the spherical Bessel functions to do this simple problem!)

(b) What is the total cross section $\sigma[\sigma = \int (d\sigma/d\Omega) d\Omega]$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section πa^2. You may assume without proof:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

$$f(\theta) = \left(\frac{1}{k}\right) \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l P_l(\cos\theta).$$

((**Solution**))

$$V(r) = \begin{cases} 0 & (r > a) \\ \infty & (r < a) \end{cases}$$

For $l = 0$,

$$\frac{d^2}{dr^2} u_0^{out} + k^2 u_0^{out} = 0$$

with

$$u_0^{out}(r) = r R_0^{out}(r)$$

$$u_0^{out}(r) = N \sin[k(r-a)]$$

since $u_0^{out}(r = a) = 0$.

In the absence of potential, $a = 0$

$$u_0^{out}(r) = N \sin kr$$

Therefore the phase shift δ_0 is given by,

$$\delta_0 = -ka$$

(b)
In the limit of $k \to 0$, only the s wave is very important.

$$\begin{aligned}
\sigma &= \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega \\
&= \int d\Omega |f(\theta)|^2 \\
&= \frac{4\pi}{k^2} \sum_{l=0} (2l+1)\sin^2\delta_l \\
&= \frac{4\pi}{k^2} \sin^2\delta_0 \\
&= \frac{4\pi}{k^2} k^2 a^2 \\
&= 4\pi a^2
\end{aligned}$$

6-8

6.8 Use $\delta_l = \Delta(b)|_{b=l/k}$ to obtain the phase shift δ_l for scattering at high energies by (a) the Gaussian potential, $V = V_0 \exp(-r^2/a^2)$, and (b) the Yukawa potential, $V = V_0 \exp(-\mu r)/\mu r$. Verify the assertion that δ_l goes to zero very rapidly with increasing l (k fixed) for $l \gg kR$, where R is the "range" of the potential. [The formula for $\Delta(b)$ is given in (6.5.14)].

((**Solution**))
We use the formula

$$\Delta(b = \frac{1}{k}) = -\frac{m}{2k\hbar^2} \int_{-\infty}^{\infty} V(r = \sqrt{b^2 + z^2}) dz$$

(a) Gaussian potential: $V = V_0 \exp(-\frac{r^2}{a^2})$

$$\Delta(b=\frac{1}{k}) = -\frac{mV_0}{2k\hbar^2}\int_{-\infty}^{\infty}\exp(-\frac{b^2+z^2}{a^2})dz$$

$$= -\frac{mV_0}{2k\hbar^2}a\sqrt{\pi}\exp(-\frac{b^2}{a^2})$$

(b) Yukawa potential: $V = V_0\frac{1}{\mu r}\exp(-\mu r)$

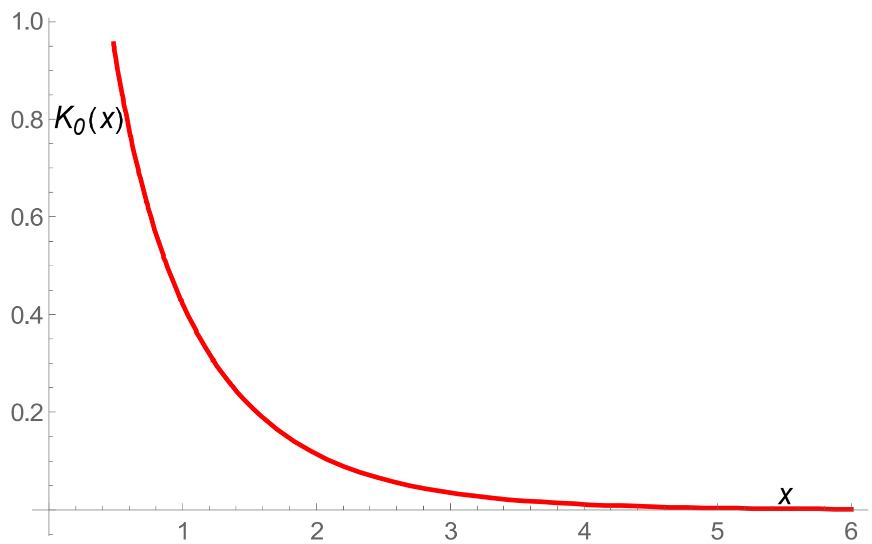
$$\Delta(b=\frac{1}{k}) = -\frac{m}{2k\hbar^2}\int_{-\infty}^{\infty}V_0\frac{\exp(-\mu\sqrt{b^2+z^2})}{\mu\sqrt{b^2+z^2}}dz$$

$$= -\frac{mV_0}{k\hbar^2\mu}\int_0^{\infty}\frac{\exp(-\mu\sqrt{b^2+z^2})}{\sqrt{b^2+z^2}}dz$$

$$= -\frac{mV_0}{k\hbar^2\mu}\int_b^{\infty}\frac{\exp(-\mu r)}{\sqrt{r^2-b^2}}dr$$

$$= -\frac{mV_0}{k\hbar^2\mu}K_0(b\mu)$$

$K_0(b\mu)$ is the modified Bessel function of the second kind.



6-9

6.9 **(a)** Prove

$$\frac{\hbar^2}{2m}\langle \mathbf{x}|\frac{1}{E-H_0+i\varepsilon}|\mathbf{x}'\rangle = -ik\sum_l\sum_m Y_l^m(\hat{\mathbf{r}})Y_l^{m*}(\hat{\mathbf{r}}')j_l(kr_<)h_l^{(1)}(kr_>),$$

where $r_<(r_>)$ stands for the smaller (larger) of r and r'.

