Quantum Entanglement: spin 1/2

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Here we discuss the physics of quantum entanglement. At first, undergraduate students who just want to know the essential points of the quantum entanglement, may encounter some difficulty in understanding the definition of technical words, such as spooky action at a distance, non-locality, locality, hidden variable theory, separability, qubit, and so on. The definition of these words is given in the APPENDIX (source: Wikipedia).

The derivation of the Bell inequality is mathematically not so complicated. It is essential for ones to verify that the Bell inequality is not satisfied for the quantum entanglement phenomena from the experimental sides with the use of entangled spins or photons. So far so many books on the quantum entanglement, quantum information, and quantum computer have been published. Even after I read these books including textbooks of quantum mechanics, I have not understood sufficiently what is going on the spooky action at a distance. In order to teach the quantum entanglement undergraduate students, I felt it necessary to understand such a weirdness of the quantum entanglement in much more detail. While I struggled to understand the spooky action at a distance (named by Einstein), I had a good opportunity to read a book entitled, Einstein: His Life and Universe (by W. Issacson). I realize that the weirdness of the behavior of the quantum entanglement can be well described in this book. Of course, physicists who want to know the essence of the weird behavior based on the mathematics, may not be satisfied with the simple and clear explanations given by Issacson.

Here the content of the book is summarized as follows.

(a) Quantum mechanics asserts that particles do not have a definite state except when observed, and two particles can be in na entangled state so that the observation of one determines a property of the other instantly. As soon as any observation is made, the system goes into a fixed state.

(b) This may be conceivable for the microscopic quantum realm, but it is baffling when one imagines the intersection between the quantum realm and observable everyday world.

(c) The EPR paper would not succeed in showing that quantum mechanics was wrong. But it did eventually become clear that quantum mechanics was incompatible with our common sense understanding of locality- our aversion to spooky action at a distance. The odd thing is that Einstein, apparently, was far more right than he hoped to be.

(d) The idea of entanglement and spooky action at a distance is the quantum weirdness in which an observation of one particle can instantly affect another one far away.

(e) The locality is not a feature of the quantum world. Spooky action at a distance or more precisely, the potential entanglement of distant particles, is a feature of the quantum world.

(f) Might the spooky action at a distance - where something that happens to a particle in one place can be instantly reflected by one that is billions of miles away - violate the speed limit of light? No, the theory of relativity still seems safe. The two particles, though distant, remain part of the same physical entity. By observing one of them, we may affect its attributes, and is correlated to what would be observed of the second particle. But no information is transmitted, no signal sent, and there is no traditional cause-and-effect relationship.

(W. Issacson, Einstein: His Life and Universe).

John Stewart Bell FRS (28 June 1928 – 1 October 1990) was a physicist from Northern Ireland (Ulster), and the originator of Bell's theorem, a significant theorem in quantum physics regarding hidden variable theories.

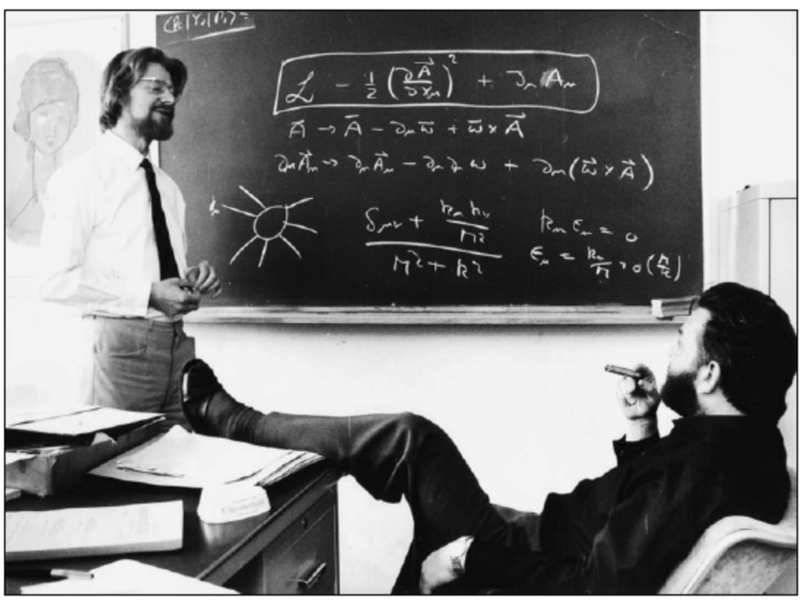

Figure 10.2: John S. Bell (on the left) with particle theorist Martinus Veltman. CERN, courtesy AIP Emilio Segre Visual Archives.

http://en.wikipedia.org/wiki/John_Stewart_Bell

The **Bell test experiments** serve to investigate the validity of the entanglement effect in quantum mechanics by using some kind of Bell inequality. John Bell published the first inequality of this kind in his paper "On the Einstein-Podolsky-Rosen Paradox". Bell's Theorem states that a Bell inequality must be obeyed under any local hidden variable theory but can in certain circumstances be violated under quantum mechanics. The term "Bell inequality" can mean any one of a number of inequalities — in practice, in real experiments, the CHSH or CH74 inequality, not the original one derived by John Bell. It places restrictions on the statistical results of experiments on sets of particles that have taken part in an interaction and then separated. A **Bell test experiment** is one designed to test whether or not the real world obeys a Bell inequality.

Alain Aspect (born 15 June 1947) is a French physicist noted for his experimental work on quantum entanglement. Aspect is a graduate of the École Normale Supérieure de Cachan (ENS Cachan). He passed the 'agrégation' in physics in 1969 and received his master's degree from Université d'Orsay. He then did his national service, teaching for three years in Cameroon.

http://en.wikipedia.org/wiki/Alain_Aspect

In the early 1980s, while working on his PhD thesis from the lesser academic rank of lecturer, he performed the elusive "Bell test experiments" that showed that Albert Einstein, Boris Podolsky and Nathan Rosen's reductio ad absurdum of quantum mechanics, namely that it implied 'ghostly action at a distance', did in fact appear to be realized when two particles were separated by an arbitrarily large distance (EPR paradox). A correlation between their wave functions remained, as they were once part of the same wave-function that was not disturbed before one of the child particles was measured.

If quantum theory is correct, the determination of an axis direction for the polarization measurement of one photon, forcing the wave function to 'collapse' onto that axis, will influence the measurement of its twin. This influence occurs despite any experimenters not knowing which axes have been chosen by their distant colleagues, and at distances that disallow any communication between the two photons, even at the speed of light.

Aspect's experiments were considered to provide overwhelming support to the thesis that Bell's inequalities are violated in its CHSH version. However, his results were not completely conclusive, since there were so-called loopholes that allowed for alternative explanations that comply with local realism.

Stated more simply, the experiment provides strong evidence that a quantum event at one location can affect an event at another location without any obvious mechanism for communication between the two locations. This has been called "spooky action at a distance" by Einstein (who doubted the physical reality of this effect). However, these experiments do not allow faster-than-light communication, as the events themselves appear to be inherently random.

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1. What is quantum entanglement? (summary of this chapter)

Entanglement is one of the strangest predictions of quantum mechanics. Two objects are entangled if their physical properties are undefined but correlated, even when the two objects are separated by a large distance. No mechanism for entanglement is known, but so far experiments universally show that nonlocal entanglement is real. Something that happens to one particle does affect, instantaneously, what happens to the second particle, no matter how far it may be from the first one.

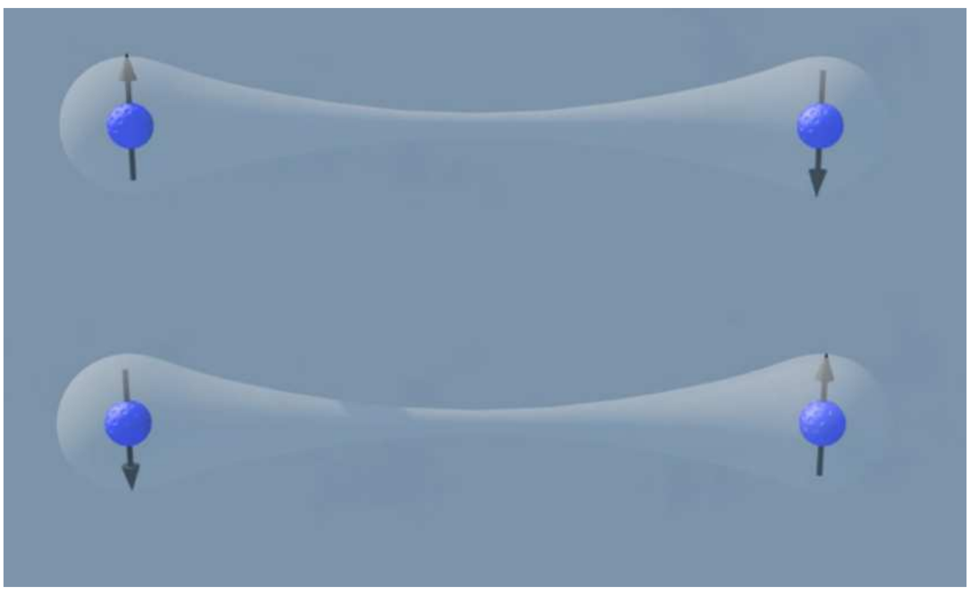


Fig. Two objects are entangled if their physical properties are undefined but correlated, even when the two objects are separated by a large distance.

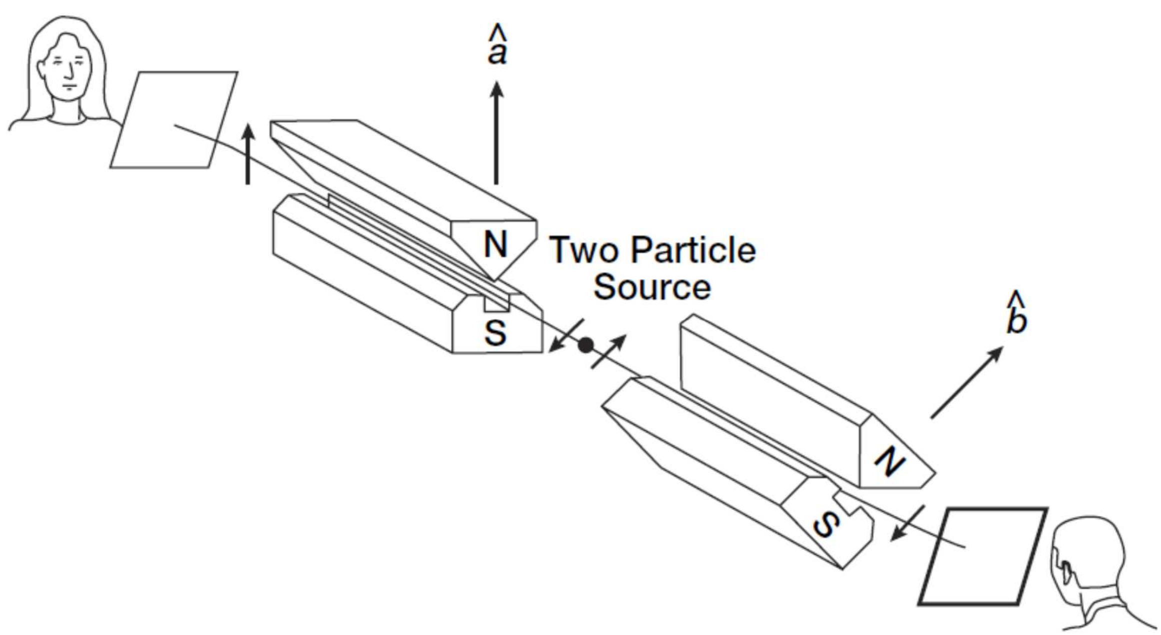
The EPR paper would not succeed in showing the quantum mechanics was wrong. But it did eventually become clear that quantum mechanics was, as Einstein argued, incompatible with our common-sense understanding of locality - our aversion to spooky action at a distance. The odd thing is that Einstein, apparently, was far more right than he hoped to be.

2. EPR (Einstein, Podolsky, and Rosen) (1935)

In 1935, Einstein, Podolsky, and Rosen (EPR) pointed out the incompleteness of quantum mechanics. This argument is one of the most remarkable attacks on quantum theory ever launched. Recall that, according to quantum theory, measurements can be made of only one of any pair of complementary variables: position *or* momentum, energy *or* time. But the simultaneous measurement of both members of such a pair is impossible. The Einstein, Podolsky and Rosen (EPR) paper argued that such measurements were quite possible, and it gave a simple description of how to carry them out. Thus, according to their analysis, it was possible to obtain a more complete description of physical reality. Their conclusion was that quantum theory was incomplete.

3. Bohm's version of EPR (1951) and Bell's inequality

Their original gedanken experiment was revised by Bohm (1951) as the form of Stern-Gerlach measurements of two spin 1/2 particles; Bohm's version of EPR. Suppose that two particles with spin 1/2 are generated from the system with spin 0. Before the measurement, we do not know the spin directions of these two particles with spin 1/2. Suppose that the spin direction (the quantized axis +*z*) of the particle-1 is determined by Alice (the observer of the SG-1). Then the spin direction of the particle -2 can be uniquely determined by Bob (the second observer) to be the -*z* axis. This result is consistent with that derived from the spin angular momentum conservation. This result is rather different from that in quantum mechanics. After the first measurement (by Alice), the original state is changed into the new state. The second measurement is influenced by the first measurement. There is some probability of finding the partice-2 having the +*z* spin direction, as well as the probability of finding the particle having the -*z* direction.



((Penrose))

In the simplest EPR situation considered by David Bohm (1951), we consider a pair of spin 1/2 particles. The particle 1 and particle 2 start together in a combined spin 0 state, and then travel away from each other to the left and right to respective detectors A (Alice) and B (Bob) at a great distance apart.

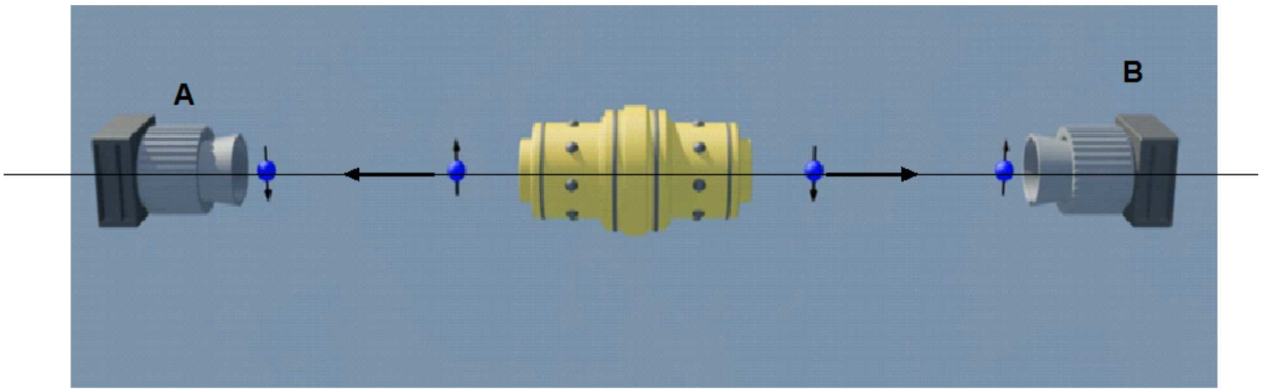


Fig. The EPR–Bohm thought experiment. A pair of spin 1/2 particles 1 and 2 originate in a combined spin 0 state, and then travel out in opposite directions, left and right, to respective widely separated detectors A (Alice) and B (Bob). Each detector is set up to measure the spin of the approaching particle, but in some direction which is decided upon only after the particles are in full flight. Bell's theorem tells us that there is no way of reproducing the expectations of quantum mechanics with a model in which the two particles can act as *classical-like independent* objects that cannot communicate after they have become separated.

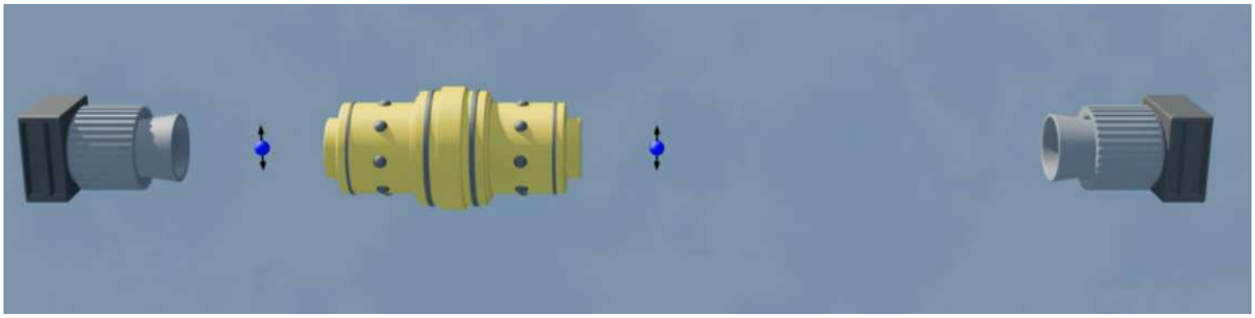
http://www.youtube.com/watch?v=qXvZpn_dnMs&list=TLwELoPwvGH1pgMhPqpa6ojOCe2M6yhUU4

Suppose that each of the detectors is capable of measuring the spin of the approaching particle in some direction that is only decided upon when the two particles are well separated from each other. The problem is to see whether it is possible to reproduce the expectations of quantum mechanics using a model in which the particles are regarded as *unconnected independent classical-like particles*, each one being unable to communicate with the other after they have separated. It turns out, because of a remarkable theorem due to the Northern Irish physicist John S. Bell, that it is not possible to reproduce the predictions of quantum theory in this way. Bell derived inequalities relating the joint probabilities of the results of two physically separated measurements that are violated by the expectations of quantum mechanics, yet which are necessarily satisfied by any model in which the two particles behave as independent entities after they have become physically separated. Thus, Bell-inequality violation demonstrates the presence of essentially quantumtheoretic effects—these being effects of quantum entanglements between physically separated particles—which cannot be explained by any model according to which the particles are treated as unconnected and independent actual things.

