Spinor function
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1. Spinor

A *spinor* is a two-dimensional vector,

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix},$$

with complex components *a* and *b*. Spinors were first applied in physics by Wolfgang Pauli; the term spinor was coined by Paul Ehrenfest.

A natural basis for the two component spinors is given by two vectors (basis)

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the general spinor is expressed by the linear commination of these two vectors as

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a|+z\rangle + b|-z\rangle$$

2. Spin orbit interaction

We consider the spin-orbit interaction given by

$$\hat{H}_{LS} = \lambda \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$$

From the addition of orbital angular momentum ($\hbar l$) and spin angular momentum ($\frac{\hbar}{2}$), we have the total angular momentum $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ since

$$D_l \times D_{1/2} = D_{l+1/2} + D_{l-1/2}$$

We know that $\left| j = l + \frac{1}{2}, m \right\rangle$ and $\left| j = l - \frac{1}{2}, m \right\rangle$ are the eigenkets of $\hat{H}_{LS}$. Using the Clebsch-Gordan coefficient,

3. The notation: $\left| j = l + \frac{1}{2}, m \right\rangle$

$$\left| j = l + \frac{1}{2}, m \right\rangle = \sqrt{\frac{l + m + 1/2}{2l + 1}} \left| l, s = \frac{1}{2}; m_l = m - \frac{1}{2}, \frac{1}{2} \right\rangle + \sqrt{\frac{l - m + 1/2}{2l + 1}} \left| l, s = \frac{1}{2}; m_l = m + \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$= \begin{pmatrix} \sqrt{\frac{l + m + 1/2}{2l + 1}} \\ \sqrt{\frac{l - m + 1/2}{2l + 1}} \end{pmatrix}$$

$$\rightarrow \frac{1}{\sqrt{2l + 1}} \begin{pmatrix} \sqrt{l + m + 1/2}\, Y_l^{m-1/2}(\theta, \phi) \\ \sqrt{l - m + 1/2}\, Y_l^{m+1/2}(\theta, \phi) \end{pmatrix}$$

or

$$\left| j = l + \frac{1}{2}, m \right\rangle \rightarrow \Phi(j, m, l = j - \frac{1}{2})$$

with

$$\Phi(j, m, l = j - \frac{1}{2}) = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j + m}\, Y_{j-1/2}^{m-1/2}(\theta, \phi) \\ \sqrt{j - m}\, Y_{j-1/2}^{m+1/2}(\theta, \phi) \end{pmatrix}$$

where

$$\left| l, s = \frac{1}{2}; m_l = m - \frac{1}{2}, \frac{1}{2} \right\rangle \rightarrow Y_l^{m-1/2}(\theta, \phi) \chi_+ = \begin{pmatrix} Y_l^{m-1/2}(\theta, \phi) \\ 0 \end{pmatrix}$$

$$\left| l, s = \frac{1}{2}; m_l = m + \frac{1}{2}, -\frac{1}{2} \right\rangle \rightarrow Y_l^{m+1/2}(\theta, \phi) \chi_- = \begin{pmatrix} 0 \\ Y_l^{m+1/2}(\theta, \phi) \end{pmatrix}$$

4. $\left| j = l - \frac{1}{2}, m \right\rangle$

$$\left|j=l-\frac{1}{2},m\right\rangle=-\sqrt{\frac{l-m+1/2}{2l+1}}\left|l,s=\frac{1}{2};m_l=m-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\sqrt{\frac{l+m+1/2}{2l+1}}\left|l,s=\frac{1}{2};m_l=m+\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$=\begin{pmatrix}-\sqrt{\frac{l-m+1/2}{2l+1}}\\ \sqrt{\frac{l+m+1/2}{2l+1}}\end{pmatrix}$$
$$\rightarrow\frac{1}{\sqrt{2l+1}}\begin{pmatrix}-\sqrt{l-m+1/2}\,Y_l^{m-1/2}(\theta,\phi)\\ \sqrt{l+m+1/2}\,Y_l^{m+1/2}(\theta,\phi)\end{pmatrix}$$

or

$$\left|j=l-\frac{1}{2},m\right\rangle\rightarrow\Phi(j,m,l=j+\frac{1}{2})$$

with

$$\Phi(j,m,l=j+\frac{1}{2})=\frac{1}{\sqrt{2(j+1)}}\begin{pmatrix}-\sqrt{j-m+1}\,Y_{j+1/2}^{m-1/2}(\theta,\phi)\\ \sqrt{j+m+1}\,Y_{j+1/2}^{m+1/2}(\theta,\phi)\end{pmatrix}$$

REFERENCES