(b) For spherically symmetrical potentials, the Lippmann-Schwinger equation can be written for *spherical* waves:

$$|Elm(+)\rangle = |Elm\rangle + \frac{1}{E-H_0+i\varepsilon}V|Elm(+)\rangle.$$

Using (a), show that this equation, written in the **x**-representation, leads to an equation for the radial function, $A_l(k;r)$, as follows:

$$A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2} \times \int_0^\infty j_l(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;r')r'^2dr'.$$

By taking r very large, also obtain

$$f_l(k) = e^{i\delta_l}\frac{\sin\delta_l}{k} = -\left(\frac{2m}{\hbar^2}\right)\int_0^\infty j_l(kr)A_l(k;r)V(r)r^2dr.$$

((**Solution**))

(a) Green function with spherical Bessel function

Free particle wave function ψ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E_k\psi,$$

where m is the mass of particle, ($E_k = \frac{\hbar^2}{2m}k^2$) is the energy of the particle, and k is the wave number. This equation can be rewritten as

$$(\nabla^2 + k^2)\psi = 0.$$

This equation is solved in a formal way as

$$\psi = \varphi_{k\ell m}(r,\theta,\phi) = \langle r\theta\phi | k\ell m\rangle$$

$$\frac{1}{2m}(p_r{}^2 + \frac{\boldsymbol{L}^2}{r^2})\varphi_{k\ell m}(r,\theta,\phi) = E_k \varphi_{k\ell m}(r,\theta,\phi)$$

(separation variables), where $\boldsymbol{L}$ is the angular momentum:

$$\varphi_{k\ell m}(r,\theta,\phi) = R_{k\ell}(r) Y_{\ell m}(\theta,\phi)$$

with

$$\boldsymbol{L}^2 Y_{\ell m}(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_{\ell m}(\theta,\phi)$$

Since $p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r$, we have

$$p_r{}^2 R_{k\ell}(r) = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r) R_{k\ell}(r) = -\hbar^2 \frac{1}{r}\frac{\partial^2}{\partial r^2}[r R_{k\ell}(r)]$$

or

$$-\frac{1}{r}\frac{\partial^2}{\partial r^2}[r R_{k\ell}(r)] + \frac{1}{r^2}\ell(\ell+1) R_{k\ell}(r) = k^2 R_{k\ell}(r)$$

or

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[r R_{k\ell}(r)] + [k^2 - \frac{1}{r^2}\ell(\ell+1)] R_{k\ell}(r) = 0 .$$

with

$$E_k = \frac{\hbar^2 k^2}{2m} .$$

We put $x = kr$ (dimensionless)

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} = k\frac{\partial}{\partial x}, \quad \frac{\partial^2}{\partial r^2} = k\frac{\partial}{\partial x}(k\frac{\partial}{\partial x}) = k^2 \frac{\partial^2}{\partial x^2}$$

$$[-\frac{k}{x}k^2 \frac{\partial^2}{\partial r^2}(\frac{x}{k}) + \frac{k^2}{x^2}\ell(\ell+1)]R = k^2 R$$

or

$$\frac{1}{x}\frac{d^2}{dx^2}(xR)+[1-\frac{1}{x^2}\ell(\ell+1)]R=0 \text{ (Spherical Bessel equation).}$$

or

$$\frac{1}{x}[xR''+2R']+[1-\frac{1}{x^2}\ell(\ell+1)]R=0$$

or

$$R''+\frac{2}{x}R'+[1-\frac{\ell(\ell+1)}{x^2}]R=0$$

or

$$\frac{d}{dx}(x^2R')+[x^2-\ell(\ell+1)]R=0\,.$$

This is a Sturm-Liouville-type differential equation.
Here we suppose that

$$R=\frac{J(x)}{\sqrt{x}}\,,$$

$$\frac{d^2J}{dx^2}+\frac{1}{x}\frac{dJ}{dx}+[1-\frac{(\ell+\frac{1}{2})^2}{x^2}]J=0\,.$$

The solution of this differential equation is

$$J(x)=J_{\ell+1/2}(x), \qquad \text{or} \qquad J(x)=N_{\ell+1/2}(x)\,.$$

Then the solutions of R are obtained as the spherical Bessel functions defined by

$$j_\ell(x)=\sqrt{\frac{\pi}{2x}}J_{\ell+1/2}(x)\,,$$

and spherical Neumann function defined by

$$n_\ell(x)=\sqrt{\frac{\pi}{2x}}N_{\ell+1/2}(x)\,.$$

Since the spherical Neumann function diverges at x =0, it cannot be chosen as a solution. Finally we have

$$\varphi_{k\ell m}(r,\theta,\phi) = \langle r,\theta,\phi | k,l,m \rangle = \sqrt{\frac{2k^2}{\pi}} j_\ell(kr) Y_{\ell m}(\theta,\phi),$$

with

$$E_k = \frac{\hbar^2 k^2}{2m},$$

and

$$\langle k'l'm' | klm \rangle = \delta(k-k')\delta_{l,l'}\delta_{m,m'}.$$

We define the spherical Hankel functions as

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+\frac{1}{2}}(x) = j_n(x) + i n_n(x)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+\frac{1}{2}}(x) = j_n(x) - i n_n(x)$$

where the spherical Bessel function and spherical Neumann function are given by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

$$n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{\pi}{2x}} J_{-n-\frac{1}{2}}(x)$$

The asymptotic values of the spherical Bessel functions and spherical Hankel functions may be obtained from the Bessel asymptotic form.

$$j_\ell(x) \approx \frac{1}{x}\sin(x - \frac{l\pi}{2}),$$

$$n_\ell(x) \approx -\frac{1}{x}\cos(x - \frac{l\pi}{2}),$$

$$h_\ell^{(1)}(x) \approx -i\frac{e^{i(x-l\pi/2)}}{x} \qquad \text{(outgoing spherical wave)}$$

$$h_\ell^{(2)}(x) \approx i\frac{e^{-i(x-l\pi/2)}}{x}$$ (incoming spherical wave)

Now we consider the Green's function given by

$$(\nabla^2 + k^2)G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

The solution of the Green's function is given by

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

with the boundary condition

$G(\boldsymbol{r},\boldsymbol{r}') \to 0$ for $r \to 0$ and for $r \to \infty$.

where $\boldsymbol{r}$ is the variable and $\boldsymbol{r}'$ is fixed.