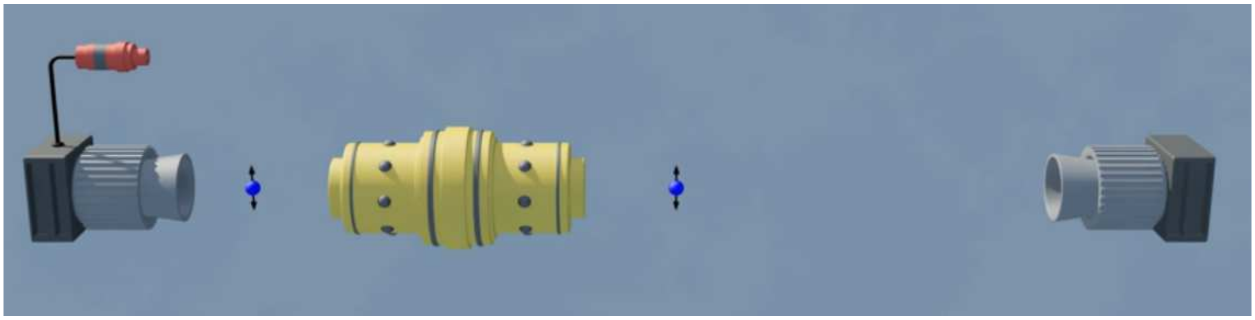
4. Locality (by Einstein)

Suppose that Alice is very far from Bob such that the propagation time for informing the result of Alice to Bob is long enough. Bob may measure the spin direction of the partricle-2, just before the result of the measurement by Alice is informed to Bob. In this case, if the spin direction of the particle 1 is measured by Alice as +*z* direction, before the second experiment (by Bob), we can conclude the spin direction of the particle-2 (measured by Bob) as -*z* direction according to the spin angular momentum conservation law. How can we explain such gedanken experiments in terms of quantum mechanics? Is there some possibility of hidden variables to determine the result of spin directions of the two particles during the production of two particles?

(a)



(b)



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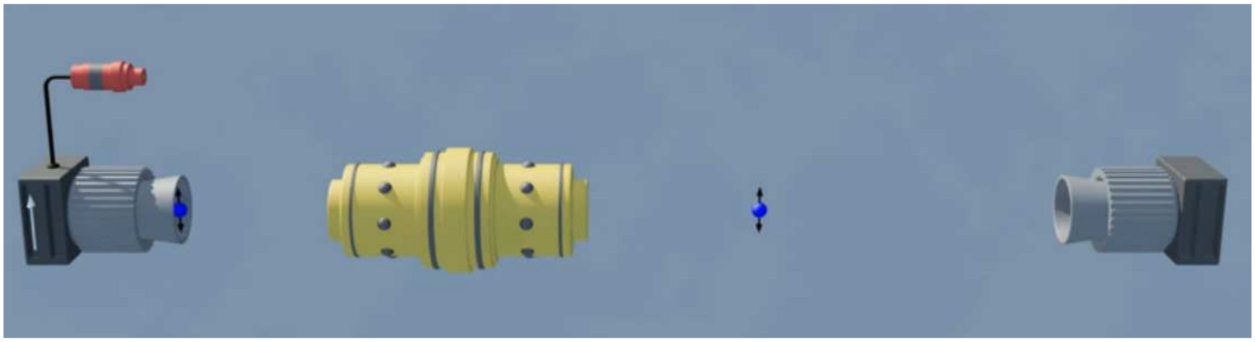

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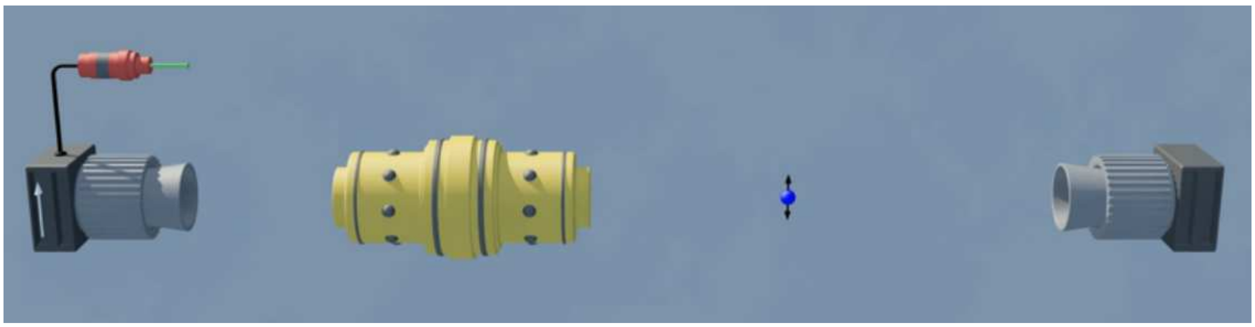

(e)

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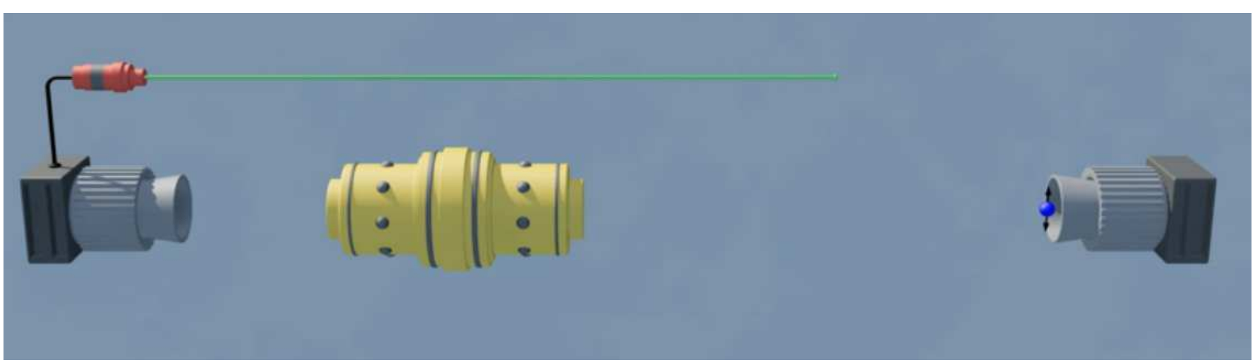
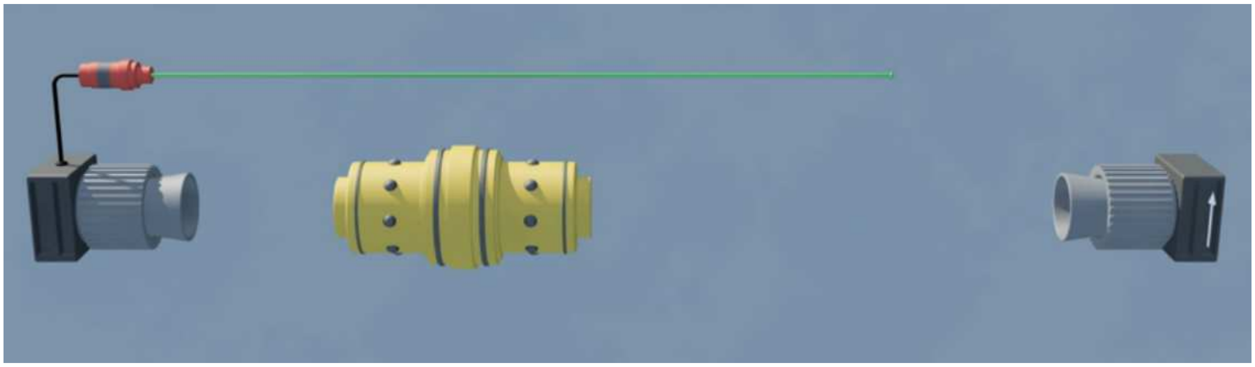


(g)



http://www.youtube.com/watch?v=qXvZpn_dnMs&list=TLwELoPwvGH1pgMhPqpa6ojOCe2M6yhUU4

Fig. Suppose that the distance between Alice and source is much shorter than that between Bob and source. The distance between Alice and Bob is long so that it takes long times for them to communicate with the speed of light. The particle 1 with up-state spin from the source moves to Alice, while the particle 2 with down-state spin from the source simultaneously moves to Bob. Because of the short distance, Alice first measures the spin direction of the particle 1 as the up-spin. At this time, the particle 2 does not reach Bob. Immediately after the measurement by Alice. Alice lets Bob know about her result at the speed of light. Before the light reaches Bob, the particle 2 reaches Bob. So Bob measures the spin direction of the particle 2 as the spin-down, before the information from Alice reaches Bob. At this moment, Bob comes to know the result of Alice without information from Alice.

5. Significance of the Bell's inequality(Aczel)

Bell's theorem concerns a very general class of local theories with hidden, or supplementary, parameters. The assumption is as follows: suppose that the quantum theory is incomplete but that Einstein's ideas about locality are preserved. We thus assume that there must be a way to complete the quantum description of the world, while preserving Einstein's requirement that what holds true here cannot affect what holds true there, unless a signal can be sent from here to there (and such a signal, by Einstein's own special theory of relativity, could not travel faster than light). In such a situation, making the theory complete means discovering the hidden variables, and describing these variables that make the particles or photons behave in a certain way.

Einstein had conjectured that correlations between distant particles are due to the fact that their common preparation endowed them with hidden variables that act locally. These hidden variables are like instruction sheets; and the particles' following the instructions, with no direct correlations between the particles, ensures that their behavior is correlated. If the universe is local in its nature (that is, there is no possibility for super-luminal communication or effect, i.e., the world is as

Einstein viewed it) then the information that is needed to complete the quantum theory must be conveyed through some pre-programmed hidden variables.

John Bell had demonstrated that any such hidden-variable theory would not be able to reproduce all of the predictions of quantum mechanics, in particular the ones related to the entanglement in Bohm's version of EPR. The conflict between a complete quantum theory and a local hidden variables universe is brought to a clash through Bell's inequality.

6. EPR argument on the simultaneous measurement (Bellac Quantum Physics)

(a) Perfect anticorrelation

Let us suppose that we are capable of making a state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|+z\rangle \otimes |-z\rangle - |-z\rangle \otimes |+z\rangle],$$

of two identical spin-1/2 particles, with the two particles traveling with equal momenta in opposite directions. For example, they could originate in the decay of an unstable particle of zero spin and zero momentum, in which case momentum conservation implies that the particles move in opposite directions. An example which is simple theoretically (but not experimentally) is the decay of a π^0 meson into an electron and a positron:

$$\pi^0 \to e^+ + e^- .$$

Two experimentalists, conventionally named Alice and Bob, measure the spin component of each particle on a certain axis when the particles are very far apart compared with the range of the force and have not interacted with each other for a long time. For clarity, in this figure the axes used for spin measurement are taken to be perpendicular to the direction of propagation, though this is not essential.

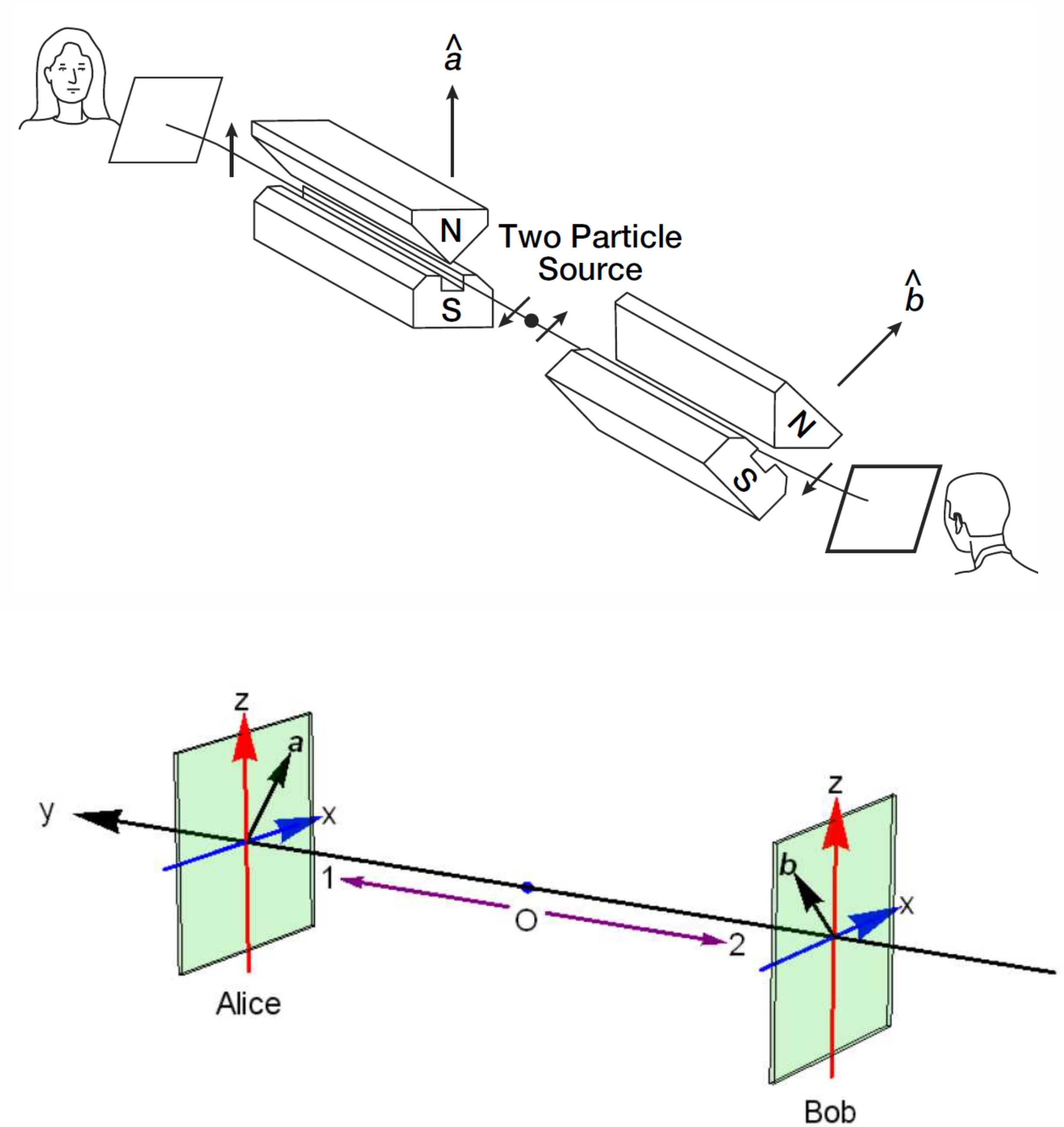


Fig. Configuration of an EPR type of experiment (using SG system and photon).

Using a Stern–Gerlach device in which the magnetic field points in the direction $\boldsymbol{a}$, Alice measures the spin component on this axis for the particle traveling to the left, particle 1, while Bob measures the component along the $\boldsymbol{b}$ axis of the particle traveling to the right, particle 2. Let us first study the case where Alice and Bob both use the z axis, $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{e}_z$. We assume that the decays are well separated in time, and that each experimentalist can know if he or she is measuring the spins of particles emitted in the same decay. In other words, each *pair* ($e+$, $e-$) is perfectly well identified in the experiment.

Using her Stern–Gerlach device, Alice measures the z component of the spin of particle 1, S_z^a, with the result $+\hbar/2$ or $-\hbar/2$, and Bob measures S_z^b of particle 2. Alice and Bob observe a random series of results $+\hbar/2$ or $-\hbar/2$. After the series of measurements has been completed, Alice and Bob meet and compare their results. They conclude that the results for each pair exhibit a perfect (anti-) correlation. When Alice has measured $+\hbar/2$ for particle 1, Bob has measured $-\hbar/2$ for particle 2 and vice versa. To explain this anticorrelation, let us calculate the result of a measurement in the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}[|+z\rangle \otimes |-z\rangle - |-z\rangle \otimes |+z\rangle],$$

of the physical property $[\hat{S}_z^a \otimes \hat{S}_z^b]$, Hermitian operator acting in the tensor product space of the two spins. It is found that $|\Phi\rangle$ is an eigenket of $[\hat{S}_z^a \otimes \hat{S}_z^b]$ with eigenvalue $-\hbar^2/4$

$$[\hat{S}_z^a \otimes \hat{S}_z^b]|\Phi\rangle = -\frac{\hbar^2}{4}|\Phi\rangle.$$

Measurement of $[\hat{S}_z^a \otimes \hat{S}_z^b]$ must then give the result $-\hbar^2/4$, which implies that Bob must measure the value $-\hbar/2$ if Alice has measured the value $+\hbar/2$ and vice versa. Within the limit of accuracy of the experimental apparatus, it is impossible that Alice and Bob both measure the value $+\hbar/2$ or $-\hbar/2$.

Upon reflection, this result is not very surprising. It is a variation of the game of the two customs inspectors. Two travelers 1 and 2, each carrying a suitcase, depart in opposite directions from the origin and eventually are checked by two customs inspectors Alice and Bob. One of the suitcases contains a red ball and the other a green ball, but the travelers have picked up their closed suitcases at random and do not know what color the ball inside is. If Alice checks the suitcase of traveler 1, she has a 50% chance of finding a green ball. But if in fact she finds a green ball, clearly Bob will find a red ball with 100% probability. Correlations between the two suitcases were introduced at the time of departure, and these correlations reappear as a correlation between the results of Alice and Bob.

However, as first noted by Einstein, Podolsky, and Rosen (EPR) in a celebrated paper (which used a different example, ours being due to Bohm), the situation becomes much less commonplace if Alice and Bob decide to use the x axis instead of the z axis for another series of measurements. Since $|\Phi\rangle$ is invariant under the rotation, if Alice and Bob orient their Stern–Gerlach devices in the x direction, they will again find that their measurements are perfectly anticorrelated, because

$$[\hat{S}_x^a \otimes \hat{S}_x^b]|\Phi\rangle = -\frac{\hbar^2}{4}|\Phi\rangle.$$

For any direction ***n*** in the *z*-*x* plane, we also have the same relation

$$[\hat{S}_{\boldsymbol{n}}^a \otimes \hat{S}_{\boldsymbol{n}}^b]|\Phi\rangle = -\frac{\hbar^2}{4}|\Phi\rangle.$$

Measurement of $[\hat{S}_{\boldsymbol{n}}^a \otimes \hat{S}_{\boldsymbol{n}}^b]$ must then give the result $-\hbar^2/4$, which implies that Bob must measure the value $-\hbar/2$ if Alice has measured the value $+\hbar/2$ and vice versa, when the magnetic field is applied along the direction ***n***.

