Translation operator
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: August 30, 2015)

Here we discuss the translation operator. The linear momentum is a generator of the translation. This is in contrast to the rotation operator where the angular momentum is a generator of the operation.

1 Definition of the translation operator

Here we discuss the transportation operator

$\hat{T}(a)$: translation operator (unitary operator)

$$|\psi'\rangle = \hat{T}(a)|\psi\rangle,$$

or

$$\langle\psi'| = \langle\psi'|\hat{T}^{+}(a).$$

(i) Analogy from classical mechanics for *x*

The average value of $\hat{x}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{x}$ in the new state $|\psi\rangle$ plus the *x*-displacement *a*.

$$\langle\psi'|\hat{x}|\psi'\rangle = \langle\psi|\hat{x}+a|\psi\rangle,$$

or

$$\langle\psi|\hat{T}^{+}(a)\hat{x}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{x}+a|\psi\rangle,$$

or

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = \hat{x} + a\hat{1}. \qquad (1)$$

Normalization condition:

$$\langle\psi'|\psi'\rangle = \langle\psi|\hat{T}^{+}(a)\hat{T}(a)|\psi\rangle = \langle\psi|\psi\rangle,$$

or

$$\hat{T}^{+}(a)\hat{T}(a) = \hat{1}, \qquad (2)$$

((**Unitary operator**))

From Eqs.(1) and (2), we have

$$\hat{x}\hat{T}(a) = \hat{T}(a)(\hat{x}+a) = \hat{T}(a)\hat{x} + a\hat{T}(a).$$

((**Commutation relation**))

$$[\hat{x}, \hat{T}(a)] = a\hat{T}(a).$$

Here we note that

$$\hat{x}\hat{T}(a)|x\rangle = \hat{T}(a)\hat{x}|x\rangle + a\hat{T}(a)|x\rangle = (x+a)\hat{T}(a)|x\rangle.$$

Thus $\hat{T}(a)|x\rangle$ is the eigenket of $\hat{x}$ with the eigenvalue (x+a)

$$\hat{T}(a)|x\rangle = |x+a\rangle.$$

or

$$\hat{T}^{+}(a)|x+a\rangle = |x\rangle$$

We note that

$$\hat{T}^{+}(a)\hat{T}(a)|x\rangle = \hat{T}^{+}(a)|x+a\rangle = |x\rangle.$$

When x is replaced by x-a in the relation $\hat{T}^{+}(a)|x+a\rangle = |x\rangle$

$$\hat{T}^{+}(a)|x\rangle = |x-a\rangle,$$

or

$$|x-a\rangle = \hat{T}^{+}(a)|x\rangle,$$

or

$$\langle x-a| = \langle x|\hat{T}(a).$$

Note that

$$\langle x|\psi'\rangle = \langle x|\hat{T}(a)|\psi\rangle = \langle x-a|\psi\rangle = \psi(x-a) .$$

(ii) Analogy from the classical mechanics for *p*

The average value of $\hat{p}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{p}$ in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{p}|\psi'\rangle = \langle\psi|\hat{p}|\psi\rangle ,$$

or

$$\langle\psi|\hat{T}^{+}(a)\hat{p}\hat{T}(a)|\psi\rangle = \langle\psi|\hat{p}|\psi\rangle$$

$$\hat{T}^{+}(a)\hat{p}\hat{T}(a) = \hat{p}$$

So we have the commutation relation

$$[\hat{T}(a), \hat{p}] = 0 .$$

From the above commutation relation, we have

$$\hat{p}\hat{T}(a)|p\rangle = \hat{T}(a)\hat{p}|p\rangle = p\hat{T}(a)|p\rangle .$$

Thus $\hat{T}(a)|p\rangle$ is the eigenket of $\hat{p}$ associated with the eigenvalue *p*.

2 Infinitesimal translation operator

We now define the infinitesimal translation operator by

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{G}dx ,$$

where $\hat{G}$ is called a generator of translation. The dimension of $\hat{G}$ is that of the linear momentum.