Within each region (region I ($0<r<r'$) and region II ($r'<r$), we have the simpler equation

$$(\nabla^2 + k^2)G(\boldsymbol{r},\boldsymbol{r}') = 0.$$

The solution of the Green's function is given by the form

$$G(\boldsymbol{r},\boldsymbol{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r,r',\theta',\phi')Y_l^m(\theta,\phi).$$

Then the differential equation of the Green's function is given by

$$\sum_{l',m'}[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) + (k^2 - \frac{l'(l'+1)}{r^2})A_{l'm'}]Y_{l'}^{m'}(\theta,\phi) = -\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu').$$

Note that

$$\delta_{l,l'}\delta_{m,m'} = \int d\Omega\langle l',m'|\mathbf{n}\rangle\langle\mathbf{n}|l,m\rangle = \iint \sin\theta d\theta d\phi Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi)$$

where

$$d\Omega = \sin\theta d\theta d\phi.$$

Then

$$\sum_{l',m'}\int d\Omega Y_l^{m^*}(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'.})+(k^2-\frac{l'(l'+1)}{r^2})A_{l'm'}]$$

$$=-\int d\Omega Y_l^{m^*}(\theta,\phi)\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\sum_{l',m'}[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'})+(k^2-\frac{l'(l'+1)}{r^2})A_{l'm'}]\delta_{l,l'}\delta_{m,m'}$$

$$=-\int d\Omega Y_l^{m^*}(\theta,\phi)\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\begin{aligned}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{lm})+[k^2-\frac{l(l+1)}{r^2}]A_{lm} &= -\frac{\delta(r-r')}{r^2}\int d\Omega Y_l^{m^*}(\theta,\phi)\delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2}Y_l^{m^*}(\theta',\phi')\int d\mu d\phi\delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2}Y_l^{m^*}(\theta',\phi')\end{aligned}$$

Since $Y_l^{m^*}(\theta',\phi')$ is constant, we put

$$G_l(r,r')=\frac{A_{lm}(r,r',\theta',\phi')}{Y_l^{m^*}(\theta',\phi')}.$$

Then we get

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG_l)+[k^2-\frac{l(l+1)}{r^2}]G_l=-\frac{\delta(r-r')}{r^2},$$

The possible solutions of G_l are $j_l(kr)$, $n_l(kr)$, $h_l^{(1)}(kr)$, $h_l^{(1)}(kr)$, or a linear combination of these functions.

$G_{l\,I}=Aj_l(kr)$ for $r<r'$ (region I)

$G_{l\,II}=Bh_l^{(1)}(kr)$ for $r>r'$ (region II)

where A and B are constant. Note that If we use the positive sign for $G(\boldsymbol{r},\boldsymbol{r}')$, we need to choose $h_l^{(1)}(kr)$;

$$h_\ell^{(1)}(kr) \approx -i\frac{e^{i(kr-l\pi/2)}}{kr} \approx \frac{e^{ikr}}{r} \qquad \text{(outgoing spherical wave)}$$

(i) The continuity of G_l at $r = r'$

$$Aj_l(kr') = Bh_l^{(1)}(kr')$$

or

$$\frac{A}{h_l^{(1)}(kr')} = \frac{B}{j_l(kr')} = C$$

(ii) The discontinuity of dG_l/dr at $r = r'$.

$$\int_{r'-\varepsilon}^{r'+\varepsilon} \{\frac{d^2}{dr^2}(rG_l) + [k^2 r - \frac{l(l+1)}{r}]G_l\}dr = -\int_{r'-\varepsilon}^{r'+\varepsilon} \frac{\delta(r-r')}{r}dr$$

or

$$[\frac{d}{dr}(rG_l)]_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{1}{r'}$$

or

$$(G_l + r\frac{dG_l}{dr})\,|_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{1}{r'}$$

$$\frac{dG_l^{II}(k,r,r')}{dr}|_{r'+\varepsilon} - \frac{dG_l^{I}(k,r,r')}{dr}|_{r'-\varepsilon} = -\frac{1}{r'^2}$$

or

$$kC[j_l(kr')h_l^{(1)}{}'(kr') - j_l'(kr')h_l^{(1)}(kr')] = -\frac{1}{r'^2}$$

We need to calculate the Wronskian

$$W = \begin{vmatrix} j_l(kr') & n_l(kr') \\ j_l'(kr') & n_l'(kr') \end{vmatrix} = \frac{i}{k^2 r'^2}$$

Thus, we get

$C = ik$

In general, we have

$$G(\boldsymbol{r},\boldsymbol{r}') = ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi').$$

This means that

$r_< = r$
$r_> = r'$ in the region I ($r<r'$)

$r_> = r$
$r_< = r'$ in the region II ($r'<r$)

We also get

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

$$G_0^{(\pm)}(\boldsymbol{r},\boldsymbol{r}') = -\frac{\hbar^2}{2\mu}\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}$$