Now we assume that

$$\boldsymbol{n}_1 = (\sin\theta_1, 0, \cos\theta_1), \qquad \boldsymbol{n}_2 = (\sin\theta_2, 0, \cos\theta_2),$$

Then we have

$$[\hat{S}_{\boldsymbol{n}_1}^a \otimes \hat{S}_{\boldsymbol{n}_2}^b]|\Phi\rangle = -\frac{\hbar^2}{4}[\sin(\theta_1 - \theta_2)|\psi\rangle + \cos(\theta_1 - \theta_2)|\Phi\rangle],$$

where

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \otimes |+z\rangle + |-z\rangle \otimes |-z\rangle).$$

When $\theta_1 = \theta_2$, it is clear that $|\Phi\rangle$ is the eigenket.

$$[\hat{S}_{\boldsymbol{n}}^a \otimes \hat{S}_{\boldsymbol{n}}^b]|\Phi\rangle = -\frac{\hbar^2}{4}|\Phi\rangle]$$

((**Mathematica**))

Proof for $[\hat{S}_{\boldsymbol{n}_1}^a \otimes \hat{S}_{\boldsymbol{n}_1}^b]|\Phi\rangle = -\frac{\hbar^2}{4}[\sin(\theta_1 - \theta_2)|\psi\rangle + \cos(\theta_1 - \theta_2)|\Phi\rangle]$

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
χ = 1/Sqrt[2] (KroneckerProduct[ψ1, ψ2] - KroneckerProduct[ψ2, ψ1]) //
   FullSimplify;
η = 1/Sqrt[2] (KroneckerProduct[ψ1, ψ1] + KroneckerProduct[ψ2, ψ2]) //
   FullSimplify;

n1 = {Sin[θ1] Cos[ϕ1], Sin[θ1] Sin[ϕ1], Cos[θ1]} /. ϕ1 → 0;
n2 = {Sin[θ2] Cos[ϕ2], Sin[θ2] Sin[ϕ2], Cos[θ2]} /. ϕ2 → 0;
σ1 = n1[[1]] σx + n1[[2]] σy + n1[[3]] σz // FullSimplify;
σ2 = n2[[1]] σx + n2[[2]] σy + n2[[3]] σz // FullSimplify;

X12 = KroneckerProduct[σ1, σ2]; X12.χ // Simplify
```

$$\left\{\left\{-\frac{\text{Sin}[\theta1-\theta2]}{\sqrt{2}}\right\}, \left\{-\frac{\text{Cos}[\theta1-\theta2]}{\sqrt{2}}\right\}, \left\{\frac{\text{Cos}[\theta1-\theta2]}{\sqrt{2}}\right\}, \left\{-\frac{\text{Sin}[\theta1-\theta2]}{\sqrt{2}}\right\}\right\}$$