The operator $\hat{T}(dx)$ satisfies the relations:

$$\hat{T}^{+}(dx)\hat{T}(dx) = \hat{1} , \qquad (1)$$

$$\hat{T}^{+}(dx)\hat{x}\hat{T}(dx) = \hat{x} + dx\hat{1} ,$$

or

$$\hat{x}\hat{T}(dx) - \hat{T}(dx)\hat{x} = dx\hat{T}(dx)\,, \qquad (2)$$

and

$$[\hat{T}(dx), \hat{p}] = 0\,, \qquad (3)$$

Using the relation (1), we get

$$(\hat{1} - \frac{i}{\hbar}\hat{G}dx)^{+}(\hat{1} - \frac{i}{\hbar}\hat{G}dx) = \hat{1}\,,$$

or

$$(\hat{1} + \frac{i}{\hbar}\hat{G}^{+}dx)(\hat{1} - \frac{i}{\hbar}\hat{G}dx) = \hat{1} + \frac{i}{\hbar}(\hat{G}^{+} - \hat{G})dx + O[(dx)^2] = \hat{1}\,,$$

or

$$\hat{G}^{+} = \hat{G}\,.$$

The operator $\hat{G}$ is a Hermitian operator. Using the relation (2), we get

$$\hat{x}(\hat{1} - \frac{i}{\hbar}\hat{G}dx) - (\hat{1} - \frac{i}{\hbar}\hat{G}dx)\hat{x} = dx(\hat{1} - \frac{i}{\hbar}\hat{G}dx) = dx\hat{1} + O(dx)^2\,,$$

or

$$-\frac{i}{\hbar}[\hat{x}, \hat{G}]dx = dx\hat{1}\,,$$

or

$$[\hat{x}, \hat{G}] = i\hbar\hat{1}\,.$$

Using the relation (3), we get

$$[\hat{1} - \frac{i}{\hbar}\hat{G}dx, \hat{p}] = 0\,.$$

Then we have

$$[\hat{G}, \hat{p}] = 0\,.$$

From these two commutation relations, we conclude that

$$\hat{G} = \hat{p} ,$$

and

$$\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{p}dx .$$

We see that the position operator and the momentum operator $\hat{p}$ obeys the commutation relation

$$[\hat{x}, \hat{p}] = i\hbar\hat{1} .$$

which leads to the Heisenberg's principle of uncertainty.

3 Momentum operator $\hat{p}$ in the position basis.

Using the relation

$$\hat{T}(\delta x)|x\rangle = |x+\delta x\rangle , \qquad \hat{T}(\delta x) = \hat{1} - \frac{i}{\hbar}\hat{p}\delta x .$$

we get

$$\begin{aligned}\hat{T}(\delta x)|\psi\rangle &= \hat{T}(\delta x)\int dx'|x'\rangle\langle x'|\psi\rangle = \int dx'|x'+\delta x\rangle\langle x'|\psi\rangle \\ &= \int dx'|x'\rangle\langle x'-\delta x|\psi\rangle = \int dx'|x'\rangle\psi(x'-\delta x)\end{aligned}$$

We apply the Taylor expansion:

$$\psi(x'-\delta x) = \psi(x') - \delta x\frac{\partial}{\partial x'}\psi(x') .$$

Substitution:

$$\begin{aligned}\hat{T}(\delta x)|\psi\rangle &= \int dx'|x'\rangle\psi(x'-\delta x) \\ &= \int dx'|x'\rangle[\psi(x') - \delta x\frac{\partial}{\partial x'}\psi(x')] \\ &= \int dx'|x'\rangle[\langle x'|\psi\rangle - \delta x\frac{\partial}{\partial x'}\langle x'|\psi\rangle] \\ &= |\psi\rangle - \delta x\int dx'|x'\rangle\frac{\partial}{\partial x'}\langle x'|\psi\rangle\end{aligned}$$

From the definition, we have

$$\hat{T}(\delta x)|\psi\rangle = (\hat{1} - \frac{i}{\hbar}\hat{p}\delta x)|\psi\rangle .$$

Comparing these two equations, we obtain the relation

$$\hat{p}|\psi\rangle = \frac{\hbar}{i}\int dx'|x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle ,$$

or

$$\begin{aligned}\langle x|\hat{p}|\psi\rangle &= \frac{\hbar}{i}\int dx'\langle x|x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle \\ &= \frac{\hbar}{i}\int dx'\,\delta(x-x')\frac{\partial}{\partial x'}\langle x'|\psi\rangle \\ &= \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle\end{aligned}$$