In summary, we have

$$\frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{\hbar^2}{2\mu}\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle$$
$$= ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

(b)
Lipmann-Schwinger equation

$$\langle\boldsymbol{r}|Elm(+)\rangle = \langle\boldsymbol{r}|Elm\rangle + \langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}\hat{V}|Elm(+)\rangle$$
$$= \langle\boldsymbol{r}|Elm\rangle + \int d^3\boldsymbol{r}'\langle\boldsymbol{r}|(E_k-\hat{H}_0+i\varepsilon)^{-1}|\boldsymbol{r}'\rangle\hat{V}(r')\langle\boldsymbol{r}'|Elm(+)\rangle$$

where

$$\langle \boldsymbol{r} | Elm(+) \rangle = c_l A_l(k;r) Y_l^m(\boldsymbol{n})$$

$$\langle \boldsymbol{r} | Elm \rangle = c_l j_l(kr) Y_l^m(\boldsymbol{n})$$

with

$$c_l = \frac{i^l}{\hbar}\sqrt{\frac{2mk}{\pi}} \qquad \langle \boldsymbol{r} | Elm \rangle = \langle r,\theta,\phi | Elm \rangle$$

Substituting these equations into the Lipmann-Schwinger equation and dividing both sides by c_l, we have

$$\begin{aligned}
A_l(k;r)Y_l^m(\boldsymbol{n}) &= j_l(kr)Y_l^m(\boldsymbol{n}) \\
&\quad -\frac{2mik}{\hbar^2}\int d^3\boldsymbol{r}'\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} Y_{l'}^{m'}(\boldsymbol{n})Y_{l'}^{m'^*}(\boldsymbol{n}')j_{l'}(kr_<)h_{l'}^{(1)}(kr_>)V(r')A_l(k;r')Y_l^m(\boldsymbol{n}') \\
&= j_l(kr)Y_l^m(\boldsymbol{n}) \\
&\quad -\frac{2mik}{\hbar^2}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} Y_{l'}^{m'}(\boldsymbol{n})\int_0^{\infty} r'^2\,dr'\, j_{l'}(kr_<)h_{l'}^{(1)}(kr_>)V(r')A_l(k;r')\int d\Omega' Y_{l'}^{m'^*}(\boldsymbol{n}')Y_l^m(\boldsymbol{n}') \\
&= j_l(kr)Y_l^m(\boldsymbol{n}) \\
&\quad -\frac{2mik}{\hbar^2}Y_l^m(\boldsymbol{n})\int_0^{\infty} r'^2\,dr'\, j_l(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;r')
\end{aligned}$$

where we use

$$\int d\Omega' Y_{l'}^{m'^*}(\boldsymbol{n}')Y_l^m(\boldsymbol{n}') = \delta_{l,l'}\delta_{m,m'}$$

Dividing both side of the above equation by $Y_l^m(\boldsymbol{n})$, we get

$$A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2}\int_0^{\infty} r'^2\,dr'\, j_l(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;r')$$

Here we note that in the limit of $r\to\infty$, $j_l(kr)$ and $h_l^{(1)}(kr_>)$ have the asymptotic forms,

$$j_l(x) \to \frac{1}{x}\cos[x - \frac{1}{2}(l+1)\pi], \qquad h_l^{(1)}(x) \to \frac{1}{x}\exp[i(x - \frac{1}{2}(l+1)\pi]$$

Since $r \to \infty$, we have $r_> = r$, and $r_< = r'$. Then we get

$$A_l(k;r) = \frac{1}{2kr}\{\exp[i(kr - \frac{1}{2}(l+1)\pi] + \exp[-i(kr - \frac{1}{2}(l+1)\pi]\}$$
$$- \frac{2mik}{\hbar^2}\frac{1}{kr}\exp[i(kr - \frac{1}{2}(l+1)\pi]\int_0^\infty r'^2\,dr'\,j_l(kr')V(r')A_l(k;r')$$
$$= \frac{1}{2kr}\exp[i(kr - \frac{1}{2}(l+1)\pi][1 - \frac{4mik}{\hbar^2}\int_0^\infty r'^2\,dr'\,j_l(kr')V(r')A_l(k;r')]$$
$$+ \frac{1}{2kr}\exp[-i(kr - \frac{1}{2}(l+1)\pi]\}$$

In the absence of the potential $(V = 0)$, we have

$$A_l(k;r)|_{V=0} = \frac{1}{2kr}\exp[i(kr - \frac{1}{2}(l+1)\pi] + \frac{1}{2kr}\exp[-i(kr - \frac{1}{2}(l+1)\pi]\}.$$

The difference between $A_l(k;r)$ and $A_l(k;r)|_{V=0}$ is given by

$$A_l(k;r) - = \frac{1}{2kr}\{\exp[i(kr - \frac{1}{2}(l+1)\pi] + \exp[-i(kr - \frac{1}{2}(l+1)\pi]\}$$
$$- \frac{2mik}{\hbar^2}\frac{1}{kr}\exp[i(kr - \frac{1}{2}(l+1)\pi]\int_0^\infty r'^2\,dr'\,j_l(kr')V(r')A_l(k;r')$$
$$= \frac{1}{r}e^{ikr}\frac{1}{i^l}\int_0^\infty r'^2\,dr'\,j_l(kr')U(r')A_l(k;r')]$$

leading to the scattering amplitude

$$f_l(\theta) = e^{i\delta_l}\frac{\sin\delta_l}{k} = -\int_0^\infty r^2 dr j_l(kr)U(r)A_l(k;r)$$

When $A_l(k;r) \approx j_l(kr)$,

$$f_l(\theta) = e^{i\delta_l}\frac{\sin\delta_l}{k} \approx -\int_0^\infty r^2 dr[j_l(kr)]^2 U(r)$$

6-10

6.10 Consider scattering by a repulsive δ-shell potential:

$$\left(\frac{2m}{\hbar^2}\right)V(r)=\gamma\delta(r-R), \quad (\gamma>0).$$

(a) Set up an equation that determines the s-wave phase shift δ_0 as a function of $k(E=\hbar^2k^2/2m)$.