```
X12.χ + Sin[θ1 - θ2] η + Cos[θ1 - θ2] χ // FullSimplify
```

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{{0}, {0}, {0}, {0}}
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(b) Possibility of simultaneous measurement: locality principle

The viewpoint underlying the EPR analysis of these results is that of "realism": EPR assume that microscopic systems possess intrinsic properties which must have a counterpart in the physical theory. More precisely, according to EPR, *if the value of a physical property can be predicted with certainty without disturbing the system in any way, there is an "element of reality" associated with this property*. For a particle of spin 1/2 in the state $|+z\rangle$. S_z is a property of this type because it can be predicted with certainty that $S_z = \hbar/2$. However, the value of S_x in this same state cannot be predicted with certainty (it can be $+\hbar/2$ or $-\hbar/2$ with 50% probability of each); $\hat{S}_x$ and $\hat{S}_z$ cannot simultaneously have a physical reality. Since the operators $\hat{S}_x$ and $\hat{S}_z$ do not commute, in quantum physics it is impossible to attribute simultaneous values to them.

In performing their analysis, EPR used a second hypothesis, the locality principle, which stipulates that if Alice and Bob make their measurements in local regions of space-time which cannot be causally connected, then it is not possible that an experimental parameter chosen by Alice, for example the orientation of her Stern–Gerlach device, can affect the properties of particle 2.

According to the preceding discussion, this implies that without disturbing particle 2 in any way, a measurement of S_z^a by Alice permits knowledge of S_z^b with certainty, and a measurement of S_x^a permits knowledge of S_x^b with certainty. If the "local realism" of EPR is accepted, the result of Alice's measurement serves only to reveal a piece of information which was already stored in the local region of space-time associated with particle 2. A theory that is more complete than quantum mechanics should contain simultaneous information on the values of S_x^b and S_z^b and be capable of predicting with certainty all the results of measurements of these two physical properties in the local region of space-time attached to particle 2. The physical properties S_x^b and S_z^b then simultaneously have a physical reality, in contrast to the quantum description of the spin of a particle by a state vector.

EPR do not dispute the fact that quantum mechanics gives predictions that are statistically correct, but quantum mechanics is not sufficient for describing the physical reality of an individual pair. Within the framework of local realism such as that defined above, the EPR argument is unassailable and the verdict incontestable: quantum mechanics is incomplete! Nevertheless, EPR do not suggest any way of "completing" it, and we shall see in what follows that local realism is in conflict with experiment.

According to local realism, even if an experiment does not permit the simultaneous measurement of S_x^b and S_z^b, these two quantities still have a simultaneous physical reality in the local region of space-time attached to particle 2, and owing to symmetry the same is true for S_x^a

and S_z^a, of particle *a*. This ineluctable consequence of local realism makes it possible to prove the *Bell inequalities*, which fix the maximum possible correlations given this hypothesis.

6. Hidden variable theory

Suppose that Alice observes the spin—up state (denoted as $\{\boldsymbol{a}+\}$ using the SG with the direction $\boldsymbol{a}$ (arbitrary). Since the spin angular momentum is conserved, she also knows that Bob will observe the spin-down without any knowledge of the result from Bob.

$\{\boldsymbol{a}+, \boldsymbol{a}-\}$.

Similarly, Alice observes the spin-down state (denoted as $\{\boldsymbol{a}-\}$ using the SG with the direction $\boldsymbol{a}$ (arbitrary). Since the spin angular momentum is conserved, she also knows that Bob will observe the spin-up $\{\boldsymbol{a}+\}$, without any knowledge of the result from Bob.

$\{\boldsymbol{a}-, \boldsymbol{a}+\}$.

7. Prediction using quantum mechanics

We consider the above argument using our knowledge of quantum mechanics. The original state before Alice measures,

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}[|+z,-z\rangle - |-z,+z\rangle].$$

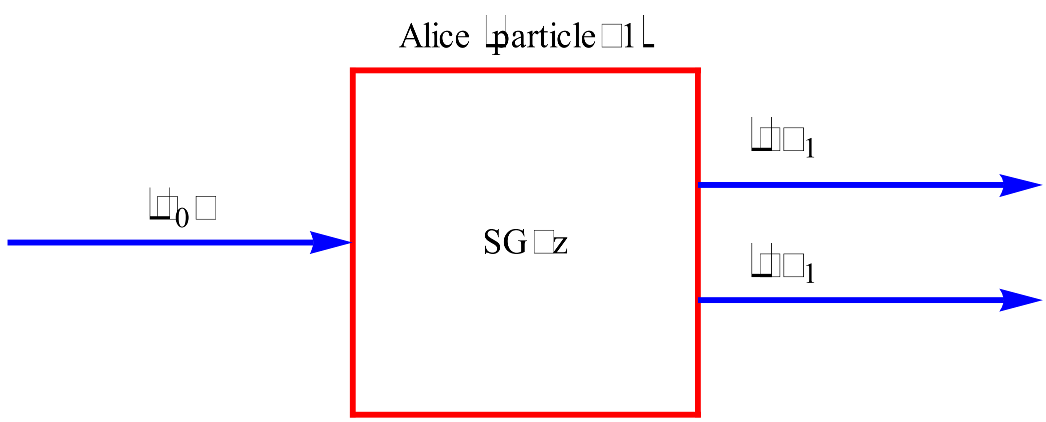


Fig. Stern-Gerlach experiment. Note that $|+\rangle = |+z\rangle$ and $|-\rangle = |-z\rangle$, for simplicity. $\boldsymbol{a} = \boldsymbol{e}_z$.

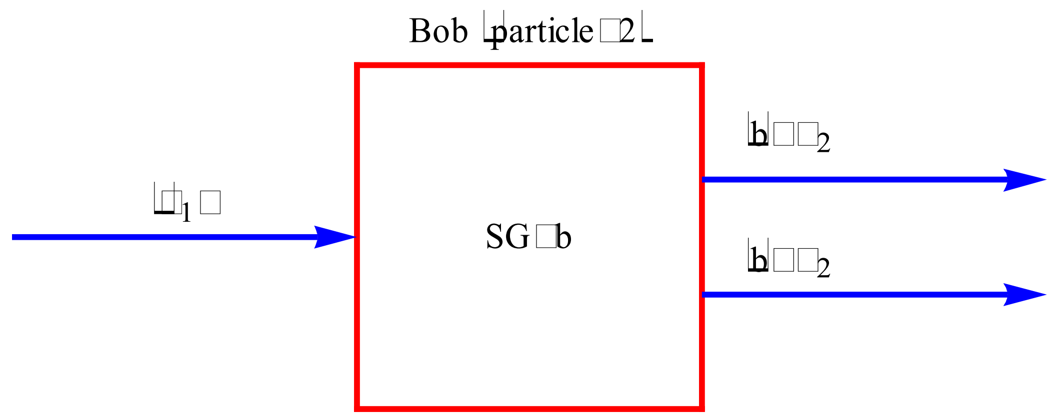
After Alice measures using the SG with the *z* direction, she finds that the particle 1 is in the spin-up state. The state is changed into

$$|\psi_1\rangle = |+z,-z\rangle = |+z\rangle_1 \otimes |-z\rangle_2 .$$

Then Bob measures using the SG with the direction ***b***.

$$|+\boldsymbol{b}\rangle_2 = \cos\frac{\theta}{2}|+z\rangle_2 + \sin\frac{\theta}{2}e^{i\phi}|-z\rangle_2 ,$$

$$|-\boldsymbol{b}\rangle_2 = \sin\frac{\theta}{2}|+z\rangle_2 - \cos\frac{\theta}{2}e^{i\phi}|-z\rangle_2 ,$$



where ***a*** is along the z direction. The angles θ and ϕ are the polar angles that give the orientation of ***b***. Note that

$$\boldsymbol{a}\cdot\boldsymbol{b} = \cos\theta ,$$

with $|\boldsymbol{a}| = |\boldsymbol{b}| = 1$.

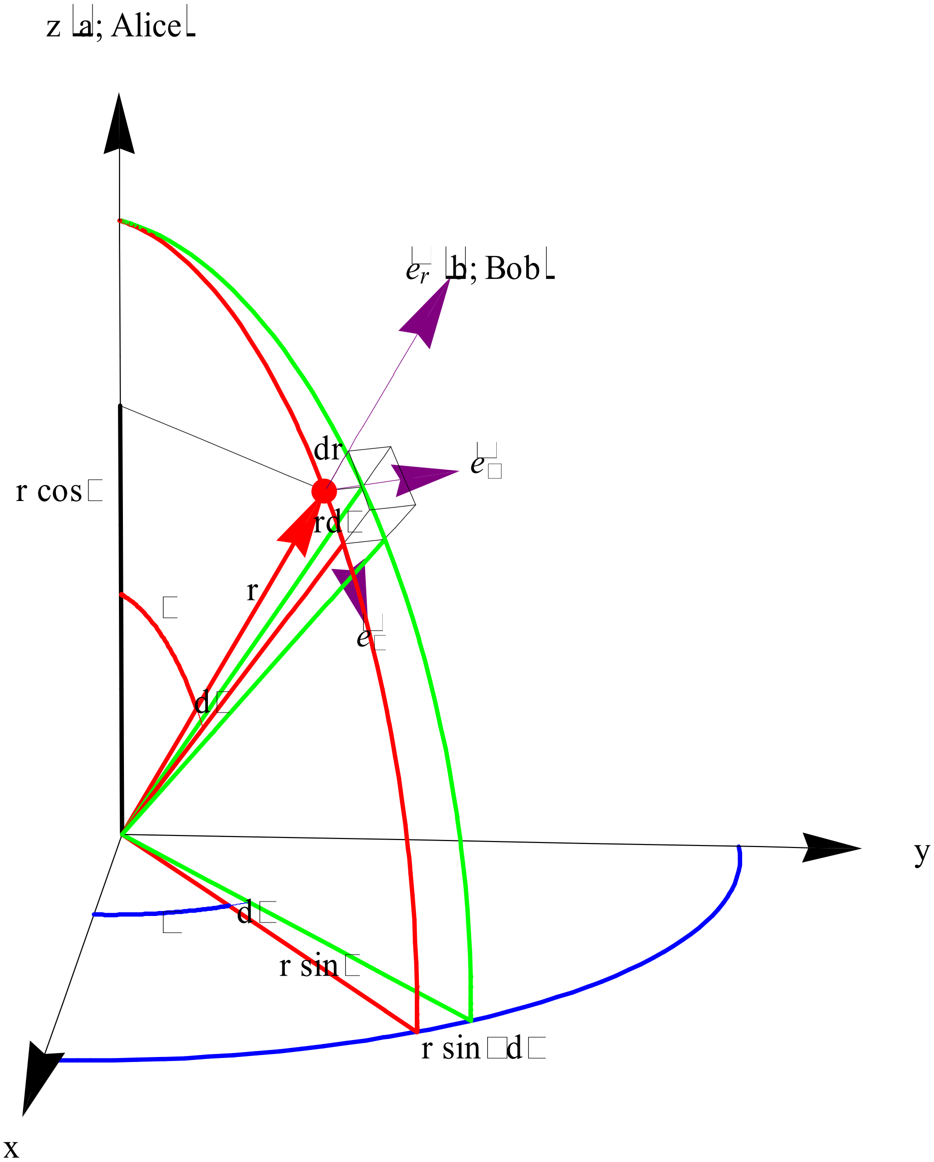


Fig. The direction of ***a*** for Alice and ***b*** for Bob. ***a*** is along the *z* direction. The angles θ and ϕ are the polar angles that give the orientation of ***b***. Note that $\boldsymbol{a} \cdot \boldsymbol{b} = \cos\theta$

The probability that Bob measures +1/2 is

$$P_B(+\boldsymbol{b}) = \left|{}_2\langle +\boldsymbol{b} | -z \rangle_2\right|^2 = \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta) .$$

The probability that Bob measures -1/2 is

$$P_B(-\boldsymbol{b}) = \left|{}_2\langle -\boldsymbol{b}|-z\rangle_2\right|^2 = \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) = \frac{1}{2}(1+\boldsymbol{a}\cdot\boldsymbol{b}).$$

In other words, these probabilities depend on the angle θ. When $\boldsymbol{a} \cong \boldsymbol{b}$,

$$P_B(-\boldsymbol{b}) = 1.$$

as Alice predicted. So the quantum mechanics is consistent with common sense.

((**Note**))

We note that the probability for finding the state $|+z, \boldsymbol{b}-\rangle$ is evaluated as

$$P = \left|\langle +z, -\boldsymbol{b}|\psi_0\rangle\right|^2 = \frac{1}{2}\left|\langle -\boldsymbol{b}|-z\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2},$$

with

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}[|+z,-z\rangle - |-z,+z\rangle].$$

When $\theta = 0$, the probability becomes $P = 1/2$ (50%), which is consistent with the result described later.

8. One type of SG system for the measurement

We consider a two-electron system in a spin singlet state, with a total spin zero.

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|+z,-z\rangle - |-z,+z\rangle].$$

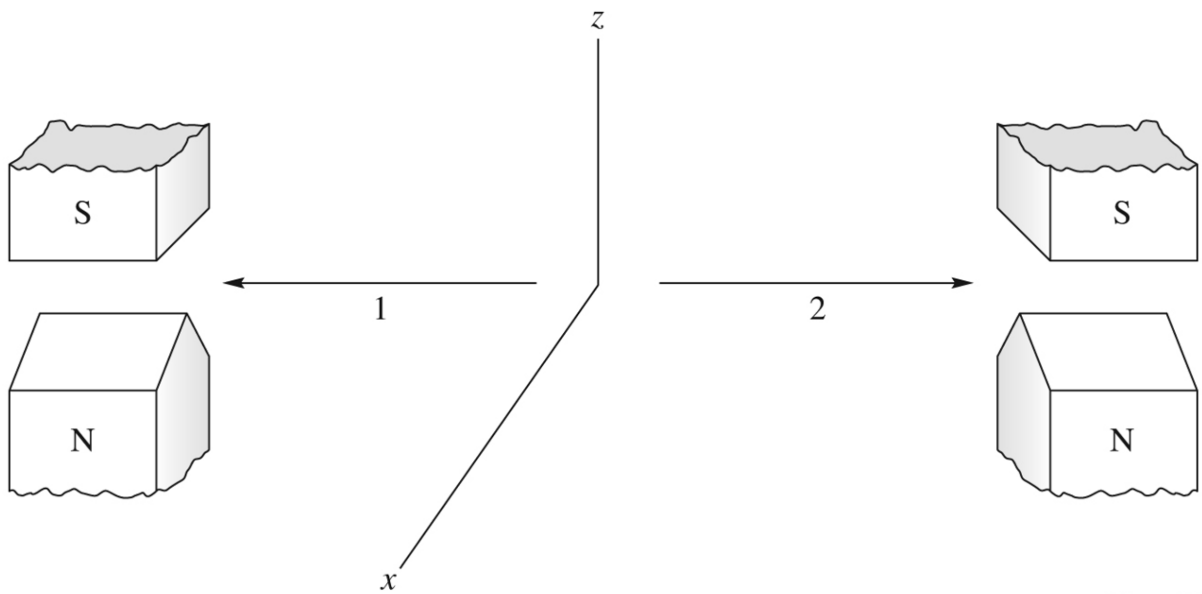
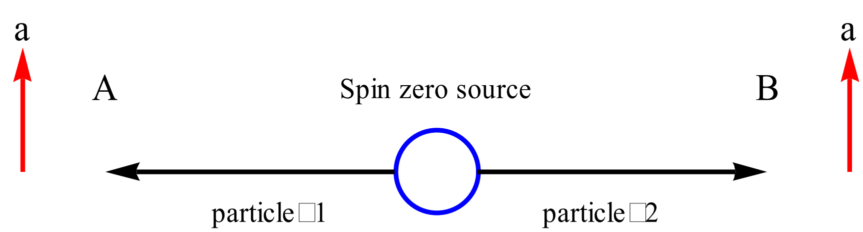


Fig. A schematic of the EPR experiment in which A measures the spin of particle 1 and B measures the spin of particle 2

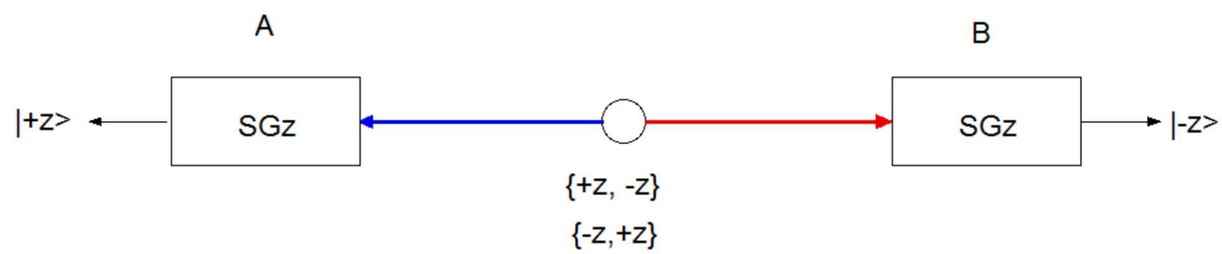
Suppose we make a measurement on the spin component of one of the electrons. If one of the spins is shown to be in the spin-up state, then the other is necessarily in the spin-down state. In this case, measurement apparatus selects the first term $\left|+z,-z>\right\rangle$, a subsequent measurement of the spin component of electron 2 must ascertain that the state ket of the system is given by $\left|-z,+z\right\rangle$. Suppose that a pair of spin 1/2 particles (two particles) originate in a combined spin 0 state, and then travel out in opposite directions, left and right, to respective widely separated detectors. Each detector is set up to measure the spin of the approaching particle, but in some direction which is decided upon only after the particles are in full flight.



((**Hidden variable theory; locality**))

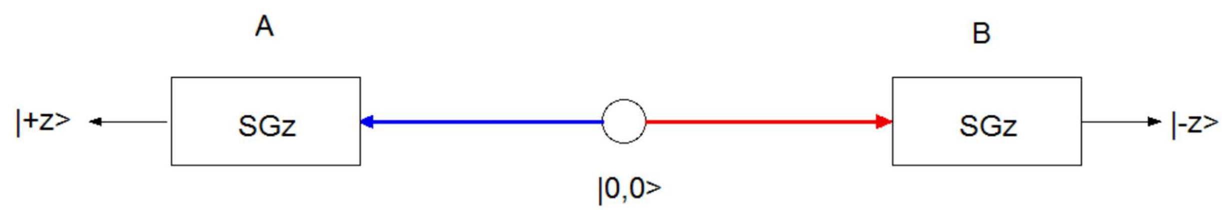
Note that the angular momentum is conserved. If Alice measures the spin up state using the SG with ***a*** direction, Bob will measure the spin-down state using the SG with ***z*** direction. If Alice measures the spin down state using the SG with ***z*** direction), Bob will measure the spin-up state using the SG with ***a*** direction: {+*z*, -*z*}; {-*z*, *z*}. Note that the event {+*z*, -*z*} means that Alice measures the state +*z* and Bob measures the state –*z*.

If A makes measurements of S_{1z} and obtain the value $\hbar/2$, and B makes measurements of S_{2z}, 50 % of B's measurements will yield $-\hbar/2$.



There are two cases: {+*z*, -*z*} and {-*z*, +*z*}. In this case the state {+*z*, -*z*} can be observed. Therefore, the probability is 50 %.

((**Quantum mechanics, non-locality**))



$$\left|0,0\right\rangle = \frac{1}{\sqrt{2}}[\left|+z,-z\right\rangle - \left|-z,+z\right\rangle],$$

$$\begin{aligned}\left\langle +z,-z\middle|0,0\right\rangle &= \frac{1}{\sqrt{2}}\left\langle +z,-z\middle|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z,-z\middle|-z,+z\right\rangle] \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

or

$$P = \left|\left\langle +z,-x\middle|0,0\right\rangle\right|^2 = \frac{1}{2}.$$

The probability is 50 %. Then both models give the same value for the probability.

((**Note**))

Suppose that Alice first does her measurement and find the state $|+z\rangle$. After that, Bob does his measurement. Before his measurement, the state of the system already collapses into the $|-z\rangle$.

$$_1\langle +z|0,0\rangle = \frac{1}{\sqrt{2}}|-z\rangle_2.$$

The probability: $$P_2 = \left|\frac{1}{\sqrt{2}}\,_2\langle -z|-z\rangle_2\right|^2 = \frac{1}{2} \qquad (50\%)$$

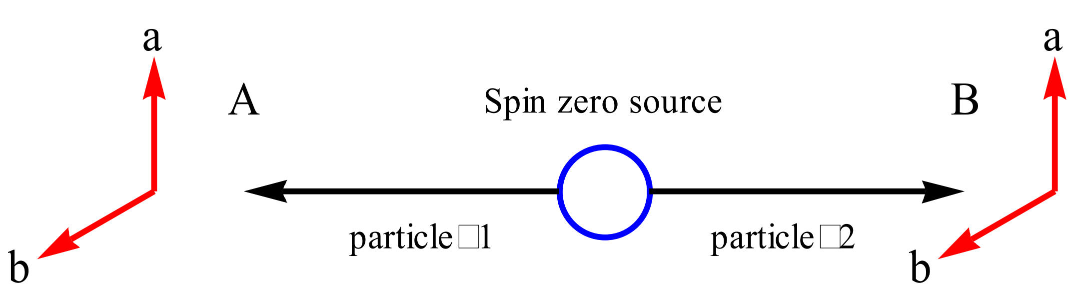
Suppose that Bob first does his measurement and find the state $|-z\rangle$. After that, Alice does her measurement. Before her measurement, the state of the system already collapses into the state $|+z\rangle$.

$$_2\langle +z|0,0\rangle = \frac{1}{\sqrt{2}}|+z\rangle_1.$$

The probability: $$P_1 = \left|\frac{1}{\sqrt{2}}\,_1\langle +z|+z\rangle_1\right|^2 = \frac{1}{2} \qquad (50\%)$$

9. Two types of SG systems for the measurement

For a particular pair, there must be a perfect matching between particle 1 and particle 2 to ensure zero total angular momentum. If the particle 1 is of type {+*z*, -*x*}, then the particle 2 must belong to type {-*z*, +*x*}, and so on. The results of correlation measurements can be reproduced if the particle 1 and particle 2 are matched, as follows.



Particle 1	Particle 2	Population	Event
{***a***+, ***b***+}	{***a***-, ***b***-}	$N_1 = 25\%$	E_1
{***a***+, ***b***-}	{***a***-, ***b***+}	$N_2 = 25\%$	E_2

{***a***-, ***b***+}	{***a***+, ***b***-}	N_3 = 25%	E_3
{***a***-, ***b***-}	{***a***+, ***b***+}	N_4 = 25%	E_4

where $\boldsymbol{a} = \boldsymbol{e}_z$ and $\boldsymbol{b} = \boldsymbol{e}_x$.

or

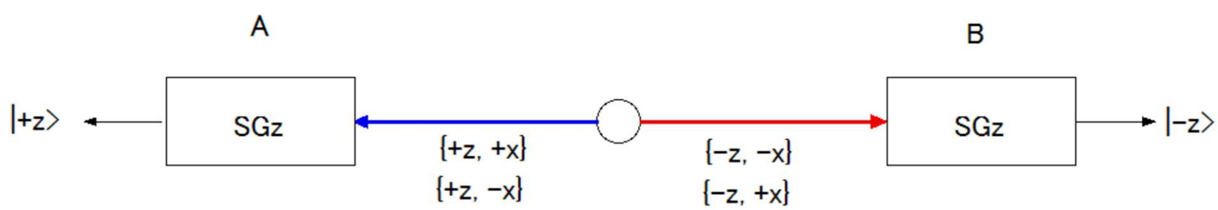
{+*z*, +*x*}	{-*z*, -*x*}	N_1 = 25%	E_1
{+*z*, -*x*}	{-*z*, +*x*}	N_2 = 25%	E_2
{-*z*, +*x*}	{+*z*, -*x*}	N_3 = 25%	E_3
{-*z*, -*x*}	{+*z*, +*x*}	N_4 = 25%	E_4

and that each of these distinct groups of particles is produced in equal number. Note that a new {..} notation provides a non-quantum description of the state of the particle.

((Hidden variable theory; locality))

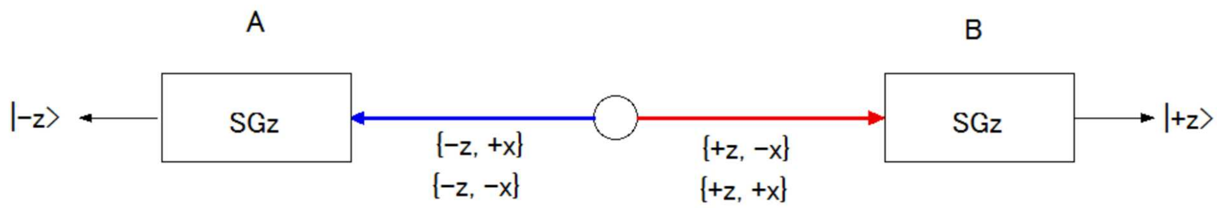
(i)

If A makes measurements of S_{1z} and obtain the value $\hbar/2$, and B makes measurements of S_{2z}, 50 % of B's measurements will yield $-\hbar/2$. The events E_1 and E_2 are allowed.



Particle-1	Particle-2	
{+*z*, +*x*}	{-*z*, -*x*}	E_1
{+*z*, -*x*}	{-*z*, +*x*}	E_2
{-*z*, +*x*}	{+*z*, -*x*}	E_3
{-*z*, -*x*}	{+*z*, +*x*}	E_4

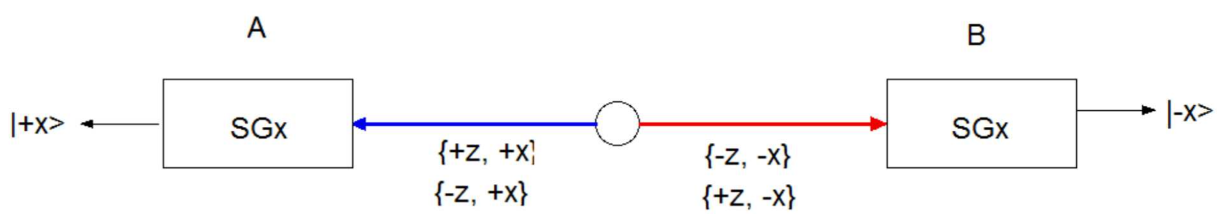
If A makes measurements of S_{1z} and obtain the value $-\hbar/2$, and B makes measurements of S_{2z}, 50 % of B's measurements will yield $\hbar/2$. The events E_3 and E_4 are allowed.



Particle-1	Particle-2
{+*z*, +*x*}	{-*z*, -*x*}
{+*z*, -*x*}	{-*z*, +*x*}
{-*z*, +*x*}	{+*z*, -*x*}
{-*z*, -*x*}	{+*z*, +*x*}

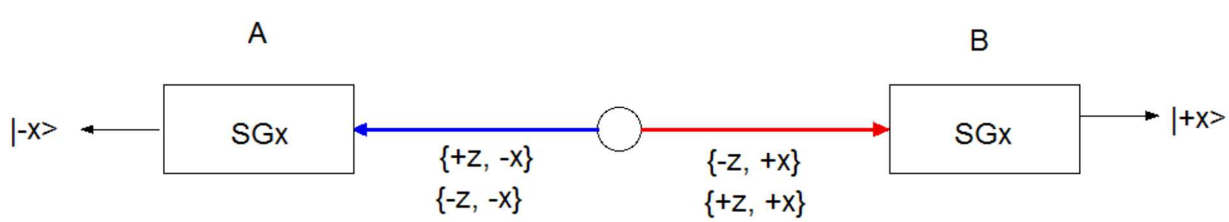
(ii)

If A makes measurements of S_{1x} and obtain the value $\hbar/2$, and B makes measurements of S_{2x}, 50 % of B's measurements will yield $-\hbar/2$. The events E_1 and E_3 are allowed.



Particle-1	Particle-2
{+*z*, +*x*}	{-*z*, -*x*}
{+*z*, -*x*}	{-*z*, +*x*}
{-*z*, +*x*}	{+*z*, -*x*}
{-*z*, -*x*}	{+*z*, +*x*}

If A makes measurements of S_{1x} and obtain the value $-\hbar/2$, and B makes measurements of S_{2x}, 50 % of B's measurements will yield $\hbar/2$. The events E_2 and E_4 are allowed.

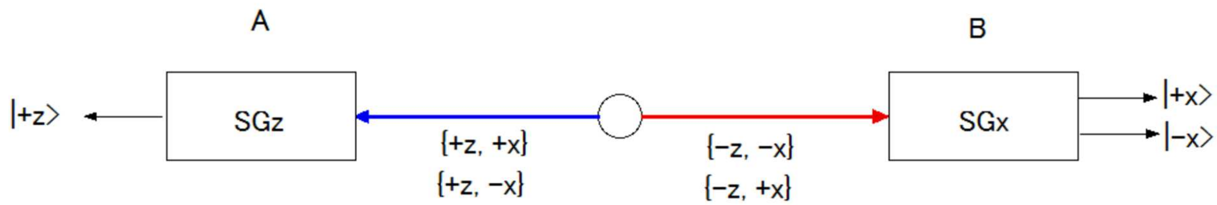


Particle-1	Particle-2

{+z, +x}	{-z, -x}
{+z, -x}	{-z, +x}
{-z, +x}	{+z, -x}
{-z, -x}	{+z, +x}

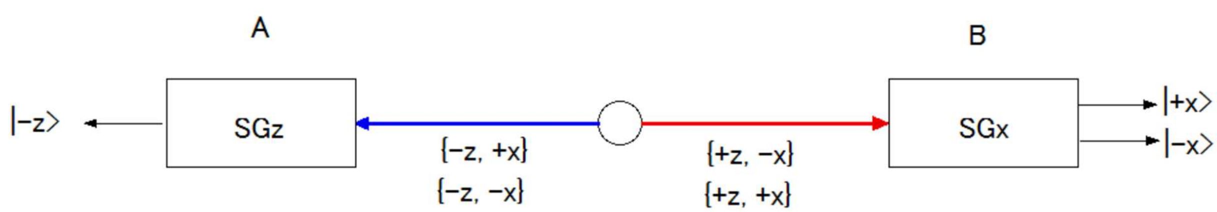
(iii)

If A makes measurements of S_{1z} and obtain the value $\hbar/2$, and B makes measurements of S_{2x}, 25 % of B's measurements will yield $\hbar/2$ and 25 % will yield $-\hbar/2$. E_1 for Bob to measure of S_{2x} ($\hbar/2$) and E_2 for Bob to measure of S_{2x} ($-\hbar/2$).



Particle-1	Particle-2
{+z, +x}	{-z, -x}
{+z, -x}	{-z, +x}
{-z, +x}	{+z, -x}
{-z, -x}	{+z, +x}

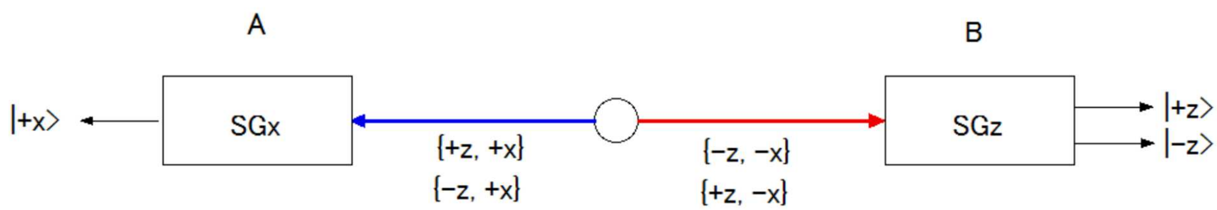
If A makes measurements of S_{1z} and obtain the value $-\hbar/2$, and B makes measurements of S_{2x}, 25 % of B's measurements will yield $\hbar/2$ and 25 % will yield $-\hbar/2$. E_4 for Bob to measure of S_{2x} ($\hbar/2$) and E_3 for Bob to measure of S_{2x} ($-\hbar/2$).



Particle-1	Particle-2
{+z, +x}	{-z, -x}
{+z, -x}	{-z, +x}
{-z, +x}	{+z, -x}
{-z, -x}	{+z, +x}

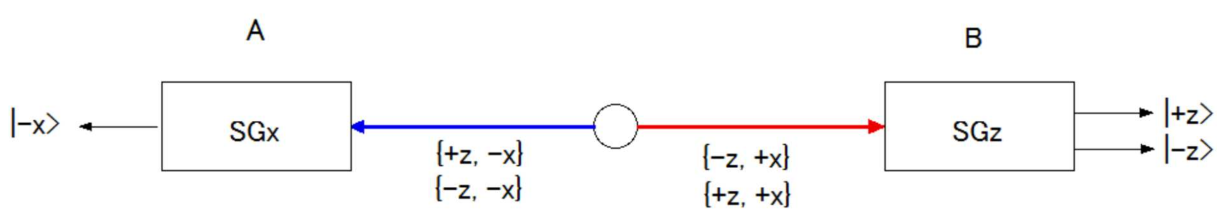
(iv)

If A makes measurements of S_{1x} and obtain the value $\hbar/2$, and B makes measurements of S_{2z}, 25 % of B's measurements will yield $\hbar/2$ and 25 % will yield $-\hbar/2$. E_3 for Bob to measure of S_{2z} ($\hbar/2$) and E_1 for Bob to measure of S_{2z} ($-\hbar/2$) and



Particle-1	Particle-2
{+*z*, +*x*}	{-*z*, -*x*}
{+*z*, -*x*}	{-*z*, +*x*}
{-*z*, +*x*}	{+*z*, -*x*}
{-*z*, -*x*}	{+*z*, +*x*}

If A makes measurements of S_{1x} and obtain the value $-\hbar/2$, and B makes measurements of S_{2z}, 25 % of B's measurements will yield $\hbar/2$ and 25 % will yield $-\hbar/2$. E_4 for Bob to measure of S_{2z} ($\hbar/2$) and E_2 for Bob to measure of S_{2z} ($-\hbar/2$).



Particle-1	Particle-2
{+*z*, +*x*}	{-*z*, -*x*}
{+*z*, -*x*}	{-*z*, +*x*}
{-*z*, +*x*}	{+*z*, -*x*}
{-*z*, -*x*}	{+*z*, +*x*}

10. Quantum mechanics (non-locality)

In quantum mechanics, we express the state $|0,0\rangle$ as

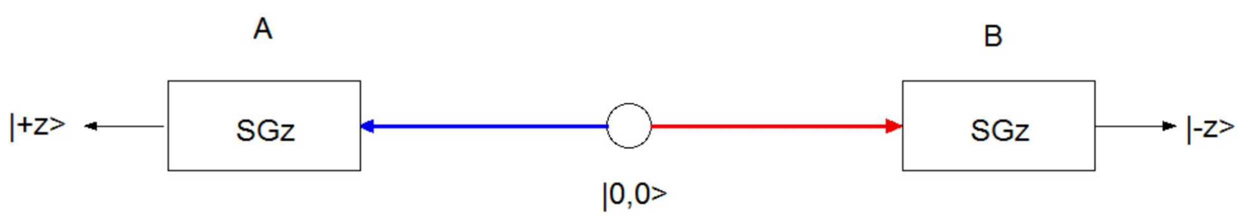
$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+z,-z\rangle - |-z,+z\rangle]$$

The ket $|+x\rangle$ and the ket $|-x\rangle$ are defined by

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$$

(a) Probability: particle 1 ($|+z\rangle_1$) and particle 2 ($|-z\rangle_2$)

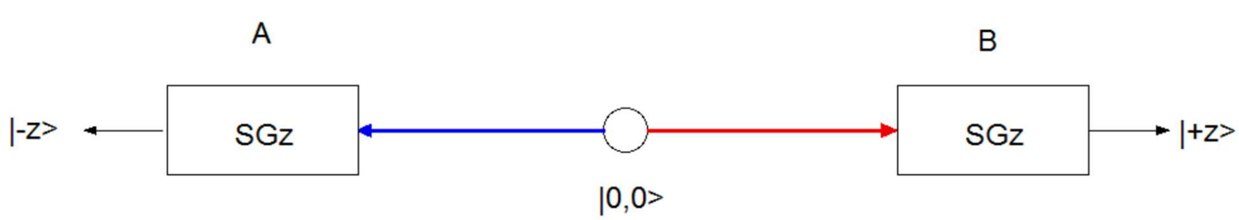


$$\langle +z,-z|0,0\rangle = \frac{1}{\sqrt{2}}\langle +z,-z|+z,-z\rangle - \frac{1}{\sqrt{2}}\langle +z,-z|-z,+z\rangle]$$
$$= \frac{1}{\sqrt{2}}$$

or

$$P = |\langle +z,-x|0,0\rangle|^2 = \frac{1}{2}$$

(b) Probability: particle 1 ($|-z\rangle_1$) and particle 2 ($|+z\rangle_2$),

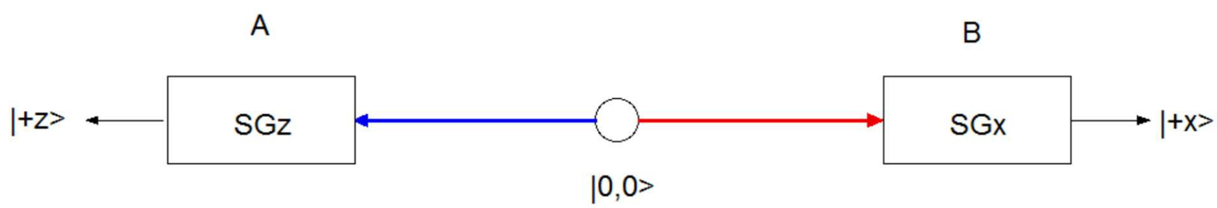


$$\langle -z,+z|0,0\rangle = \frac{1}{\sqrt{2}}\langle -z,+z|+z,-z\rangle - \frac{1}{\sqrt{2}}\langle -z,+z|-z,+z\rangle]$$
$$= -\frac{1}{\sqrt{2}}$$

or

$$P = |\langle -z,+z|0,0\rangle|^2 = \frac{1}{2}.$$

(c) Probability: particle 1 ($|+z\rangle_1$) and particle 2 ($|+x\rangle_2$)

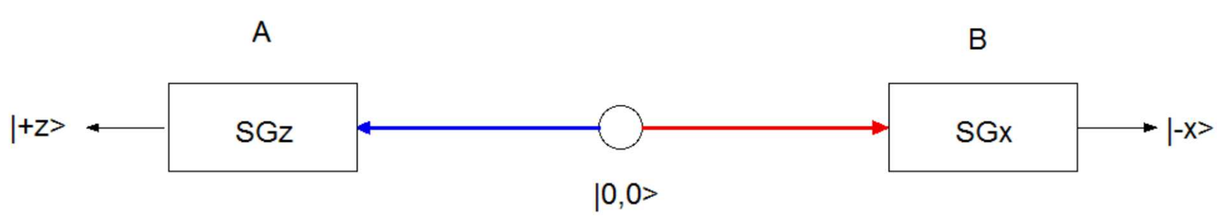


$$\langle +z,+x|0,0\rangle = \frac{1}{\sqrt{2}}\langle +z,+x|+z,-z\rangle - \frac{1}{\sqrt{2}}\langle +z,+x|-z,+z\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle +x|-z\rangle$$
$$= \frac{1}{2}$$

or

$$P = |\langle +z,+x|0,0\rangle|^2 = \frac{1}{4}.$$

(d) Probability: particle 1 ($|+z\rangle_1$) and particle 2 ($|-x\rangle_2$)



$$\langle +z,-x|0,0\rangle = \frac{1}{\sqrt{2}}\langle +z,-x|+z,-z\rangle - \frac{1}{\sqrt{2}}\langle +z,-x|-z,+z\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle -x|-z\rangle = -\frac{1}{2}$$

or

$$P = |\langle +z,-x|0,0\rangle|^2 = \frac{1}{4}$$

In conclusion, the non-quantum model in which each of the particles in the two-particle system has definite attributes, is able to reproduce the results of quantum mechanics.

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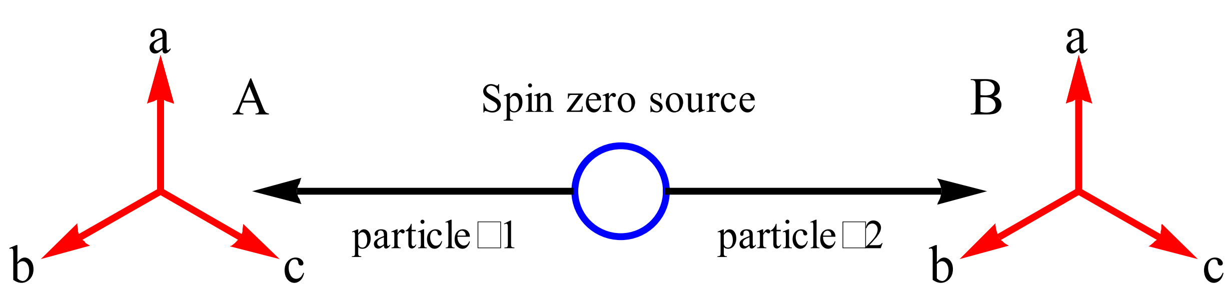
12. Three types of SG systems: Bell's inequality (Townsend, Sakurai)

We now consider more complicated situation where the model leads to predictions different from the usual quantum-mechanical predictions. We start with three-unit vectors, ***a***, ***b***, and ***c***, which are, in general, not mutually orthogonal. We define {***a***-, ***b***+, ***c***+} as follows. The Stern-Gerlach experiment along the ***a*** direction yields $-\hbar/2$. The Stern-Gerlach experiment along the ***b*** direction yields $\hbar/2$. The Stern-Gerlach experiment along the ***c*** direction yields $\hbar/2$. There must be a perfect matching in the sense that the other particles necessarily belong to type {***a***-, ***b***+, ***c***+} to ensure zero angular momentum.



((**Hidden variable theory**))

Note that the angular momentum is conserved. If Alice measures the spin up state using the SG with ***a*** direction), Bob will measure the spin-down state using the SG with ***a*** direction.

(i)

Particle 1	Particle 2	Population	Events
{***a***+, ***b***+, ***c***+}	{***a***-, ***b***-, ***c***-}	N_1	E_1

{***a***+, ***b***+, ***c***-}	{***a***-, ***b***-, ***c***+}	N_2	E_2
{***a***+, ***b***-, ***c***+}	{***a***-, ***b***+, ***c***-}	N_3	E_3
{***a***+, ***b***-, ***c***-}	{***a***-, ***b***+, ***c***+}	N_4	E_4
{***a***-, ***b***+, ***c***+}	{***a***+, ***b***-, ***c***-}	N_5	E_5
{***a***-, ***b***+, ***c***-}	{***a***+, ***b***-, ***c***+}	N_6	E_6
{***a***-, ***b***-, ***c***+}	{***a***+, ***b***+, ***c***-}	N_7	E_7
{***a***-, ***b***-, ***c***-}	{***a***+, ***b***+, ***c***+}	N_8	E_8

Probability: particle 1 ($\left|+\boldsymbol{a}\right\rangle_1$) and particle 2 ($\left|+\boldsymbol{b}\right\rangle_2$),

$$P(+\boldsymbol{a},+\boldsymbol{b}) = \frac{N_3 + N_4}{\sum_{i=1}^{8} N_i}$$

Probability: particle 1 ($\left|+\boldsymbol{a}\right\rangle_1$) and particle 2 ($\left|+\boldsymbol{c}\right\rangle_2$),

$$P(+\boldsymbol{a},+\boldsymbol{c}) = \frac{N_2 + N_4}{\sum_{i=1}^{8} N_i}$$

Probability: particle 1 ($\left|+\boldsymbol{c}\right\rangle_1$) and particle 2 ($\left|+\boldsymbol{b}\right\rangle_2$),

$$P(+\boldsymbol{c},+\boldsymbol{b}) = \frac{N_3 + N_7}{\sum_{i=1}^{8} N_i}$$

We note that

$$N_3 + N_4 \le (N_2 + N_4) + (N_3 + N_7)$$

since N_i is positive. Then we have

$$P(+\boldsymbol{a},+\boldsymbol{c}) - P(+\boldsymbol{a},+\boldsymbol{b}) + P(+\boldsymbol{c},+\boldsymbol{b}) \ge 0.$$

This is the Bell's inequality.

(ii)

Similarly, we get the Bell’s inequality as

$$P(+\boldsymbol{a},+\boldsymbol{b}) - P(+\boldsymbol{a},+\boldsymbol{c}) + P(+\boldsymbol{b},+\boldsymbol{c}) \geq 0.$$

This inequality can be derived as follows. The probability: particle 1 ($\left|+\boldsymbol{b}\right\rangle_1$) and particle 2 ($\left|+\boldsymbol{c}\right\rangle_2$) is given by

$$P(+\boldsymbol{b},+\boldsymbol{c}) = \frac{N_2 + N_6}{\sum_{i=1}^{8} N_i}$$

We note that

$$N_2 + N_4 \leq (N_3 + N_4) + (N_2 + N_6)$$

since N_i is positive. This leads to the above inequality.

Before we discuss this case, we note that

$$\left|0,0\right\rangle = \frac{1}{\sqrt{2}}[\left|+z,-z\right\rangle - \left|-z,+z\right\rangle] = -\frac{1}{\sqrt{2}} e^{-i\phi}[\left|+\boldsymbol{n}\right\rangle_1 \left|-\boldsymbol{n}\right\rangle_2 - \left|-\boldsymbol{n}\right\rangle_1 \left|+\boldsymbol{n}\right\rangle_2]$$

Here we use the notations

$$\left|+\boldsymbol{n}\right\rangle = \cos\frac{\theta}{2}\left|+z\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|-z\right\rangle, \qquad \left|-\boldsymbol{n}\right\rangle = \sin\frac{\theta}{2}\left|+z\right\rangle - e^{i\phi}\cos\frac{\theta}{2}\left|-z\right\rangle$$

and

$$\left|+z\right\rangle = \cos\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle - \sin\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle$$

$$\left|-z\right\rangle = e^{-i\phi}\sin\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle - e^{-i\phi}\cos\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle$$

Then we have

$$
\begin{aligned}
\left|0,0\right\rangle &= \frac{1}{\sqrt{2}}[\left|+z,-z\right\rangle - \left|-z,+z\right\rangle] \\
&= \frac{1}{\sqrt{2}}\{[\cos\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1 + \sin\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1][e^{-i\phi}\sin\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_2 - e^{-i\phi}\cos\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_2]\} \\
&\quad -\frac{1}{\sqrt{2}}\{[e^{-i\phi}\sin\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1 - e^{-i\phi}\cos\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1][\cos\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_2 + \sin\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_2] \\
&= \frac{1}{\sqrt{2}}\{[e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2 - e^{-i\phi}\cos^2\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2 + e^{-i\phi}\sin^2\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2 \\
&\quad - e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2\} - \frac{1}{\sqrt{2}}\{[e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2 \\
&\quad + e^{-i\phi}\sin^2\frac{\theta}{2}\left|+\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2 - e^{-i\phi}\cos^2\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2 - e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left|-\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2\} \\
&= \frac{1}{\sqrt{2}}e^{-i\phi}(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})[\left|+\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2 - \left|-\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2] \\
&= -\frac{1}{\sqrt{2}}e^{-i\phi}[\left|+\boldsymbol{n}\right\rangle_1\left|-\boldsymbol{n}\right\rangle_2 - \left|-\boldsymbol{n}\right\rangle_1\left|+\boldsymbol{n}\right\rangle_2]
\end{aligned}
$$

((**Mathematica**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψpn = (Cos[θ/2]
       E^(I φ) Sin[θ/2]);
ψmn = (-Sin[θ/2]
       E^(I φ) Cos[θ/2]);
B1 =
  1/Sqrt[2] (KroneckerProduct[ψpn, ψmn] -
     KroneckerProduct[ψmn, ψpn]) // FullSimplify;
B1 // MatrixForm
```

