Time Reversal in Scattering

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Based on the properties of the time reversal operator and the parity opereator, we analyze the symmetric properties of the transition operator in the scattering.

We start with the Lippmann-Schwinger equation given by

$$\left|\psi^{(+)}\right\rangle = \left|\boldsymbol{k}\right\rangle + (E_k - \hat{H}_0 + i\varepsilon)^{-1}\hat{V}\left|\psi^{(+)}\right\rangle = \left|\boldsymbol{k}\right\rangle + (E_k - \hat{H}_0 + i\varepsilon)^{-1}\hat{T}\left|\boldsymbol{k}\right\rangle,$$

where the transition operator $\hat{T}$ is defined as

$$\hat{V}\left|\psi^{(+)}\right\rangle = \hat{T}\left|\boldsymbol{k}\right\rangle$$

or

$$\hat{T}\left|\boldsymbol{k}\right\rangle = \hat{V}\left|\psi^{(+)}\right\rangle = \hat{V}\left|\boldsymbol{k}\right\rangle + \hat{V}(E_k - \hat{H}_0 + i\varepsilon)^{-1}\hat{T}\left|\boldsymbol{k}\right\rangle$$

This is supposed to hold for any $\left|\boldsymbol{k}\right\rangle$ taken to be any plane-wave state.

$$\hat{T} = \hat{V} + \hat{V}(E_k - \hat{H}_0 + i\varepsilon)^{-1}\hat{T} .$$

(a) Parity operator

$$\hat{\pi}\left|\boldsymbol{k}\right\rangle = \left|-\boldsymbol{k}\right\rangle, \qquad \hat{\pi}\left|-\boldsymbol{k}\right\rangle = \left|\boldsymbol{k}\right\rangle$$

or

$$\left\langle\boldsymbol{k}\right|\hat{\pi}^{+} = \left\langle-\boldsymbol{k}\right|, \qquad \left\langle\boldsymbol{k}\right| = \left\langle-\boldsymbol{k}\right|\hat{\pi}$$

Suppose that $\hat{H}_0$ and $\hat{V}$ are invariant under the parity operator.

$$\hat{\pi}\hat{H}_0\hat{\pi}^{+} = \hat{H}_0, \qquad \hat{\pi}\hat{V}\hat{\pi}^{+} = \hat{V}$$

This implies that $\hat{T}$ is also invariant under the parity operator.

$$\hat{\pi}\hat{T}\hat{\pi}^{+} = \hat{T}, \qquad \text{or} \qquad \hat{\pi}\hat{T}\hat{\pi} = \hat{T}$$

Thus we have

$$\langle -\boldsymbol{k}'|\hat{T}|-\boldsymbol{k}\rangle = \langle -\boldsymbol{k}'|\hat{\pi}\hat{T}\hat{\pi}|-\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle$$

(b) Time reversal operator

We note that the requirement that $\hat{H}_0$ and $\hat{V}$ are invariant under the time reversal operator, requires that

$$\hat{\Theta}\hat{T}\hat{\Theta}^{-1} = \hat{T}^{+}$$

We note that

$$\begin{aligned}\hat{\Theta}\hat{T}\hat{\Theta}^{-1} &= \hat{U}(\hat{K}\hat{T}\hat{K}^{-1})\hat{U}^{+}\\ &= \hat{U}\hat{T}^{*}\hat{U}^{+}\\ &= \hat{U}[\hat{V}+\hat{V}(E_k-\hat{H}_0-i\varepsilon)^{-1}\hat{T}]\hat{U}^{+}\\ &= \hat{T}^{+}\end{aligned}$$

since

$$\hat{U}\hat{T}\hat{U}^{+} = \hat{T}$$

Here we recall that

$$\langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle\beta|\alpha\rangle$$

where

$$|\tilde{\alpha}\rangle = \hat{\Theta}|\alpha\rangle, \qquad |\tilde{\beta}\rangle = \hat{\Theta}|\beta\rangle$$