(b) Assume now that γ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

Show that if $\tan kR$ is *not* close to zero, the s-wave phase shift resembles the hard-sphere result discussed in the text. Show also that for $\tan kR$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot\delta_0$ goes through zero from the positive side as k increases. Determine approximately the positions of the resonances keeping terms of order $1/\gamma$; compare them with the bound-state energies for a particle confined *inside* a spherical wall of the same radius,

$$V=0, \quad r<R; \qquad V=\infty, \quad r>R.$$

Also obtain an approximate expression for the resonance width Γ defined by

$$\Gamma=\frac{-2}{[d(\cot\delta_0)/dE]|_{E=E_r}},$$

and notice, in particular, that the resonances become extremely sharp as γ becomes large. (*Note:* For a different, more sophisticated approach to this problem, see Gottfried 1966, pp. 131–41, who discusses the analytic properties of the D_l-function defined by $A_l=j_l/D_l$.)

((**Solution**))

$$f_l(\theta)=e^{i\delta_l}\frac{\sin\delta_l}{k}=-\int_0^\infty r^2 dr j_l(kr)U(r)A_l(k;r)$$

When

$$U(r)=\frac{2m}{\hbar^2}V(r)=\gamma\delta(r-R) \qquad \text{for } \gamma>0$$

we have

$$f_0(\theta) = e^{i\delta_0}\frac{\sin\delta_0}{k} = -\int_0^\infty r^2 dr j_0(kr) A_0(k;r)\gamma\delta(r-R) = -\gamma R^2 j_0(kR) A_0(k;R)$$

Here we note that

$$\begin{aligned} A_0(k;R) &= j_0(kR) - ikh_0^{(1)}(kR)\int_0^\infty r'^2\, dr'\, j_0(kr')\gamma\delta(r'-R) A_0(k;r') \\ &= j_0(kR) - ik\gamma h_0^{(1)}(kR) j_0(kR) A_0(k;R) R^2 \end{aligned}$$

Then we get

$$A_0(k;R) = \frac{j_0(kR)}{1 + ik\gamma h_0^{(1)}(kR) j_0(kR) R^2}$$

Here we use

$$h_0(x) = j_0(x) + in_0(x) = -i\frac{e^{ix}}{x}, \qquad j_0(x) = \frac{\sin x}{x}$$

$$\begin{aligned} A_0(k;R) &= \frac{j_0(kR)}{1 + k\gamma\frac{1}{kR}e^{ikR} j_0(kR) R^2} \\ &= \frac{\frac{\sin(kR)}{kR}}{1 + \gamma e^{ikR}\frac{\sin(kR)}{kR}R} \\ &= \frac{1}{kR}\frac{\sin(kR)}{1 + \frac{\gamma}{k}e^{ikR}\sin(kR)} \end{aligned}$$

Then we have

$$e^{i\delta_0}\frac{\sin\delta_0}{k}=-\gamma R^2 j_0(kR)A_0(k;R)$$

$$=-\gamma R^2\frac{\sin(kR)}{kR}\frac{1}{kR}\frac{\sin(kR)}{1+\frac{\gamma}{k}e^{ikR}\sin(kR)}$$

$$=-\gamma R^2\frac{\sin^2(kR)}{(kR)^2}\frac{1}{1+\frac{\gamma}{k}e^{ikR}\sin(kR)}$$

$$=-\gamma R^2\frac{\sin^2(kR)}{(kR)^2}\left(\frac{1+\frac{\gamma}{k}\sin(kR)\cos(kR)-i\frac{\gamma}{k}\sin^2(kR)}{1+\frac{\gamma^2}{k^2}\sin^2(kR)+2\frac{\gamma}{k}\sin(kR)\cos(kR)}\right)$$

Then we get

$$\tan\delta_0=-\frac{\frac{\gamma}{k}\sin^2(kR)}{1+\frac{\gamma}{k}\sin(kR)\cos(kR)}=\frac{1}{\cot\delta_0}$$

(i) $\gamma >> k$ ($kR << 1$, $\gamma R >> 1$)

$$1+\frac{\gamma}{k}\sin(kR)\cos(kR)\approx 1+\frac{\gamma}{k}kR\cos(kR)\approx 1+\gamma R\approx\gamma R$$

Then we get

$$\tan\delta_0\approx-\frac{\frac{\gamma}{k}\sin^2(kR)}{\frac{\gamma}{k}\sin(kR)\cos(kR)}=-\tan(kR)$$

leading to

$$\delta_0=-kR+n\pi$$

(ii) Resonance condition

$$\cot\delta_0 = \frac{1+\frac{\gamma}{2k}\sin(2kR)}{-\frac{\gamma}{k}\sin^2(kR)}$$

The denominator is always negative. $\frac{\gamma}{2k} >> 1$. The resonance occurs when

$$1+\frac{\gamma}{k}\sin(kR)\cos(kR) = 1+\frac{\gamma}{2k}\sin(2kR) = 0$$

or

$$\sin(2kR) = -\frac{2k}{\gamma} \approx \sin(-\frac{2k}{\gamma})$$

or

$$2kR = 2n\pi - \frac{2k}{\gamma}$$

6-11

6.11 A spinless particle is scattered by a time-dependent potential

$$\mathcal{V}(\mathbf{r},t) = V(\mathbf{r})\cos\omega t.$$

Show that if the potential is treated to first order in the transition amplitude, the energy of the scattered particle is increased or decreased by $\hbar\omega$. Obtain $d\sigma/d\Omega$. Discuss qualitatively what happens if the higher-order terms are taken into account.