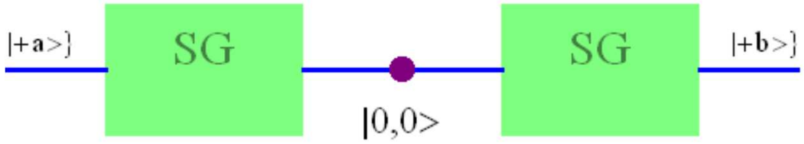
$$\begin{pmatrix} 0 \\ \frac{e^{i\phi}}{\sqrt{2}} \\ -\frac{e^{i\phi}}{\sqrt{2}} \\ 0 \end{pmatrix}$$

In quantum mechanics, we express the state $|0,0\rangle$ as

$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+\boldsymbol{a},-\boldsymbol{a}\rangle - |-\boldsymbol{a},+\boldsymbol{a}\rangle], \qquad |0,0\rangle = \frac{1}{\sqrt{2}}[|+\boldsymbol{b},-\boldsymbol{b}\rangle - |-\boldsymbol{b},+\boldsymbol{b}\rangle],$$

except for the phase factor.

(1)



$$\langle +\boldsymbol{a},+\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},+\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},+\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle$$

$$P(+\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}|\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle|^2 = \frac{1}{2}|\langle -\boldsymbol{a}|+\boldsymbol{b}\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$$

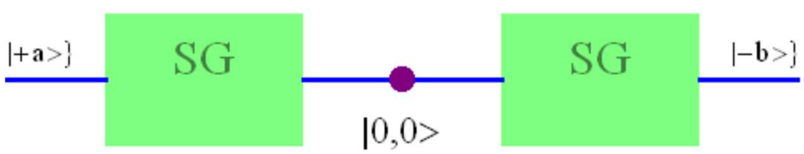
where

$$|+\boldsymbol{b}\rangle = \cos\frac{\theta_{ab}}{2}|+\boldsymbol{a}\rangle + \sin\frac{\theta_{ab}}{2}|-\boldsymbol{a}\rangle,$$

$$|-\boldsymbol{b}\rangle = \sin\frac{\theta_{ab}}{2}|+\boldsymbol{a}\rangle - \cos\frac{\theta_{ab}}{2}|-\boldsymbol{a}\rangle.$$

where θ_{ab} is the angle between two unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

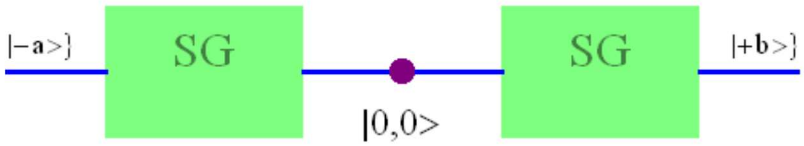
(2)



$$\langle +\boldsymbol{a},-\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},-\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},-\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle$$

$$P(+\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}|\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle|^2 = \frac{1}{2}|\langle -\boldsymbol{a}|-\boldsymbol{b}\rangle|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}.$$

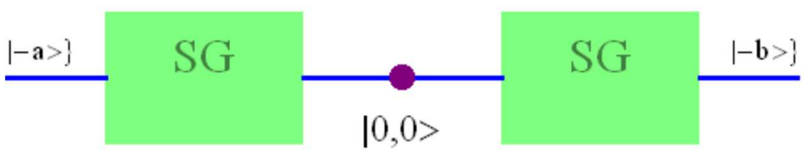
(3)



$$\langle -\boldsymbol{a},+\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},+\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},+\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$P(-\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}|\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle|^2 = \frac{1}{2}|\langle +\boldsymbol{a}|+\boldsymbol{b}\rangle|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}.$$

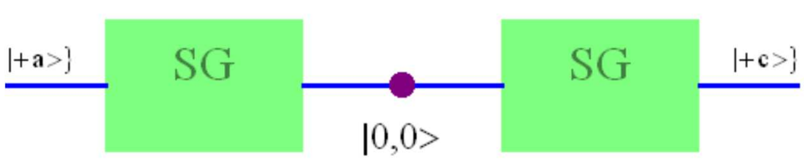
(4)



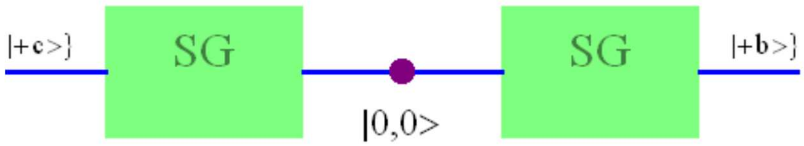
$$\langle -\boldsymbol{a},-\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},-\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},-\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= -\frac{1}{\sqrt{2}}\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$P(-\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}|\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle|^2 = \frac{1}{2}|\langle +\boldsymbol{a}|-\boldsymbol{b}\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}.$$

Similarly, we have



$$P(+\boldsymbol{a},+\boldsymbol{c}) = \frac{1}{2}|\langle \boldsymbol{a}+|\boldsymbol{c}+\rangle|^2 = \frac{1}{2}\sin^2\frac{\theta_{ac}}{2}.$$



$$P(+\boldsymbol{c},+\boldsymbol{b}) = \frac{1}{2}\left|\langle \boldsymbol{c}+|\boldsymbol{b}+\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{cb}}{2}.$$

Since

$$P(+\boldsymbol{a},+\boldsymbol{b}) \le P(+\boldsymbol{a},+\boldsymbol{c}) + P(+\boldsymbol{c},+\boldsymbol{b}) \qquad \text{(Bell's inequality)}$$

we have

$$\sin^2\frac{\theta_{ab}}{2} \le \sin^2\frac{\theta_{ac}}{2} + \sin^2\frac{\theta_{cb}}{2},$$

or

$$1-\cos\theta_{ab} \le 1-\cos\theta_{ac} + 1-\cos\theta_{cb},$$

or

$$\cos\theta_{ac} + \cos\theta_{cb} \le \cos\theta_{ab} + 1, \qquad \text{(Bell's inequality)}$$

Suppose that $\theta_{ab} = 2\theta$, $\theta_{bc} = \theta_{ac} = \theta$,

$$f(\theta) = 1 + \cos(2\theta) - 2\cos\theta.$$

We make a plot of $f(\theta)$ as a function of θ.

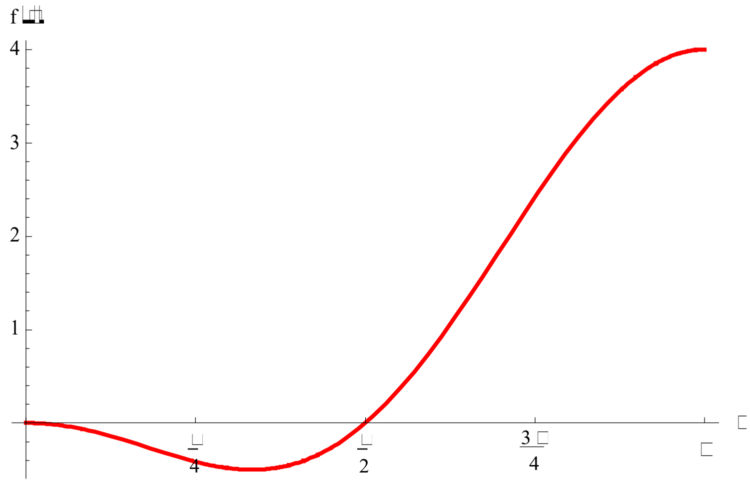


Fig. $f(\theta)<0$ for $0<\theta<\pi/2$, which means that the Bell's inequality is violated. So, the quantum mechanical predictions are not compatible with the Bell's inequality.