We obtain a very important formula

$$\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle .$$

Note that

$$\begin{aligned}\langle\psi|\hat{p}|\psi\rangle &= \int dx\langle\psi|x\rangle\langle x|\hat{p}|\psi\rangle \\ &= \int dx\langle\psi|x\rangle\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle \\ &= \int dx\langle x|\psi\rangle^{*}\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle\end{aligned}$$

These results suggest that in position space the momentum operator takes the form

$$\hat{p} \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial x} .$$

4. Position operator $\hat{x}$ in the momentum basis.

$$
\begin{aligned}
\langle p|\hat{x}|\psi\rangle &= \int dx \langle p|x\rangle\langle x|\hat{x}|\psi\rangle \\
&= \int dx x \langle p|x\rangle\langle x|\psi\rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}}\int dx x e^{-\frac{ipx}{\hbar}}\langle x|\psi\rangle \\
&= i\hbar\frac{\partial}{\partial p}\frac{1}{\sqrt{2\pi\hbar}}\left(\int dx e^{-\frac{ipx}{\hbar}}\langle x|\psi\rangle\right) \\
&= i\hbar\frac{\partial}{\partial p}\int dx \langle p|x\rangle\langle x|\psi\rangle \\
&= i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle
\end{aligned}
$$

Then we have

$$
\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle .
$$

Using this result, we get

$$
\begin{aligned}
\langle \phi|\hat{x}|\psi\rangle &= \int dp \langle \phi|p\rangle\langle p|\hat{x}|\psi\rangle \\
&= \int dp \langle \phi|p\rangle i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle \\
&= \int dp \langle p|\phi\rangle^* i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle
\end{aligned}
$$

These results suggest that in momentum space the position operator takes the form

$$
\hat{x} \to i\hbar\frac{\partial}{\partial p} .
$$

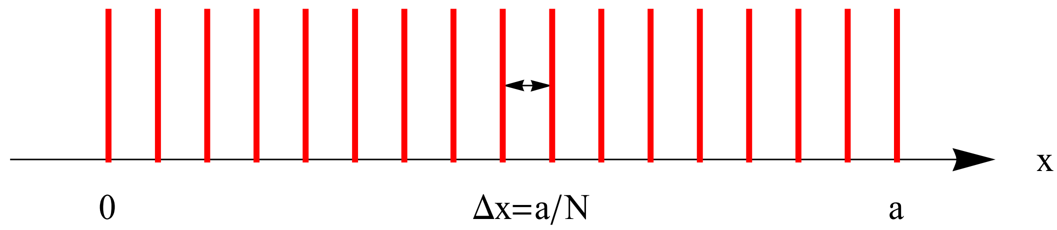
5. The finite translation operator

What is the operator $\hat{T}(a)$ corresponding to a finite translation *a*? We find it by the following procedure. We divide the interval into *N* parts of size d*x* = *a*/*N*. As *N*→∞, *a*/*N* becomes infinitesimal.

$$
\hat{T}(dx) = \hat{1} - \frac{i}{\hbar}\hat{p}(\frac{a}{N}) .
$$

Since a translation by *a* equals *N* translations by *a*/*N*, we have

$$\hat{T}(a) = \lim_{N\to\infty}[\hat{1} - \frac{i}{\hbar}\hat{p}(\frac{a}{N})]^N = \exp(-\frac{i}{\hbar}\hat{p}a).$$



Here we use the formula

$$\lim_{N\to\infty}(1+\frac{1}{N})^N = e, \qquad \lim_{N\to\infty}(1-\frac{1}{N})^N = e^{-1}$$

$$\lim_{N\to\infty}[(1-\frac{ax}{N})^{\frac{N}{ax}}]^{ax} = \lim_{N\to\infty}(1-\frac{ax}{N})^N = (e^{-1})^{ax} = e^{-ax}$$

In summary, we have

$$\hat{T}(a) = \exp(-\frac{i}{\hbar}\hat{p}a).$$

6. Discussion on the commutation relation

It is interesting to calculate

$$\hat{T}^+(a)\hat{x}\hat{T}(a) = e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a},$$

by using the Baker-Hausdorff theorem:

$$\exp(\hat{A}x)\hat{B}\exp(-\hat{A}x) = \hat{B} + \frac{x}{1!}[\hat{A},\hat{B}] + \frac{x^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{x^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