((**Solution**))

The initial state is given by $|\boldsymbol{k}\rangle$. The time dependent potential is given by

$$V = v(r)\cos(\omega t)e^{\eta t}$$

$$\langle \boldsymbol{k}'|\hat{U}_I(t,t_0=-\infty)|\boldsymbol{k}\rangle = I_1 + I_2$$

$$= (-\frac{i}{\hbar})\int_{-\infty}^{t}\langle \boldsymbol{k}'|\hat{V}_I(t')|\boldsymbol{k}\rangle dt' + \lambda^2(-\frac{i}{\hbar})^2\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\langle \boldsymbol{k}'|\hat{V}_I(t')\hat{V}_I(t'')|\boldsymbol{k}\rangle$$

$$I_1 = (-\frac{i}{\hbar})\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\int_{-\infty}^{t}dt'\cos(\omega t')e^{\eta t'}\exp[\frac{i(E'-E)t'}{\hbar}]dt'$$

$$= (-\frac{i}{\hbar})\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\int_{-\infty}^{t}dt'\cos(\omega t')e^{\eta t'}\exp[\frac{i(E'-E)t'}{\hbar}]dt'$$

$$= (-\frac{i}{\hbar})\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\frac{1}{2}[\frac{\exp[\eta t + i(E'-E+\hbar\omega)t/\hbar}{\eta + \frac{i}{\hbar}(E'-E+\hbar\omega)}$$

$$+\frac{\exp[\eta t + i(E'-E-\hbar\omega)t/\hbar}{\eta + \frac{i}{\hbar}(E'-E-\hbar\omega)}]$$

The probability of finding the system in the state $|\boldsymbol{k}'\rangle$

$$P_1 = \frac{1}{\hbar^2}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2\frac{1}{4}[\frac{e^{2\eta t}}{\eta^2 + \frac{1}{\hbar^2}(E'-E+\hbar\omega)^2} + \frac{e^{2\eta t}}{\eta^2 + \frac{1}{\hbar^2}(E'-E-\hbar\omega)^2}$$

$$+2\,\mathrm{Re}\frac{\exp(\eta t + 2i\omega t)}{[\eta + \frac{i}{\hbar}(E'-E+\hbar\omega)][\eta - \frac{i}{\hbar}(E'-E-\hbar\omega)]}]$$

When $\eta \to 0$ (t is finite), the transition rate

$$w_{\boldsymbol{k}\to\boldsymbol{k}'} = \frac{d}{dt}P_1$$

$$= \frac{1}{\hbar^2}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2\frac{1}{4}[\frac{2\eta e^{2\eta t}}{\eta^2 + \frac{1}{\hbar^2}(E'-E+\hbar\omega)^2} + \frac{2\eta e^{2\eta t}}{\eta^2 + \frac{1}{\hbar^2}(E'-E-\hbar\omega)^2}]$$

Here the oscillatory part (proportional to $e^{2i\omega t}$ Using the formula

$$\lim_{\eta\to 0}\frac{\eta}{x^2+\eta^2} = \pi\delta(x)$$

we have

$$w_{k\to k'} = \frac{2\pi}{\hbar}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2 \frac{1}{4}[\delta(E'-E+\hbar\omega)+\delta(E'-E-\hbar\omega)]$$

((**Density of states**))

$$E_{k'} = \frac{\hbar^2}{2m}\boldsymbol{k}'^2 \qquad \varepsilon' = \frac{\hbar^2}{2m}k'^2$$

$$d\varepsilon' = \frac{\hbar^2}{m}k'dk'$$

The density of states is obtained as

$$\rho(\varepsilon')d\varepsilon' = \frac{1}{\left(\frac{2\pi}{L}\right)^3}k'^2 dk' d\Omega = \frac{L^3}{(2\pi)^3}\frac{m}{\hbar^2}k' d\varepsilon' d\Omega$$

$$\begin{aligned} w &= \frac{2\pi}{\hbar}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2 \frac{1}{4}\int[\delta(E'-E+\hbar\omega)+\delta(E'-E-\hbar\omega)]dE'\frac{L^3}{(2\pi)^3}\frac{mk'}{\hbar^2}d\Omega \\ &= \frac{2\pi}{\hbar}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2 \frac{1}{4}\frac{L^3}{(2\pi)^3}\frac{m}{\hbar^2}(k_+ + k_-)d\Omega \end{aligned}$$

From the Dirac function in the first term,

$$E' = E - \hbar\omega, \qquad k' = k_- = \sqrt{\frac{2m}{\hbar^2}(E-\hbar\omega)}$$

From the Dirac function in the second term,

$$E' = E + \hbar\omega \qquad k' = k_+ = \sqrt{\frac{2m}{\hbar^2}(E+\hbar\omega)}$$

We consider the wave function given by

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{L^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad \left|\langle \boldsymbol{r}|\boldsymbol{k}\rangle\right| = \frac{1}{L^3}$$

We consider the number of incident particles. Since $\left|\langle \boldsymbol{r}|\boldsymbol{k}\rangle\right|^2 = \frac{1}{L^3}$ and the velocity of particle is $v = \frac{\hbar k}{m}$. The number of incident particles passing per unit area per unit second is $v\frac{1}{L^3} = \frac{\hbar k}{mL^3}$.