((**Note**))

$$P_{++}(\mathbf{a},\mathbf{b}) = \left|\langle +\mathbf{a},+\mathbf{b}|\Phi_B\rangle\right|^2 = \frac{1}{2}\left|\langle +\mathbf{a}|-\mathbf{b}\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$$

$$P_{--}(\mathbf{a},\mathbf{b}) = \left|\langle -\mathbf{a},-\mathbf{b}|\Phi_B\rangle\right|^2 = \frac{1}{2}\left|\langle -\mathbf{a}|+\mathbf{b}\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$$

$$P_{-+}(\mathbf{b},\mathbf{a}) = \left|\langle \text{-}\mathbf{b},+\mathbf{a}|\mathbf{\Phi_B}\rangle\right|^2 = \frac{1}{2}\left|\langle -\mathbf{b}|-\mathbf{a}\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}$$

$$P_{+-}(\mathbf{a},\mathbf{b}) = \left|\langle +\mathbf{a},-\mathbf{b}|\mathbf{\Phi_B}\rangle\right|^2 = \frac{1}{2}\left|\langle +\mathbf{a}|+\mathbf{b}\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}$$

$$P_{+-}(\mathbf{a},\mathbf{a}) = \left|\langle +\mathbf{a},-\mathbf{a}|\mathbf{\Phi_B}\rangle\right|^2 = \frac{1}{2}$$

$$P_{+-}(\mathbf{b},\mathbf{b}) = \left|\langle +\mathbf{b},-\mathbf{b}|\mathbf{\Phi_B}\rangle\right|^2 = \frac{1}{2}$$

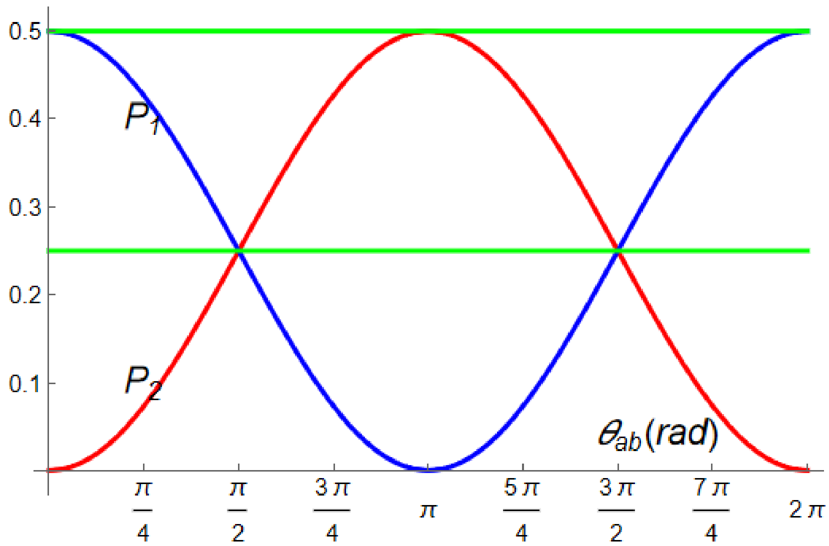


Fig. P_1: $\cos^2\left(\frac{\theta_{ab}}{2}\right)$. P_2: $\sin^2\left(\frac{\theta_{ab}}{2}\right)$. θ_{ab} is the angle between the unit vectors ***a*** and ***b***. The green lines denote the values predicted from the hidden variable theory.

13. Probability for finding two particles in the specified state

We assume the population N_i (i = 1, 2, 3, …., 8). For example, N_1 is the population for the case when the particle 1 (spin 1/2) is in the states {***a***+, ***b***+, ***c***+} and the particle 2 (spin 1/2) is in the state {***a***-, ***b***-, ***c***-}.

Particle-1	Particle-2	Population
{***a***+, ***b***+, ***c***+}	{***a***-, ***b***-, ***c***-}	N_1
{***a***+, ***b***+, ***c***-}	{***a***-, ***b***-, ***c***+}	N_2
{***a***+, ***b***-, ***c***+}	{***a***-, ***b***+, ***c***-}	N_3
{***a***+, ***b***-, ***c***-}	{***a***-, ***b***+, ***c***+}	N_4
{***a***-, ***b***+, ***c***+}	{***a***+, ***b***-, ***c***-}	N_5
{***a***-, ***b***+, ***c***-}	{***a***+, ***b***-, ***c***+}	N_6
{***a***-, ***b***-, ***c***+}	{***a***+, ***b***+, ***c***-}	N_7
{***a***-, ***b***-, ***c***-}	{***a***+, ***b***+, ***c***+}	N_8

Then we get the probability for each case,

$$P(+\boldsymbol{a},+\boldsymbol{b}) = \frac{N_3 + N_4}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{a},-\boldsymbol{b}) = \frac{N_1 + N_2}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{a},+\boldsymbol{c}) = \frac{N_2 + N_4}{\sum_{i=1}^{8} N_i},$$

$$P(+\boldsymbol{a},-\boldsymbol{c}) = \frac{N_1 + N_3}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{a},+\boldsymbol{b}) = \frac{N_7 + N_8}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{a},-\boldsymbol{b}) = \frac{N_5 + N_6}{\sum_{i=1}^{8} N_i},$$

$$P(-\boldsymbol{a},+\boldsymbol{c}) = \frac{N_6 + N_8}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{a},-\boldsymbol{c}) = \frac{N_5 + N_7}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{b},+\boldsymbol{a}) = \frac{N_5 + N_6}{\sum_{i=1}^{8} N_i},$$

$$P(+\boldsymbol{b},-\boldsymbol{a}) = \frac{N_1 + N_2}{\sum_{i=1}^{8} N_i}, \qquad P(\boldsymbol{b}+,\boldsymbol{c}+) = \frac{N_2 + N_6}{\sum_{i=1}^{8} N_i}, \qquad P(\boldsymbol{b}+,\boldsymbol{c}-) = \frac{N_1 + N_5}{\sum_{i=1}^{8} N_i},$$

$$P(-\boldsymbol{b},+\boldsymbol{a}) = \frac{N_7 + N_8}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{b},-\boldsymbol{a}) = \frac{N_3 + N_4}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{b},+\boldsymbol{c}) = \frac{N_4 + N_8}{\sum_{i=1}^{8} N_i},$$

$$P(-\boldsymbol{b},-\boldsymbol{c}) = \frac{N_3 + N_7}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{c},+\boldsymbol{a}) = \frac{N_5 + N_7}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{c},-\boldsymbol{a}) = \frac{N_1 + N_3}{\sum_{i=1}^{8} N_i},$$

$$P(+\boldsymbol{c},+\boldsymbol{b}) = \frac{N_3 + N_7}{\sum_{i=1}^{8} N_i}, \qquad P(+\boldsymbol{c},-\boldsymbol{b}) = \frac{N_1 + N_5}{\sum_{i=1}^{8} N_i}, \quad P(-\boldsymbol{c},+\boldsymbol{a}) = \frac{N_6 + N_8}{\sum_{i=1}^{8} N_i},$$

$$P(-\boldsymbol{c},-\boldsymbol{a}) = \frac{N_2 + N_4}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{c},+\boldsymbol{b}) = \frac{N_4 + N_8}{\sum_{i=1}^{8} N_i}, \qquad P(-\boldsymbol{c},-\boldsymbol{b}) = \frac{N_2 + N_6}{\sum_{i=1}^{8} N_i}.$$

Note that for example, $P(+\boldsymbol{a},+\boldsymbol{b})$ is the probability of finding the particle 1 in the state ***a***+ and the particle 2 in the state ***b***+. Then we have the inequalities such as

$$P(+\boldsymbol{a},+\boldsymbol{b}) - P(+\boldsymbol{a},+\boldsymbol{c}) + P(+\boldsymbol{b},+\boldsymbol{c}) = \frac{N_3 + N_4 - (N_2 + N_4) + N_1 + N_5}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_4 + N_5}{\sum_{i=1}^{8} N_i} > 0$$

$$P(+\boldsymbol{a},+\boldsymbol{c}) - P(+\boldsymbol{a},+\boldsymbol{b}) + P(+\boldsymbol{c},+\boldsymbol{b}) = \frac{N_2 + N_4 - (N_3+N_4) + N_3 + N_7}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_2 + N_7}{\sum_{i=1}^{8} N_i} > 0$$

$$P(+\boldsymbol{a},-\boldsymbol{b}) - P(+\boldsymbol{a},+\boldsymbol{c}) + P(-\boldsymbol{b},+\boldsymbol{c}) = \frac{N_1 + N_2 - (N_2+N_4) + N_4 + N_8}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_1 + N_8}{\sum_{i=1}^{8} N_i} > 0 ,$$

$$P(+\boldsymbol{a},-\boldsymbol{c}) - P(+\boldsymbol{a},+\boldsymbol{b}) + P(-\boldsymbol{c},+\boldsymbol{b}) = \frac{N_1 + N_3 - (N_3+N_4) + N_4 + N_8}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_1 + N_8}{\sum_{i=1}^{8} N_i} > 0$$

$$P(+\boldsymbol{c},-\boldsymbol{a}) - P(+\boldsymbol{a},+\boldsymbol{b}) + P(-\boldsymbol{b},+\boldsymbol{c}) = \frac{N_1 + N_3 - (N_3+N_4) + N_4 + N_8}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_1 + N_8}{\sum_{i=1}^{8} N_i} > 0$$

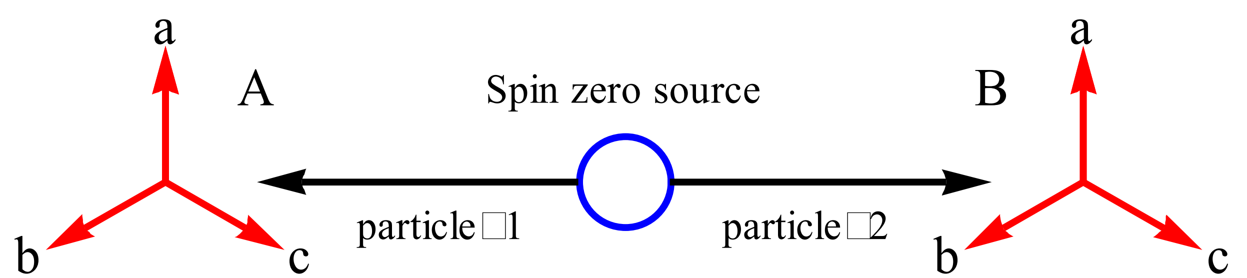
It is noted that the sign of the following equation cannot be determined,

$$P(+\boldsymbol{a},+\boldsymbol{b}) - P(+\boldsymbol{a},+\boldsymbol{c}) + P(-\boldsymbol{b},+\boldsymbol{c}) = \frac{N_3 + N_4 - (N_2+N_4) + N_4 + N_5}{\sum_{i=1}^{8} N_i}$$

$$= \frac{N_3 + 2N_4 + N_5 - N_4}{\sum_{i=1}^{8} N_i}$$

14. Bell's inequality (I): McIntyre, Quantum Mechanics

Bell's argument relies on observers A (Alice) and B (Bob) making measurements along a set of different directions. We consider three directions ***a***, ***b***, and ***c***. Each observer makes measurements of the spin projection along one of these three directions, chosen randomly. Any single observer's result can be only spin up or spin down along that direction, but we record the results independent of the direction of the SG analyzers, so we denote one observer's result simply as + or -, without noting the axis of measurement. The results of the pair of measurements from one correlated pair of particles are denoted + -, for example, which means observer A recorded ***a***+ and observer B recorded ***a***-. There are only four possible system results: ++, +-, -+, or --. Even more simply, we classify the results as either the same, ++, --, or opposite, +-, or -+.



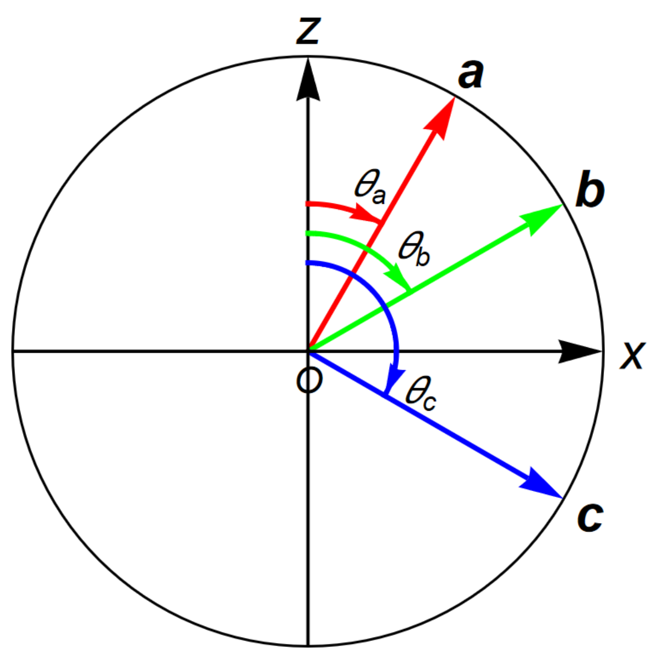


Fig. The direction of position vectors ***a***, ***b***, and ***c*** on the unit circle in the *z*-*x* plane. θ_a, is the angle of vector a from the z axis.

((**Hidden variable theory**))

Note that the angular momentum is conserved. If Alice measures the spin up state using the SG with ***a*** direction, Bob will measure the spin-down state using the SG with ***a*** direction. There are eight types measurements.

Particle 1	Particle 2	Population	
{***a***+, ***b***+, ***c***+}	{***a***-, ***b***-, ***c***-}	N_1	Type-1
{***a***+, ***b***+, ***c***-}	{***a***-, ***b***-, ***c***+}	N_2	Type-2
{***a***+, ***b***-, ***c***+}	{***a***-, ***b***+, ***c***-}	N_3	Type-3
{***a***+, ***b***-, ***c***-}	{***a***-, ***b***+, ***c***+}	N_4	Type-4
{***a***-, ***b***+, ***c***+}	{***a***+, ***b***-, ***c***-}	N_5	Type-5
{***a***-, ***b***+, ***c***-}	{***a***+, ***b***-, ***c***+}	N_6	Type-6
{***a***-, ***b***-, ***c***+}	{***a***+, ***b***+, ***c***-}	N_7	Type-7
{***a***-, ***b***-, ***c***-}	{***a***+, ***b***+, ***c***+}	N_8	Type-8

There are nine different combinations of measurement directions for the pair of observers: ***aa***, ***ab***, ***ac***, ***ba***, ***bb***, ***bc***, ***ca***, ***cb***, ***cc***.

Type-1: N_1

{***a***+, ***b***+, ***c***+}, {***a***-, ***b***-, ***c***-},

The probability of observing the same spin directions: P_{same}

The probability of observing the opposite spin directions: P_{opp}

$P_{\text{opp}}=1$, $P_{\text{same}} = 0$,

{***a***+, ***a***-}, {***a***+, ***b***-}, {***a***+, ***c***-}, {***b***+, ***a***-}, {***b***+, ***b***-}, {***b***+, ***c***-}, {***c***+, ***a***-}
{***c***+, ***b***-}, {***c***+, ***c***-},

Type-8: N_8

{***a***-, ***b***-, ***c***-} {***a***+, ***b***+, ***c***+}

$P_{\text{opp}}=1$, $P_{\text{same}} = 0$,

{***a***-, ***a***+}, {***a***-, ***b***+}, {***a***-, ***c***+}, {***b***-, ***a***+}, {***b***-, ***b***+}, {***b***-, ***c***+}, {***c***-, ***a***+}

{***c***-, ***b***+}, {***c***-, ***c***+},

Type-2: N_2

{***a***+, ***b***+, ***c***-} {***a***-, ***b***-, ***c***+} N_2

P_{opp}=5/9, {***a***+, ***a***-},{***a***+, ***b***-}, {***b***+, ***a***-},{***b***+, ***b***-}, {***c***-, ***c***+}
P_{same} = 4/9 {***a***+, ***c***+},{***b***+, ***c***+}, {***c***-, ***a***-},{***c***-, ***b***-},

Type-3 – 7 (similar to Type-2), N_3, N_4, N_5, N_6, N_7

{***a***+, ***b***-, ***c***+}	{***a***-, ***b***+, ***c***-}	Type-3
{***a***+, ***b***-, ***c***-}	{***a***-, ***b***+, ***c***+}	Type-4
{***a***-, ***b***+, ***c***+}	{***a***+, ***b***-, ***c***-}	Type-5
{***a***-, ***b***+, ***c***-}	{***a***+, ***b***-, ***c***+}	Type-6
{***a***-, ***b***-, ***c***+}	{***a***+, ***b***+, ***c***-}	Type-7

P_{opp}=5/9, P_{same} = 4/9

Then we have

$$P_{same} = \frac{0(N_1+N_8)+\frac{4}{9}(N_2+N_3+N_4+N_5+N_6+N_7)}{\sum_{i=1}^{8} N_i}$$

$$= \frac{\frac{4}{9}(N_1+N_2+N_3+N_4+N_5+N_6+N_7+N_8)-\frac{4}{9}(N_1+N_8)}{\sum_{i=1}^{8} N_i}$$

$$= \frac{4}{9}-\frac{4}{9}\frac{(N_1+N_8)}{\sum_{i=1}^{8} N_i} \le \frac{4}{9}$$

or

$$P_{same} \le \frac{4}{9},$$

and

$$P_{opp} = \frac{1(N_1 + N_8) + \frac{5}{9}(N_2 + N_3 + N_4 + N_5 + N_6 + N_7)}{\sum_{i=1}^{8} N_i}$$

$$= \frac{\frac{4}{9}(N_1 + N_8) + \frac{5}{9}(N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8)}{\sum_{i=1}^{8} N_i}$$

$$= \frac{5}{9} + \frac{4}{9}\frac{(N_1 + N_8)}{\sum_{i=1}^{8} N_i} \geq \frac{5}{9}$$

or

$$P_{opp} \geq \frac{5}{9}.$$

((**Prediction by Quantum mechanics**))

$$P(+\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}\left|\langle +\boldsymbol{b} | -\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\left|\langle -\boldsymbol{a} | +\boldsymbol{b}\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2},$$

$$P(+\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}\left|\langle -\boldsymbol{b} | -\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\left|\langle -\boldsymbol{a} | -\boldsymbol{b}\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2},$$

$$P(-\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}\left|\langle +\boldsymbol{b} | +\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\left|\langle +\boldsymbol{a} | +\boldsymbol{b}\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2},$$

$$P(-\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}\left|\langle -\boldsymbol{b} | +\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\left|\langle +\boldsymbol{a} | -\boldsymbol{b}\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}.$$

Then we get

$$P_{same} = P(+\boldsymbol{a},+\boldsymbol{b}) + P(-\boldsymbol{a},-\boldsymbol{b}) = 2\frac{1}{2}\sin^2\frac{\theta_{ab}}{2} = \sin^2\frac{\theta_{ab}}{2},$$

$$P_{opp} = P(+\boldsymbol{a},-\boldsymbol{b}) + P(+\boldsymbol{a},-\boldsymbol{b}) = 2\frac{1}{2}\cos^2\frac{\theta_{ab}}{2} = \cos^2\frac{\theta_{ab}}{2}.$$

for {***a***, ***b***}.