When $x = 1$, we have

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + ...$$

Then we have

$$\hat{T}^+(a)\hat{x}\hat{T}(a) = e^{\frac{i}{\hbar}\hat{p}a}\hat{x}e^{-\frac{i}{\hbar}\hat{p}a} = \hat{x} + [\frac{i}{\hbar}\hat{p}a,\hat{x}] = \hat{x} + \frac{i}{\hbar}a[\hat{p},\hat{x}] = \hat{x} + \frac{i}{\hbar}a\frac{\hbar}{i} = \hat{x} + a\hat{1}.$$

So we confirmed that the relation

$$\hat{T}^{+}(a)\hat{x}\hat{T}(a) = \hat{x} + a\hat{1},$$

holds for any finite translation operator.

7. Invariance of Hamiltonian under the translation

Now we consider the condition for the invariance of Hamiltonian $\hat{H}$ under the translation.

The average value of $\hat{H}$ in the new state $|\psi'\rangle$ is equal to the average value of $\hat{H}$ in the new state $|\psi\rangle$.

$$\langle\psi'|\hat{H}|\psi'\rangle = \langle\psi|\hat{H}|\psi\rangle,$$

or

$$\hat{T}^{+}(dx)\hat{H}\hat{T}(dx) = \hat{H}, \qquad \text{or} \qquad \hat{H}\hat{T}(dx) = \hat{T}(dx)\hat{H},$$

or

$$\hat{H}(\hat{1} - \frac{i}{\hbar}\hat{p}dx) = (\hat{1} - \frac{i}{\hbar}\hat{p}dx)\hat{H}.$$

Then we have

$$[\hat{H}, \hat{p}] = 0.$$

8. ((Sakurai 1-28))

(a) Let x and p_x be the coordinate and linear momentum in one dimension. Evaluate the classical Poisson bracket.

$$[x, F(p_x)]_{classical}.$$

(b) Let $\hat{x}$ and $\hat{p}_x$ be the corresponding quantum-mechanical operators this time. Evaluate the commutator

$$[\hat{x}, \exp(\frac{i\hat{p}_x a}{\hbar})].$$

(c) Using the result obtained in (b), prove that

$$\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle, \qquad \hat{x}|x'\rangle = x'|x'\rangle$$

is an eigenstate of the coordinate operator x, What is the corresponding eigenvalue?

((**Solution**))

(a)

$$[x, F(p_x)]_{classical} = \frac{\partial x}{\partial x}\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial x}{\partial p_x}\frac{\partial F(p_x)}{\partial x} = \frac{\partial F(p_x)}{\partial p_x}$$

(b)
We use the Gottfried's result

$$[\hat{x}, \exp(\frac{i\hat{p}_x a}{\hbar})] = i\hbar\frac{\partial}{\partial \hat{p}_x}\exp(\frac{i\hat{p}_x a}{\hbar}) = -a\exp(\frac{i\hat{p}_x a}{\hbar})$$

(c)

$$\hat{x}\exp(\frac{i\hat{p}_x a}{\hbar}) = \exp(\frac{i\hat{p}_x a}{\hbar})\hat{x} - a\exp(\frac{i\hat{p}_x a}{\hbar})$$

Then we have

$$\begin{aligned}\hat{x}\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle &= \exp(\frac{i\hat{p}_x a}{\hbar})\hat{x}|x'\rangle - a\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle \\ &= (x'-a)\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle\end{aligned}$$

The ket $\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle$ is the eigenket of $\hat{x}$ with an eigenvalue (x'-a).

$$\exp(\frac{i\hat{p}_x a}{\hbar})|x'\rangle = |x'-a\rangle$$

Therefore $\hat{T}_x(a) = \exp(\frac{i\hat{p}_x a}{\hbar})$ is a translation operator.

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9. ((Sakurai 1-29))

(a) Gottfried (1966) states that

$$\left[\hat{x}_i, G(\hat{\boldsymbol{p}})\right] = i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\boldsymbol{p}}), \qquad \left[\hat{p}_i, F(\hat{\boldsymbol{x}})\right] = -i\hbar \frac{\partial}{\partial \hat{x}_i} F(\hat{\boldsymbol{x}})$$

can be easily derived from the fundamental commutation relations for all functions of F and G can be expressed as power series in their arguments. Verify this statement.