$$
\begin{aligned}
d\sigma &= \frac{w}{\frac{\hbar k}{mL^3}} \\
&= \frac{2\pi}{\hbar}\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2 \frac{1}{4}\frac{L^3}{(2\pi)^3}\frac{m}{\hbar^2}(k_+ + k_-)\frac{d\Omega}{\frac{\hbar k}{mL^3}} \\
&= 2\pi\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2 \frac{L^6}{(2\pi)^3}\frac{m^2}{\hbar^4}\left(\frac{k_+}{k} + \frac{k_-}{k}\right)d\Omega \\
&= \frac{1}{4}\left(\frac{mL^3}{2\pi\hbar^2}\right)^2\left|\langle \boldsymbol{k}'|\hat{V}|\boldsymbol{k}\rangle\right|^2\left(\frac{k_+}{k} + \frac{k_-}{k}\right)d\Omega
\end{aligned}
$$

6-12

6.12 Show that the differential cross section for the elastic scattering of a fast electron by the ground state of the hydrogen atom is given by

$$
\frac{d\sigma}{d\Omega} = \left(\frac{4m^2e^4}{\hbar^4 q^4}\right)\left\{1 - \frac{16}{[4+(qa_0)^2]^2}\right\}^2 .
$$

(Ignore the effect of identity.)

((**Solution**))

$$
f(\boldsymbol{k}',\boldsymbol{k}) = -\frac{1}{4\pi}(2\pi)^3\frac{2\mu}{\hbar^2}\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle
$$

The differential cross section:

$$
\frac{\partial\sigma}{\partial\Omega} = \left|f(\boldsymbol{k}',\boldsymbol{k})\right| = \frac{16\pi^4\mu^2}{\hbar^4}\left|\langle \boldsymbol{k}',1s|V(\hat{\boldsymbol{r}},\hat{\boldsymbol{r}}')|\boldsymbol{k},1s\rangle\right|^2
$$

The potential energy is given by

$$V(\hat{\boldsymbol{r}},\hat{\boldsymbol{r}}') = \frac{e^2}{|\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}'|} - \frac{e^2}{|\hat{\boldsymbol{r}}|}$$

We calculate the matrix element

$$\begin{aligned}
\langle \boldsymbol{k}',1s|\frac{1}{|\hat{\boldsymbol{r}}|}|\boldsymbol{k},1s\rangle &= \int d\boldsymbol{r}\int d\boldsymbol{r}'\langle \boldsymbol{k}',1s|\boldsymbol{r},\boldsymbol{r}'\rangle\langle \boldsymbol{r},\boldsymbol{r}'|\frac{1}{|\hat{\boldsymbol{r}}|}|\boldsymbol{k},1s\rangle \\
&= \int d\boldsymbol{r}\int d\boldsymbol{r}'\langle \boldsymbol{k}'|\boldsymbol{r}\rangle\langle 1s|\boldsymbol{r}'\rangle\frac{1}{|\boldsymbol{r}|}\langle \boldsymbol{r}|\boldsymbol{k}\rangle\langle \boldsymbol{r}'|1s\rangle \\
&= \frac{1}{(2\pi)^3}\frac{1}{\pi a_0^{\ 3}}\int d\boldsymbol{r}\int d\boldsymbol{r}'e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\frac{1}{r}e^{-2r'/a_0} \\
&= \frac{1}{(2\pi)^3}\frac{1}{\pi a_0^{\ 3}}\int d\boldsymbol{r}\int d\boldsymbol{r}'e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{1}{r}e^{-2r'/a_0} \\
&= \frac{1}{(2\pi)^3}\frac{1}{\pi a_0^{\ 3}}\int d\boldsymbol{r}'e^{-2r'/a_0}\int e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{1}{r}d\boldsymbol{r}
\end{aligned}$$

where $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$

$$\langle \boldsymbol{r}|1s\rangle = \frac{1}{\sqrt{\pi}a_0^{\ 3/2}}e^{-r/a_0} \qquad \text{(1s state wave function)}$$

$$I_1 = \int d\boldsymbol{r}'e^{-2r'/a_0} = 4\pi\int_0^\infty r'^2 e^{-2r'/a_0}dr' = 4\pi\frac{a_0^{\ 3}}{4}$$

$$\begin{aligned}
I_2 &= \int d\boldsymbol{r}\frac{1}{r}e^{1\boldsymbol{q}\cdot\boldsymbol{r}}e^{-\varepsilon r} \\
&= 2\pi\int_0^\infty \frac{1}{r}r^2 e^{-\varepsilon r}dr\int_0^\pi d\theta\sin\theta e^{iqr\cos\theta}
\end{aligned}$$

where $\varepsilon \to 0$.

$$\int_0^{\pi} d\theta \sin\theta e^{iqr\cos\theta} = [-\frac{1}{iqr} e^{iqr\cos\theta}]_0^{\pi}$$

$$= \frac{1}{iqr}(e^{iqr} - e^{-iqr})$$

$$= \frac{2}{qr}\sin(qr)$$

Then we have

$$I_2 = 2\pi \int_0^{\infty} \frac{1}{r} r^2 e^{-\varepsilon r} \frac{2}{qr} \sin(qr) dr$$

$$= \frac{4\pi}{q} \int_0^{\infty} e^{-\varepsilon r} \sin(qr) dr$$

$$= \frac{4\pi}{q} \mathrm{Im}[\int_0^{\infty} e^{-\varepsilon r} e^{iqr} dr]$$

$$= \frac{4\pi}{q} \mathrm{Im}(\frac{1}{\varepsilon - iq})$$

$$= \frac{4\pi}{q} \mathrm{Im}(\frac{\varepsilon + iq}{\varepsilon^2 + q^2})$$

$$= \frac{4\pi}{(\varepsilon^2 + q^2)}$$

In the limit ($\varepsilon \to 0$), we get

$$I_2 = \frac{4\pi}{q^2}$$

$$\langle \boldsymbol{k}', 1s | \frac{1}{|\hat{\boldsymbol{r}}|} | \boldsymbol{k}, 1s \rangle = \frac{1}{(2\pi)^3} \frac{1}{\pi a_0^3} \pi a_0^3 \frac{4\pi}{q^2} = \frac{1}{(2\pi)^3} \frac{4\pi}{q^2}$$