The angle θ_{ab} between the measurement directions of the observers A and B is $\theta = 0$ in 1/3 of the measurements, $\theta_{bc} = 2\pi/3$ in 1/3 and $\theta_{ca} = 4\pi/3$ in 1/3. So the average probabilities are

$$\begin{aligned} P_{same} &= \frac{1}{3}\sin^2\frac{\theta_{ab}}{2} + \frac{1}{3}\sin^2\frac{\theta_{bc}}{2} + \frac{1}{3}\sin^2\frac{\theta_{ca}}{2} \\ &= \frac{1}{3}\sin^2\frac{0}{2} + \frac{2}{3}\sin^2\frac{\pi}{3} \\ &= \frac{1}{2} > \frac{4}{9} \end{aligned}$$

$$\begin{aligned} P_{opp} &= \frac{1}{3}\cos^2\frac{\theta_{ab}}{2} + \frac{1}{3}\cos^2\frac{\theta_{bc}}{2} + \frac{1}{3}\cos^2\frac{\theta_{ca}}{2} \\ &= \frac{1}{3}\cos^2\frac{0}{2} + \frac{2}{3}\cos^2\frac{\pi}{3} \\ &= \frac{1}{2} < \frac{5}{9} \end{aligned}$$

These predictions of quantum mechanics are inconsistent with the range of possibilities that we derived for local hidden variable theories.

15. Bell's inequality (II)

We have shown the inequality based on the same argument which is derived above

$$P(\boldsymbol{a}+,\boldsymbol{b}-) - P(+\boldsymbol{a},+\boldsymbol{c}) + P(-\boldsymbol{b},+\boldsymbol{c}) \geq 0\,.$$

This can be rewritten as

$$\frac{1}{2}\cos^2\frac{\theta_{ab}}{2} - \frac{1}{2}\sin^2\frac{\theta_{ac}}{2} + \frac{1}{2}\cos^2\frac{\theta_{bc}}{2} \geq 0\,,$$

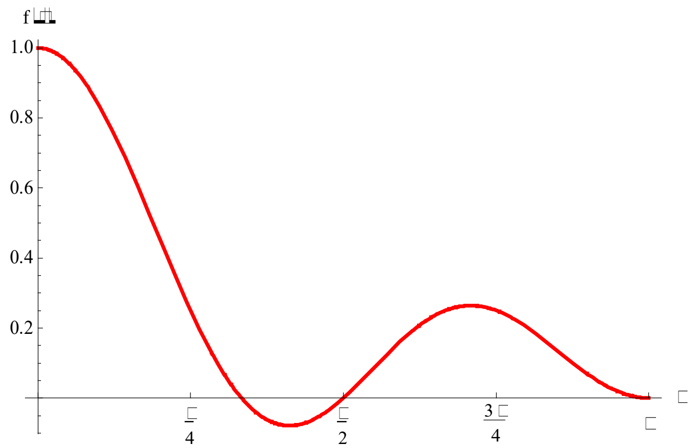
When

$$\theta_{ab} + \theta_{bc} = \theta_{ac}\,, \qquad \theta_{ac} = 2\theta_{ab}\,, \qquad \theta_{ab} = \theta$$

then we have

$$f(\theta)=\frac{1}{2}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{3\theta}{2}+\cos^2\theta\right)\geq 0$$

We make a plot of $f(\theta)$ as a function of θ. It is clearly shown that $f(\theta)$ becomes negative between $\pi/4$ and $\pi/2$.



16. Spin correlation function (I)

Measurements of the spin components along two arbitrary directions ***a*** and ***b*** are performed on two spin 1/2 particles in the singlet $\left|0,0\right\rangle$. The results of each measurement find the parallel spin-up or spin-down along that particular axis. Denoting $P(\boldsymbol{a}\pm,\boldsymbol{b}\pm)$ the probabilities of obtaining ±1 along ***a*** for particle 1 and ±1 along ***b*** for particle 2, the average value for the product of spins is given by

$$\begin{aligned}E(\boldsymbol{a},\boldsymbol{b}) &= P_{same}-P_{oppo}\\ &= P(+\boldsymbol{a},+\boldsymbol{b})+P(-\boldsymbol{a},-\boldsymbol{b})-P(+\boldsymbol{a},-\boldsymbol{b})-P(-\boldsymbol{a},+\boldsymbol{b})\end{aligned}$$

Since

$$P(+\boldsymbol{a},+\boldsymbol{b})=\frac{1}{2}\sin^2\frac{\theta_{ab}}{2}, \qquad P(-\boldsymbol{a},-\boldsymbol{b})=\frac{1}{2}\sin^2\frac{\theta_{ab}}{2},$$

$$P(+\boldsymbol{a},-\boldsymbol{b})=\frac{1}{2}\cos^2\frac{\theta_{ab}}{2}, \qquad P(-\boldsymbol{a},+\boldsymbol{b})=\frac{1}{2}\cos^2\frac{\theta_{ab}}{2},$$

we get

$$\begin{aligned} E(\boldsymbol{a},\boldsymbol{b}) &= P(+\boldsymbol{a},+\boldsymbol{b}) + P(-\boldsymbol{a},-\boldsymbol{b}) - P(+\boldsymbol{a},-\boldsymbol{b}) - P(-\boldsymbol{a},+\boldsymbol{b}) \\ &= \sin^2\frac{\theta_{ab}}{2} - \cos^2\frac{\theta_{ab}}{2} \\ &= -\cos\theta_{ab} \\ &= -(\boldsymbol{a}\cdot\boldsymbol{b}) \end{aligned}$$

if $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors. $E(\boldsymbol{a}, \boldsymbol{b})$ is the spin correlation function and defined by

$$E(\boldsymbol{a},\boldsymbol{b}) = \langle \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})\rangle = -(\boldsymbol{a}\cdot\boldsymbol{b}).$$

17. Derivation of the spin correlation function (II): $E(\boldsymbol{a},\boldsymbol{b}) = \langle \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})\rangle$

Here we show that

$$\langle 0,0|(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})(\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b})|0,0\rangle = -\frac{\hbar^2}{4}\cos\theta_{ab},$$

or

$$\langle 0,0|(\hat{\boldsymbol{\sigma}}_1\cdot\boldsymbol{a})(\hat{\boldsymbol{\sigma}}_2\cdot\boldsymbol{b})|0,0\rangle = -\cos\theta_{ab}$$

where θ_{ab} is the angle between the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Therefore, we have the spin correlation function which is defined by

$$\begin{aligned} E(\boldsymbol{a},\boldsymbol{b}) &= \frac{4}{\hbar^2}\langle 0,0|(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})(\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b})|0,0\rangle \\ &= \langle 0,0|\hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})|0,0\rangle \\ &= \langle \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})\rangle \end{aligned}$$

where

$$\hat{\sigma}_1(\boldsymbol{a}) = \frac{2}{\hbar}\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a}, \qquad \hat{\sigma}_2(\boldsymbol{b}) = \frac{2}{\hbar}\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}.$$

((**Proof**))

$$(\hat{\boldsymbol{S}}\cdot\boldsymbol{n}) = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix},$$

$$\langle +z|\hat{\boldsymbol{S}}\cdot\boldsymbol{n}|+z\rangle = \frac{\hbar}{2}\cos\theta$$

$$\langle -z|\hat{\boldsymbol{S}}\cdot\boldsymbol{n})|-z\rangle = -\frac{\hbar}{2}\cos\theta$$

Therefore

$$\langle +\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle = \frac{\hbar}{2}\cos\theta_{ab}$$

$$\langle -\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle = -\frac{\hbar}{2}\cos\theta_{ab}$$

where θ_{ab} is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (see the APPENDIX for the derivation in detail). Here we write the state $|0,0\rangle$ as

$$|0,0\rangle = \frac{1}{\sqrt{2}}[|+\boldsymbol{a},-\boldsymbol{a}\rangle - |-\boldsymbol{a},+\boldsymbol{a}\rangle].$$

We see that

$$\begin{aligned}(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})|0,0\rangle &= \frac{1}{\sqrt{2}}[(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})|+\boldsymbol{a},-\boldsymbol{a}\rangle - (\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})|-\boldsymbol{a},+\boldsymbol{a}\rangle]\\ &= \frac{\hbar}{2}\frac{1}{\sqrt{2}}[|+\boldsymbol{a},-\boldsymbol{a}\rangle + |-\boldsymbol{a},+\boldsymbol{a}\rangle]\end{aligned}$$

or

$$\langle 0,0|(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a}) = \frac{\hbar}{2}\frac{1}{\sqrt{2}}[\langle +\boldsymbol{a},-\boldsymbol{a}| + \langle -\boldsymbol{a},+\boldsymbol{a}|],$$

where

$$(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})|+\boldsymbol{a},-\boldsymbol{a}\rangle = \frac{\hbar}{2}|+\boldsymbol{a},-\boldsymbol{a}\rangle, \qquad (\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})|-\boldsymbol{a},+\boldsymbol{a}\rangle = -\frac{\hbar}{2}|-\boldsymbol{a},+\boldsymbol{a}\rangle.$$

Then, we have

$$\begin{aligned}
\langle 0,0|(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})(\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b})|0,0\rangle &= \frac{\hbar}{4}[\langle +\boldsymbol{a},-\boldsymbol{a}| + \langle -\boldsymbol{a},+\boldsymbol{a}|][\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{4}[\langle +\boldsymbol{a},-\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle + \langle -\boldsymbol{a},+\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle \\
&\quad -\langle +\boldsymbol{a},-\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle - \langle -\boldsymbol{a},+\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{4}[\langle +\boldsymbol{a},-\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \langle -\boldsymbol{a},+\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{4}[\langle +\boldsymbol{a}|+\boldsymbol{a}\rangle\langle -\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle - \langle -\boldsymbol{a}|-\boldsymbol{a}\rangle\langle +\boldsymbol{a}|\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{4}[\langle -\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle - \langle +\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle] \\
&= \frac{\hbar^2}{8}(-\cos\theta_{ab} - \cos\theta_{ab}) \\
&= -\frac{\hbar^2}{4}\cos\theta_{ab}
\end{aligned}$$

since $\langle +\boldsymbol{a}|+\boldsymbol{a}\rangle = 1$, and $\langle -\boldsymbol{a}|-\boldsymbol{a}\rangle = 1$. Thus, we have the spin correlation function,

$$\begin{aligned}
E(\boldsymbol{a},\boldsymbol{b}) &= \langle \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})\rangle \\
&= \frac{4}{\hbar^2}\langle 0,0|(\hat{\boldsymbol{S}}_1\cdot\boldsymbol{a})(\hat{\boldsymbol{S}}_2\cdot\boldsymbol{b})|0,0\rangle \\
&= -\cos\theta_{ab} \\
&= -\boldsymbol{a}\cdot\boldsymbol{b}
\end{aligned}$$

where

$$\hat{\sigma}_1(\boldsymbol{a}) = \hat{\boldsymbol{\sigma}}_1\cdot\boldsymbol{a}, \qquad \hat{\sigma}_2(\boldsymbol{b}) = \hat{\boldsymbol{\sigma}}_2\cdot\boldsymbol{b}$$