(b) Evaluate $\left[\hat{x}^2, \hat{p}^2\right]$. Compare your result with the classical Poisson bracket $[x^2, p^2]_{classic}$.

((**Solution**))

(a)

(i)

$$\begin{aligned}
\langle \boldsymbol{p} | [\hat{x}_i, G(\hat{\boldsymbol{p}})] | \alpha \rangle &= [i\hbar \frac{\partial}{\partial p_i} G(\boldsymbol{p}) - G(\boldsymbol{p}) i\hbar \frac{\partial}{\partial p_i}] \langle \boldsymbol{p} | \alpha \rangle \\
&= i\hbar \frac{\partial}{\partial p_i} [G(\boldsymbol{p}) \langle \boldsymbol{p} | \alpha \rangle] - i\hbar G(\boldsymbol{p}) \frac{\partial}{\partial p_i} \langle \boldsymbol{p} | \alpha \rangle \\
&= i\hbar \left(\frac{\partial}{\partial p_i} G(\boldsymbol{p}) \right) \langle \boldsymbol{p} | \alpha \rangle + i\hbar G(\boldsymbol{p}) \frac{\partial}{\partial p_i} \langle \boldsymbol{p} | \alpha \rangle - i\hbar G(\boldsymbol{p}) \frac{\partial}{\partial p_i} \langle \boldsymbol{p} | \alpha \rangle \\
&= i\hbar \left(\frac{\partial}{\partial p_i} G(\boldsymbol{p}) \right) \langle \boldsymbol{p} | \alpha \rangle \\
&= \langle \boldsymbol{p} | i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\boldsymbol{p}}) | \alpha \rangle
\end{aligned}$$

Thus we have the final result

$$[\hat{x}_i, G(\hat{\boldsymbol{p}})] = i\hbar \frac{\partial}{\partial \hat{p}_i} G(\hat{\boldsymbol{p}})$$

(ii)

$$\langle \mathbf{r}|[\hat{p}_i, F(\hat{\boldsymbol{r}})]|\alpha\rangle = [\frac{\hbar}{i}\frac{\partial}{\partial x_i}F(\boldsymbol{r}) - F(\boldsymbol{r})\frac{\hbar}{i}\frac{\partial}{\partial x_i}]\langle \mathbf{r}|\alpha\rangle$$
$$= \frac{\hbar}{i}\frac{\partial}{\partial x_i}[F(\mathbf{r})\langle \mathbf{r}|\alpha\rangle] - \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle$$
$$= \frac{\hbar}{i}\left(\frac{\partial}{\partial x_i}F(\mathbf{r})\right)\langle \mathbf{r}|\alpha\rangle + \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle] - \frac{\hbar}{i}F(\mathbf{r})\frac{\partial}{\partial x_i}\langle \mathbf{r}|\alpha\rangle$$
$$= \frac{\hbar}{i}\left(\frac{\partial}{\partial x_i}F(\mathbf{r})\right)\langle \mathbf{r}|\alpha\rangle$$
$$= \langle \mathbf{r}|\frac{\hbar}{i}\frac{\partial}{\partial \hat{x}_i}F(\hat{\mathbf{r}})|\alpha\rangle$$

or

$$[\hat{p}_i, F(\hat{\mathbf{r}})] = \frac{\hbar}{i}\frac{\partial}{\partial \hat{x}_i}F(\hat{\mathbf{r}})$$

(b)

$$[\hat{x}^2, \hat{p}^2] = \hat{x}[\hat{x}, \hat{p}^2] + [\hat{x}, \hat{p}^2]\hat{x}$$
$$= \hat{x}i\hbar\frac{\partial}{\partial \hat{p}}\hat{p}^2 + i\hbar\left(\frac{\partial}{\partial \hat{p}}\hat{p}^2\right)\hat{x}$$
$$= 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})$$

The classical Poisson bracket is defined by

$$[x^2, p^2]_{classic} = \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial x^2}{\partial p}\frac{\partial p^2}{\partial x}$$
$$= 4xp$$
$$= 2(xp + px)$$

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10. ((Sakurai 1-30))

The translation operator for a finite (spatial) displacement is given by

$$\hat{T}(\boldsymbol{l}) = \exp(-\frac{i\hat{\boldsymbol{p}}\cdot\boldsymbol{l}}{\hbar}),$$

where $\hat{\boldsymbol{p}}$ is the momentum operator.