Next we calculate another matric element

$$\left\langle \boldsymbol{k}',1s\right|\frac{1}{|\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}'|}\left|\boldsymbol{k},1s\right\rangle = \int d\boldsymbol{r}\int d\boldsymbol{r}'\left\langle \boldsymbol{k}',1s\middle|\boldsymbol{r},\boldsymbol{r}'\right\rangle\left\langle \boldsymbol{r},\boldsymbol{r}'\right|\frac{1}{|\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}'|}\left|\boldsymbol{k},1s\right\rangle$$

$$= \int d\boldsymbol{r}\int d\boldsymbol{r}'\left\langle \boldsymbol{k}'\middle|\boldsymbol{r}\right\rangle\left\langle 1s\middle|\boldsymbol{r}'\right\rangle\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left\langle \boldsymbol{r}\middle|\boldsymbol{k}\right\rangle\left\langle \boldsymbol{r}'\middle|1s\right\rangle$$

$$= \frac{1}{(2\pi)^3}\frac{1}{\pi a_0{}^3}\int d\boldsymbol{r}\int d\boldsymbol{r}' e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{-2r'/a_0}$$

with

$$\boldsymbol{q} = \boldsymbol{k}-\boldsymbol{k}', \qquad \boldsymbol{\xi} = \boldsymbol{r}-\boldsymbol{r}'$$

$$\left\langle \boldsymbol{k}',1s\right|\frac{1}{|\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}'|}\left|\boldsymbol{k},1s\right\rangle = \frac{1}{(2\pi)^3}\frac{1}{\pi a_0{}^3}\int \frac{1}{\xi}d\boldsymbol{\xi}\int d\boldsymbol{r}' e^{i\boldsymbol{q}\cdot\boldsymbol{r}}e^{-2r'/a_0}$$

$$= \frac{1}{(2\pi)^3}\frac{1}{\pi a_0{}^3}\int \frac{1}{\xi}e^{i\boldsymbol{q}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}\int d\boldsymbol{r}' e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}e^{-2r'/a_0}$$

$$I_3 = \int \frac{1}{\xi}e^{i\boldsymbol{q}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} = \frac{4\pi}{q^2}$$

$$I_4 = \int d\boldsymbol{r}' e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}e^{-2r'/a_0}$$

where

$$\int d\boldsymbol{r}' e^{i\boldsymbol{q}\cdot\boldsymbol{r}'}e^{-2r'/a_0} = \int_0^\infty 2\pi r'^2\, e^{-2r'/a_0}dr'\int_0^\pi \sin\theta d\theta e^{iqr'\cos}$$

$$= \int_0^\infty 2\pi r'^2\, e^{-2r'/a_0}\,\frac{2\sin(qr')}{qr'}dr'$$

$$= \frac{4\pi}{q}\int_0^\infty r' e^{-2r'/a_0}\sin(qr')dr'$$

$$= \frac{4\pi}{q}\frac{4a^3q}{(4+a^2q^2)^2}$$

$$= \frac{16\pi a^3}{(4+a^2q^2)^2}$$

where

$$\int_0^{\pi} \sin\theta d\theta e^{iqr'\cos} = \frac{2\sin(qr')}{qr'}$$

$$\begin{aligned}\left\langle \boldsymbol{k}',1s\right|\frac{1}{\left|\hat{\boldsymbol{r}}-\hat{\boldsymbol{r}}'\right|}\left|\boldsymbol{k},1s\right\rangle &= \frac{1}{(2\pi)^3}\frac{1}{\pi a^3}\frac{4\pi}{q^2}\frac{16\pi a^3}{(4+a^2q^2)^2} \\ &= \frac{1}{(2\pi)^3}\frac{4\pi}{q^2}\frac{16}{(4+a^2q^2)^2}\end{aligned}$$

Thus we have

$$\left\langle \boldsymbol{k}',1s\right|V(\hat{\boldsymbol{x}},\hat{\boldsymbol{x}}')\left|\boldsymbol{k},1s\right\rangle = \frac{1}{(2\pi)^3}\frac{4\pi}{q^2}[1-\frac{16}{(4+a^2q^2)^2}]$$

Thus we have

$$\begin{aligned}\frac{\partial\sigma}{\partial\Omega} &= \frac{16\pi^4\mu^2}{\hbar^4}\left|\left\langle \boldsymbol{k}',1s\right|V(\hat{\boldsymbol{r}},\hat{\boldsymbol{r}}')\left|\boldsymbol{k},1s\right\rangle\right|^2 \\ &= \frac{16\pi^4\mu^2}{\hbar^4}\frac{e^4}{(2\pi)^6}\frac{16\pi^2}{q^4}[1-\frac{16}{(4+a^2q^2)^2}]^2 \\ &= \frac{4\mu^2e^4}{\hbar^4q^4}[1-\frac{16}{(4+a^2q^2)^2}]^2\end{aligned}$$