Here we assume that

$$|\alpha\rangle = \hat{T}|\boldsymbol{k}\rangle, \qquad\qquad |\beta\rangle = |\boldsymbol{k}'\rangle$$

Then we get

$$|\tilde{\alpha}\rangle = \hat{\Theta}\hat{T}|\boldsymbol{k}\rangle = \hat{\Theta}\hat{T}\hat{\Theta}^{-1}\hat{\Theta}|\boldsymbol{k}\rangle = \hat{T}^{+}|-\boldsymbol{k}\rangle$$

and

$$|\tilde{\beta}\rangle = \hat{\Theta}|\beta\rangle = \hat{\Theta}|\boldsymbol{k}'\rangle = |-\boldsymbol{k}'\rangle$$

Then we have

$$\langle\beta|\alpha\rangle = \langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle = \langle\tilde{\alpha}|\tilde{\beta}\rangle = \langle -\boldsymbol{k}|\hat{T}|-\boldsymbol{k}'\rangle$$

The initial and final momenta are interchanged, in addition to the fact that the directions of the momenta are reversed. This equation expresses the equality of two scattering processes obtained by reversing the path of the particle and is known as the reciprocity relation.

(c) Requirement of time reversal and parity

Suppose that $\hat{H}_0$ and $\hat{V}$ are invariant under the time reversal operator as well as the parity. Thus we get the two conditions,

$$\langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle = \langle -\boldsymbol{k}|\hat{T}|-\boldsymbol{k}'\rangle \qquad \text{under the time reversal}$$

$$\langle -\boldsymbol{k}|\hat{T}|-\boldsymbol{k}'\rangle = \langle \boldsymbol{k}|\hat{T}|\boldsymbol{k}'\rangle \qquad \text{under the parity}$$

leading to the condition that

$$\langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle = \langle \boldsymbol{k}|\hat{T}|\boldsymbol{k}'\rangle$$

or

$$f(\boldsymbol{k}',\boldsymbol{k}) = f(\boldsymbol{k},\boldsymbol{k}')$$

which is known as detailed balance. Note that

$$f(\boldsymbol{k}',\boldsymbol{k}) = -\frac{2\mu}{\hbar^2}\frac{1}{4\pi}(2\pi)^3\langle \boldsymbol{k}'|\hat{T}|\boldsymbol{k}\rangle$$

(d) Spin

Suppose that the particle has a spin s. The spin state of the particle is given by $|s, m_s\rangle$. The initial state of incident free particle is denoted by $|\boldsymbol{k}, m_s\rangle$, while the final state of the outgoing particle is denoted by $|\boldsymbol{k}', m_s'\rangle$.

$$\begin{aligned}\langle \boldsymbol{k}', m_s'|\hat{T}|\boldsymbol{k}, m_s\rangle &= i^{-2m_s+2m_s'}\langle -\boldsymbol{k}, -m_s|\hat{T}|-\boldsymbol{k}', -m_s'\rangle \\ &= i^{-2m_s+2m_s'}\langle \boldsymbol{k}, -m_s|\hat{T}|\boldsymbol{k}', -m_s'\rangle\end{aligned}$$

where

$$\hat{\Theta}|j, m\rangle = i^{2m}|j, -m\rangle$$

For unpolarized initial states, we sum over the initial spin states and divide by (2s+1); if the final polarization is not conserved, we must sum over final states. We then obtain detailed balance in the form

$$\overline{\frac{d\sigma}{d\Omega}}(\boldsymbol{k} \to \boldsymbol{k}') = \overline{\frac{d\sigma}{d\Omega}}(\boldsymbol{k}' \to \boldsymbol{k}), \qquad (1)$$

where we understand the bar on the top of $\frac{d\sigma}{d\Omega}$ in Eq.(1) to mean that we average over the initial spin states and sum over the final spin states (**Sakurai and Napolitano**).

REFERENCES

J.J. Sakurai and J. Napolitano, Modern Quantum Mechanics, second edition (Pearson, 2011).
E. Merzbacher, Quantum Mechanics third edition (John Wiley & Sons, 1998).