(a) Evaluate

$$[\hat{x}, \hat{T}(\boldsymbol{l})]$$

(b) Using (a) (or otherwise), demonstrate how the expectation value $\langle x \rangle$ changes under translation.

((**Solution**))

(a)

The translation operator is defined by

$$\hat{T}(\boldsymbol{l}) = \exp(-\frac{i\hat{\boldsymbol{p}} \cdot \boldsymbol{l}}{\hbar})$$

$$[\hat{x}_i, \hat{T}(\boldsymbol{l})] = i\hbar \frac{\partial}{\partial \hat{p}_i} \hat{T}(\boldsymbol{l}) = l_i \exp(-\frac{i\hat{\boldsymbol{p}} \cdot \boldsymbol{l}}{\hbar}) = l_i \hat{T}(\boldsymbol{l})$$

or

$$[\hat{\boldsymbol{r}}, \hat{T}(\boldsymbol{l})] = \boldsymbol{l}\hat{T}(\boldsymbol{l})$$

(b)

$$|\alpha'\rangle = \hat{T}(\boldsymbol{l})|\alpha\rangle$$

$$\begin{aligned}\langle \alpha' | \hat{\boldsymbol{r}} | \alpha' \rangle &= \langle \alpha | \hat{T}^{+}(\boldsymbol{l}) \hat{\boldsymbol{r}} \hat{T}(\boldsymbol{l}) | \alpha' \rangle \\ &= \langle \alpha | \hat{T}^{+}(\boldsymbol{l}) [\hat{T}(\boldsymbol{l}) \hat{\boldsymbol{r}} + \boldsymbol{l}\hat{T}(\boldsymbol{l})] | \alpha' \rangle \\ &= \langle \alpha | \hat{\boldsymbol{r}} + \boldsymbol{l} | \alpha' \rangle\end{aligned}$$

or

$$\langle \alpha' | \hat{\boldsymbol{r}} | \alpha' \rangle = \langle \alpha | \hat{\boldsymbol{r}} | \alpha \rangle + \boldsymbol{l}$$

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11. ((Sakurai 1-31))

Prove

$$\langle \boldsymbol{r} \rangle \to \langle \boldsymbol{r} \rangle + d\boldsymbol{r}', \qquad \langle \boldsymbol{p} \rangle \to \langle \boldsymbol{p} \rangle$$

under infinitesimal translation.

((**Solution**))

We use the commutation relations

$$[\hat{\boldsymbol{r}}, \hat{T}(d\boldsymbol{r})] = d\boldsymbol{r}\hat{T}(d\boldsymbol{r})$$

and

$$[\hat{\boldsymbol{p}}, \hat{T}(d\boldsymbol{r})] = 0$$

We have

$$\begin{aligned}\langle \alpha | \hat{T}^+(d\boldsymbol{r})\hat{\boldsymbol{r}}\hat{T}(d\boldsymbol{r}) | \alpha \rangle &= \langle \alpha | \hat{T}^+(d\boldsymbol{r})[\hat{T}(d\boldsymbol{r})\hat{\boldsymbol{r}} + d\boldsymbol{r}\hat{T}(d\boldsymbol{r}) | \alpha \rangle \\ &= \langle \alpha | \hat{\boldsymbol{r}} + d\boldsymbol{r} | \alpha \rangle\end{aligned}$$

or

$$\langle \alpha' | \hat{\mathbf{r}} | \alpha' \rangle = \langle \alpha | \hat{\mathbf{r}} | \alpha \rangle + d\mathbf{r}$$

Similarly

$$\langle \alpha | \hat{T}^+(d\boldsymbol{r})\hat{\boldsymbol{p}}\hat{T}(d\boldsymbol{r}) | \alpha \rangle = \langle \alpha | \hat{T}^+(d\boldsymbol{r})\hat{T}(d\boldsymbol{r})\hat{\boldsymbol{p}} | \alpha \rangle = \langle \alpha | \hat{\boldsymbol{p}} | \alpha \rangle$$

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12. ((Sakurai 1-33))

(a) Prove the following:

(i)

$$\langle p' | \hat{x} | \alpha \rangle = i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle$$

(ii)

$$\langle \beta | \hat{x} | \alpha \rangle = \int dp \langle p' | \beta \rangle^* i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle$$

(b) What is the physical significance of

$$\exp(\frac{i\hat{x}p_0}{\hbar}),$$

where $\hat{x}$ is the position operator and p_0 is some number with the dimension of momentum? Justify your answer.

((**Solution**))

(a)

(i)

$$\begin{aligned}
\langle p' | \hat{x} | \alpha \rangle &= \int dx' \langle p' | x' \rangle \langle x' | \hat{x} | \alpha \rangle \\
&= \int dx' x' \langle p' | x' \rangle \langle x' | \alpha \rangle \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int dx' x' e^{-\frac{ip'x'}{\hbar}} \langle x' | \alpha \rangle \\
&== i\hbar \frac{\partial}{\partial p'} \frac{1}{\sqrt{2\pi\hbar}} \left(\int dx' e^{-\frac{ip'x'}{\hbar}} \langle x' | \alpha \rangle \right) \\
&= i\hbar \frac{\partial}{\partial p'} \int dx' \langle p' | x' \rangle \langle x' | \alpha \rangle \\
&= i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle
\end{aligned}$$

(ii)

$$\begin{aligned}
\langle \beta | \hat{x} | \alpha \rangle &= \int dp' \langle \beta | p' \rangle \langle p' | \hat{x} | \alpha \rangle \\
&= \int dp' \langle \beta | p' \rangle \langle p' | \hat{x} | \alpha \rangle \\
&= \int dp' \langle \beta | p' \rangle i\hbar \frac{\partial}{\partial p'} \langle p' | \alpha \rangle
\end{aligned}$$

(b)

$$\hat{p}\exp(\frac{ip_0\hat{x}}{\hbar})|p'\rangle = \left\{\left[\hat{p}, \exp(\frac{ip_0\hat{x}}{\hbar})\right] + \exp(\frac{ip_0\hat{x}}{\hbar})\hat{p}\right\}|p'\rangle$$

$$= \{p_0 \exp(\frac{ip_0\hat{x}}{\hbar}) + \exp(\frac{ip_0\hat{x}}{\hbar}) p'\}|p'\rangle$$

or

$$\hat{p}\exp(\frac{ip_0\hat{x}}{\hbar})|p'\rangle = (p_0 + p')\exp(\frac{ip_0\hat{x}}{\hbar})|p'\rangle .$$

Therefore $\exp(\frac{ip_0\hat{x}}{\hbar})|p'\rangle$ is the eigenket of $\hat{p}$ with an eigenvalue of (p' + p_0).

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John S. Townsend , *A Modern Approach to Quantum Mechanics*, second edition (University Science Books, 2012).

APPENDIX

Properties of the translation operator

(i) $\hat{T}(a+b) = \hat{T}(a)\hat{T}(b) = \hat{T}(b)\hat{T}(a)$

((proof))

$$\hat{T}(b)|x\rangle = |x+b\rangle ,$$

$$\hat{T}(a)\hat{T}(b)|x\rangle = \hat{T}(a)|x+b\rangle = |x+a+b\rangle$$

$$\hat{T}(a+b)|x\rangle = |x+a+b\rangle$$

Then we have

$$\hat{T}(a+b) = \hat{T}(a)\hat{T}(b) = \hat{T}(b)\hat{T}(a)$$

(ii) $\hat{T}(0) = \hat{1}$

((**Proof**))

For any $|x\rangle$, we have

$$\hat{T}(0)|x\rangle = |x\rangle$$

leading to the relation

$$\hat{T}(0) = \hat{1}.$$

(iii) $\hat{T}(a)\hat{T}(-a) = \hat{T}(-a)\hat{T}(a) = \hat{1}$

((**Proof**))
In the relation

$$\hat{T}(a+b) = \hat{T}(a)\hat{T}(b) = \hat{T}(b)\hat{T}(a),$$

we assume that $a+b=0$. Then we have

$$\hat{T}(a)\hat{T}(-a) = \hat{T}(-a)\hat{T}(a) = \hat{T}(0) = \hat{1}$$

leading to the relation

$$\hat{T}(-a) = \hat{T}^{-1}(a) = \hat{T}^{+}(a)$$