((**Mathematica-1**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
ψB = 1/√2 (KroneckerProduct[ψ1, ψ2] - KroneckerProduct[ψ2, ψ1]) //
   FullSimplify;
ψB // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
A1 = ħ/2 (ax σx + ay σy + az σz);
B1 = ħ/2 (bx σx + by σy + bz σz);
K1 = KroneckerProduct[A1, B1] // FullSimplify;
F1 = Transpose[ψB*].K1.ψB // FullSimplify;
F1 // MatrixForm
```

$$\left(-\frac{1}{4} (ax\ bx + ay\ by + az\ bz)\ \hbar^2 \right)$$

((**Mathematica-2**))

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1z = (1
       0);
ψ2z = (0
       1);
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
nx = Sin[θ] Cos[ϕ];
ny = Sin[θ] Sin[ϕ];
nz = Cos[θ];
```

```
χ1 =
  1/Sqrt[2] (KroneckerProduct[ψ1z, ψ2z] -
      KroneckerProduct[ψ2z, ψ1z]) // FullSimplify;
χ1 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
σn = nx σx + ny σy + nz σz // FullSimplify;
K1 = KroneckerProduct[σz, σn] // FullSimplify;
K1 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\theta] & e^{-i\phi}\,\text{Sin}[\theta] & 0 & 0 \\ e^{i\phi}\,\text{Sin}[\theta] & -\text{Cos}[\theta] & 0 & 0 \\ 0 & 0 & -\text{Cos}[\theta] & -e^{-i\phi}\,\text{Sin}[\theta] \\ 0 & 0 & -e^{i\phi}\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

18. Proof of Bell's theorem in terms of the hidden variables

Suppose that the complete state of the system (such as the state of electron-positron system) is characterized by the hidden variables $\boldsymbol{v}$. The spin-up and down states are described by

$$\sigma_1(\boldsymbol{v},\boldsymbol{a}) = \pm 1, \qquad \sigma_2(\boldsymbol{v},\boldsymbol{b}) = \pm 1,$$

where $\boldsymbol{v}$ is n-vector of hidden variables. Since the angular momentum is conserved,

$$-\sigma_1(\boldsymbol{v},\boldsymbol{b}) = \sigma_2(\boldsymbol{v},\boldsymbol{b}).$$

The spin correlation is defined by

$$E(\boldsymbol{a},\boldsymbol{b}) = \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle = \int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_2(\boldsymbol{v},\boldsymbol{b}),$$

where $\rho(\boldsymbol{v})$ is the probability density. Using the angular momentum conservation, we have

$$\langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle = -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b}).$$

Then we get

$$\begin{aligned}\langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle - \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{c})\rangle &= -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})[\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b}) - \sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{c})] \\ &= -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})[\sigma_1(\boldsymbol{v},\boldsymbol{b}) - \sigma_1(\boldsymbol{v},\boldsymbol{c})]\end{aligned}$$

Since $\sigma_1^{\,2}(\boldsymbol{v},\boldsymbol{b}) = 1$,

$$\begin{aligned}\langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle - \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{c})\rangle &= -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1^{\,2}(\boldsymbol{v},\boldsymbol{b})[\sigma_1(\boldsymbol{v},\boldsymbol{b}) - \sigma_1(\boldsymbol{v},\boldsymbol{c})] \\ &= -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b})[\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{b}) - \sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})] \\ &= -\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b})[1 - \sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})]\end{aligned}$$

Then

$$
\begin{aligned}
\left|\langle\sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle-\langle\sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{c})\rangle\right| &= \left|\int d^n\boldsymbol{v}\rho(\boldsymbol{v})\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b})[1-\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})]\right| \\
&\le \int d^n\boldsymbol{v}\rho(\boldsymbol{v})\left|\sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_1(\boldsymbol{v},\boldsymbol{b})\right|[1-\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})] \\
&\le \int d^n\boldsymbol{v}\rho(\boldsymbol{v})[1-\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})] \\
&= \int d^n\boldsymbol{v}\rho(\boldsymbol{v})[1+\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_2(\boldsymbol{v},\boldsymbol{c})] \\
&= 1+\langle\sigma_1(\boldsymbol{b})\sigma_2(\boldsymbol{c})\rangle
\end{aligned}
$$

where

$$\rho(\boldsymbol{v})[1-\sigma_1(\boldsymbol{v},\boldsymbol{b})\sigma_1(\boldsymbol{v},\boldsymbol{c})]\ge 0$$

$$-\sigma_1(\boldsymbol{v},\boldsymbol{c})=\sigma_2(\boldsymbol{v},\boldsymbol{c})\,,$$

Thus we have

$$\left|E(\boldsymbol{a},\boldsymbol{b})-E(\boldsymbol{a},\boldsymbol{c})\right|\le 1+E(\boldsymbol{b},\boldsymbol{c}),\qquad \textbf{(Bell's theorem)}$$

since

$$\sigma_2(\boldsymbol{v},\boldsymbol{c})=-\sigma_1(\boldsymbol{v},\boldsymbol{c})\,,$$

where the spin correlation function,

$$E(\boldsymbol{a},\boldsymbol{b})=\langle\sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle\,,\qquad E(\boldsymbol{a},\boldsymbol{c})=\langle\sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{c})\rangle\,,$$

and

$$E(\boldsymbol{b},\boldsymbol{c})=\langle\sigma_1(\boldsymbol{b})\sigma_2(\boldsymbol{c})\rangle\,.$$

The Bell’s inequality holds for any local hidden variable theory. Here we put

$$E(\boldsymbol{a},\boldsymbol{b})=-\boldsymbol{a}\cdot\boldsymbol{b}\,.$$

Then we have

$$LHS=\left|E(\boldsymbol{a},\boldsymbol{b})-E(\boldsymbol{a},\boldsymbol{c})\right|=\left|\boldsymbol{a}\cdot(\boldsymbol{b}-\boldsymbol{c})\right|,$$

$$RHS = 1 - \boldsymbol{b}\cdot\boldsymbol{c}\,.$$

We now choose that

$$\boldsymbol{a}\cdot\boldsymbol{b} = 0\,, \qquad \boldsymbol{c} = \boldsymbol{a}\sin\psi + \boldsymbol{b}\cos\psi\,.$$

Thus

$$LHS = \left|-\boldsymbol{a}\cdot\boldsymbol{c}\right| = \left|\sin\psi\right|, \qquad RHS = 1 - \cos\psi\,.$$

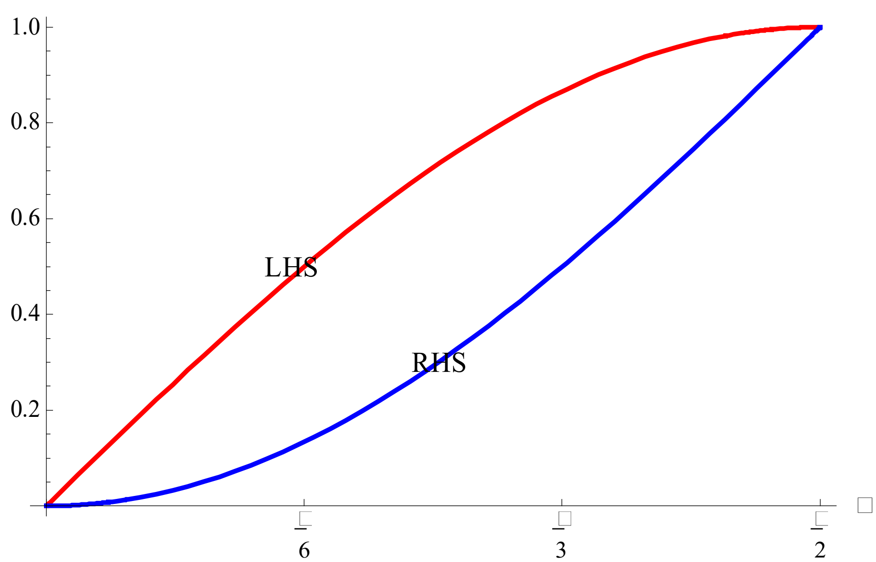


Fig. LHS and RHS as a function of the angle ψ between $\boldsymbol{b}$ and $\boldsymbol{c}$.

LHS>RHS except for $\psi = 0$ and $\pi/2$. This means that the quantum mechanics is inconsistent with hidden variable theory (locality).

((**Note**)) **The proof of the Sell's inequality using the model used in Section 14.**

Here we show the Bell's inequality

$$\left|E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c})\right| \le 1 + E(\mathbf{b},\mathbf{c})$$

using the above simple model, where two observers A (Alice) and B (Bob) make measurements along a set of three different directions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Particle 1 Particle 2 Population

{***a***+, ***b***+, ***c***+}	{***a***-, ***b***-, ***c***-}	N_1	Type-1
{***a***+, ***b***+, ***c***-}	{***a***-, ***b***-, ***c***+}	N_2	Type-2
{***a***+, ***b***-, ***c***+}	{***a***-, ***b***+, ***c***-}	N_3	Type-3
{***a***+, ***b***-, ***c***-}	{***a***-, ***b***+, ***c***+}	N_4	Type-4
{***a***-, ***b***+, ***c***+}	{***a***+, ***b***-, ***c***-}	N_5	Type-5
{***a***-, ***b***+, ***c***-}	{***a***+, ***b***-, ***c***+}	N_6	Type-6
{***a***-, ***b***-, ***c***+}	{***a***+, ***b***+, ***c***-}	N_7	Type-7
{***a***-, ***b***-, ***c***-}	{***a***+, ***b***+, ***c***+}	N_8	Type-8

We note that the total number N is

$$N = N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8$$

We have

$$\begin{aligned} E(\mathbf{a},\mathbf{b}) &= P(+\mathbf{a},+\mathbf{b}) - P(+\mathbf{a},-\mathbf{b}) - P(-\mathbf{a},+\mathbf{b}) + P(-\mathbf{a},-\mathbf{b}) \\ &= \frac{1}{N}[(N_3+N_4)-(N_1+N_2)-(N_7+N_8)+(N_5+N_6)] \end{aligned},$$

$$\begin{aligned} E(\mathbf{a},\mathbf{c}) &= P(+\mathbf{a},+\mathbf{c}) - P(+\mathbf{a},-\mathbf{c}) - P(-\mathbf{a},+\mathbf{c}) + P(-\mathbf{a},-\mathbf{c}) \\ &= \frac{1}{N}[(N_2+N_4)-(N_1+N_3)-(N_6+N_8)+(N_5+N_7)] \end{aligned},$$

$$\begin{aligned} E(\mathbf{b},\mathbf{c}) &= P(+\mathbf{b},+\mathbf{c}) - P(+\mathbf{b},-\mathbf{c}) - P(-\mathbf{b},+\mathbf{c}) + P(-\mathbf{b},-\mathbf{c}) \\ &= \frac{1}{N}[(N_2+N_6)-(N_5+N_1)-(N_4+N_8)+(N_3+N_7)] \end{aligned},$$

$$\begin{aligned} E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{c}) &= \frac{1}{N}[(N_3+N_4)-(N_1+N_2)-(N_7+N_8)+(N_5+N_6)] \\ &\quad -\frac{1}{N}[(N_2+N_4)-(N_1+N_3)-(N_6+N_8)+(N_5+N_7)] \\ &= \frac{1}{N}(-2N_2+2N_3+2N_6-2N_7) \end{aligned}$$

Since

$$1 + E(\mathbf{b},\mathbf{c}) = \frac{1}{N}[2N_2+2N_3+2N_6+2N_7)$$

we have

$$1+E(\mathbf{b},\mathbf{c})-[E(\mathbf{a},\mathbf{b})-E(\mathbf{a},\mathbf{c})]=\frac{1}{N}[2N_2+2N_3+2N_6+2N_7)$$
$$-\frac{1}{N}(-2N_2+2N_3+2N_6-2N_7)$$
$$=\frac{4}{N}(N_2+N_7)>0$$

and

$$1+E(\mathbf{b},\mathbf{c})+[E(\mathbf{a},\mathbf{b})-E(\mathbf{a},\mathbf{c})]=\frac{1}{N}[2N_2+2N_3+2N_6+2N_7)$$
$$+\frac{1}{N}(-2N_2+2N_3+2N_6-2N_7)$$
$$=\frac{4}{N}(N_3+N_6)>0$$

So that, we have the inequality

$$\left|E(\mathbf{a},\mathbf{b})-E(\mathbf{a},\mathbf{c})\right|\le 1+E(\mathbf{b},\mathbf{c}) \qquad \text{(Bell's inequality)}$$

Q.E.D.

19. RegionPlot of the Bell's inequality

Bell's inequality (by Bell)

$$\left|E(\mathbf{a},\mathbf{b})-E(\mathbf{a},\mathbf{c})\right|\le 1+E(\mathbf{b},\mathbf{c})$$

where

$$E(\mathbf{a},\mathbf{b})=-\mathbf{a}\cdot\mathbf{b}=-\cos(\theta_{ab})=-\cos(\theta_a-\theta_b)$$

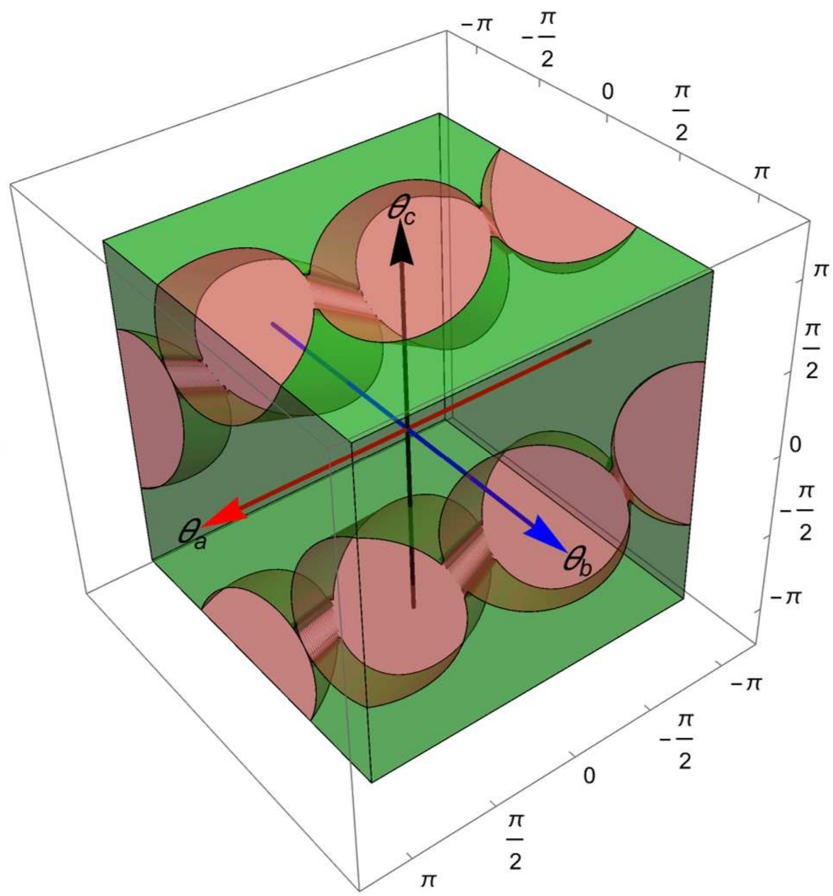
$$E(\mathbf{a},\mathbf{c})=-\mathbf{a}\cdot\mathbf{c}=-\cos(\theta_{ac})=-\cos(\theta_a-\theta_c)$$

$$E(\mathbf{b},\mathbf{c})=-\mathbf{b}\cdot\mathbf{c}=-\cos\theta_{bc}=-\cos(\theta_b-\theta_c)$$

We find the region of $(\theta_a, \theta_b, \theta_c)$ satisfying the inequality

$$f(\theta_a, \theta_b, \theta_c) = \left|\cos(\theta_a - \theta_b) - \cos(\theta_a - \theta_c)\right| - \cos(\theta_b - \theta_c) \leq 1$$

using the RegionPlot3D of as a function of θ_a, θ_b and θ_c, where $0 \leq \theta_a \leq \pi$, $0 \leq \theta_b \leq \pi$, and $0 \leq \theta_c \leq \pi$.



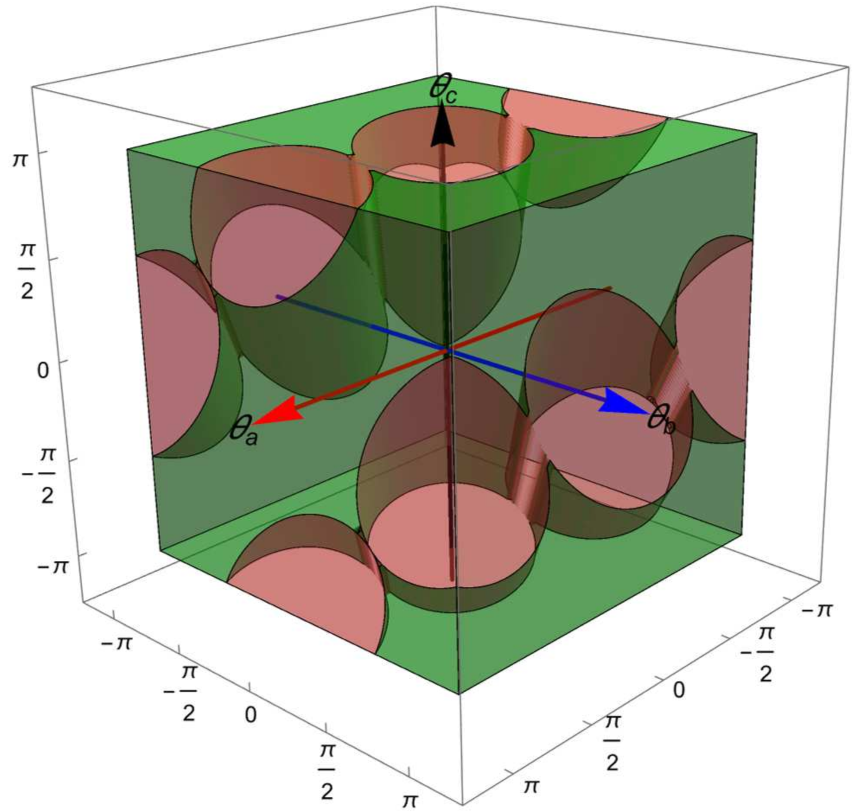


Fig. RegionPlot3D in the 3D space denoted by the angles $(\theta_a, \theta_b, \theta_c)$. The regions denoted by green for $f < 1$ (satisfying the Bell's inequality). The regions denoted by pink for $f > 1$. (violation of the Bell inequality)

For simplicity we assume that $\theta_a = 0$. In this case, we need to consider the RegionPlot in the 2D space denoted by the angles (θ_b, θ_c)

$$f(\theta_a = 0, \theta_b, \theta_c) = \left|\cos(\theta_b) - \cos(\theta_c)\right| - \cos(\theta_b - \theta_c) \leq 1$$

where $-\pi \leq \theta_b \leq \pi$ and $-\pi \leq \theta_b \leq \pi$.

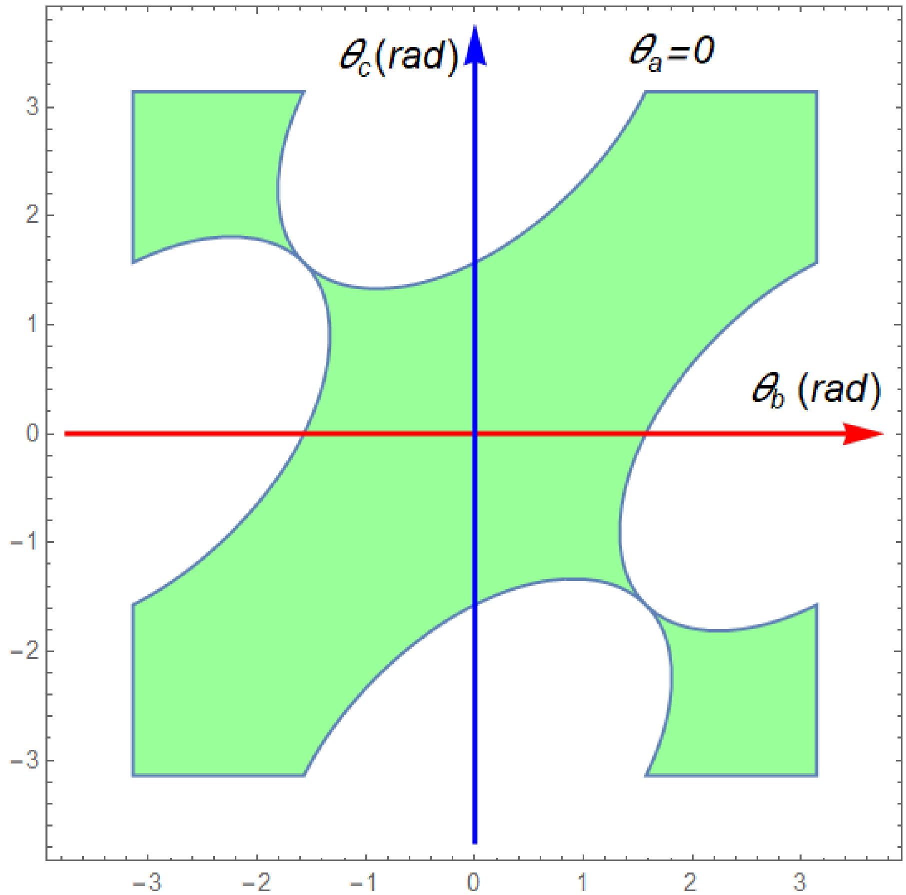


Fig. RegionPlot of $f(\theta_a = 0, \theta_b, \theta_c) = \left|\cos(\theta_b) - \cos(\theta_c)\right| - \cos(\theta_b - \theta_c) \le 1$. The region (green) for the inequality $f \le 1$ (satisfying the Bell's inequality)

20. Bell's states for the spin 1/2 system

The Bell's states are defined by

$$\left|\Phi^{+}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|+z\right\rangle_2 + \left|-z\right\rangle_1\left|-z\right\rangle_2)$$

$$\left|\Phi^{-}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|+z\right\rangle_2 - \left|-z\right\rangle_1\left|-z\right\rangle_2)$$

$$\left|\Psi^{+}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|-z\right\rangle_2 + \left|-z\right\rangle_1\left|+z\right\rangle_2)$$

$$\left|\Psi^{-}\right\rangle_{12} = \frac{1}{\sqrt{2}}(\left|+z\right\rangle_1\left|-z\right\rangle_2 - \left|-z\right\rangle_1\left|+z\right\rangle_2)$$

Using the total spin angular momentum along the z axis, which is given by

$$\hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z} = \hat{S}_{1z} \otimes \hat{1}_2 + \hat{1}_1 \otimes \hat{S}_{2z} .$$

we have

$$\begin{aligned}
\hat{S}_z \left|\Phi^+\right\rangle_{12} &= (\hat{S}_{1z} + \hat{S}_{2z}) \frac{1}{\sqrt{2}} (\left|+z\right\rangle_1 \left|+z\right\rangle_2 + \left|-z\right\rangle_1 \left|-z\right\rangle_2) \\
&= \frac{\hbar}{\sqrt{2}} (\left|+z\right\rangle_1 \left|+z\right\rangle_2 - \left|-z\right\rangle_1 \left|-z\right\rangle_2) \\
&= \left|\Phi^-\right\rangle_{12}
\end{aligned}$$

$$\begin{aligned}
\hat{S}_z \left|\Phi^-\right\rangle_{12} &= (\hat{S}_{1z} + \hat{S}_{2z}) \frac{1}{\sqrt{2}} (\left|+z\right\rangle_1 \left|+z\right\rangle_2 - \left|-z\right\rangle_1 \left|-z\right\rangle_2) \\
&= \frac{\hbar}{\sqrt{2}} (\left|+z\right\rangle_1 \left|+z\right\rangle_2 + \left|-z\right\rangle_1 \left|-z\right\rangle_2) \qquad , \\
&= \left|\Phi^+\right\rangle_{12}
\end{aligned}$$

$$\hat{S}_z \left|\Psi^+\right\rangle_{12} = (\hat{S}_{1z} + \hat{S}_{2z}) \frac{1}{\sqrt{2}} (\left|+z\right\rangle_1 \left|-z\right\rangle_2 + \left|-z\right\rangle_1 \left|+z\right\rangle_2) = 0 ,$$

$$\hat{S}_z \left|\Psi^-\right\rangle_{12} = (\hat{S}_{1z} + \hat{S}_{2z}) \frac{1}{\sqrt{2}} (\left|+z\right\rangle_1 \left|-z\right\rangle_2 - \left|-z\right\rangle_1 \left|+z\right\rangle_2) = 0 .$$

So that $\left|\Psi^+\right\rangle_{12}$ and $\left|\Psi^-\right\rangle_{12}$ are the eigenkets of $\hat{S}_z$ with the eigenvalue 0. On the other hands, $\left|\Phi^+\right\rangle_{12}$ and $\left|\Phi^-\right\rangle_{12}$ are not the eigenkets of $\hat{S}_z$.

((**Mathematica**))

```
Clear["Global`*"];
exp_* :=
 exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
I2 = IdentityMatrix[2];

ψB1 =
  1/√2 (KroneckerProduct[ψ1, ψ1] +
     KroneckerProduct[ψ2, ψ2]) // FullSimplify;
ψB1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψB2 =
  1/√2 (KroneckerProduct[ψ1, ψ1] -
     KroneckerProduct[ψ2, ψ2]) // FullSimplify;
ψB2 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
ψB3 =
  1/√2 (KroneckerProduct[ψ1, ψ2] +
     KroneckerProduct[ψ2, ψ1]) // FullSimplify;
ψB3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
ψB4 =
  1/√2 (KroneckerProduct[ψ1, ψ2] -
      KroneckerProduct[ψ2, ψ1]) // FullSimplify;
ψB4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
Sz = KroneckerProduct[σz, I2] +
  KroneckerProduct[I2, σz]

{{2, 0, 0, 0}, {0, 0, 0, 0},
 {0, 0, 0, 0}, {0, 0, 0, -2}}
```

```
Transpose[ψB1*].Sz.ψB1
```

```
{{0}}
```

```
Transpose[ψB2*].Sz.ψB2
```

```
{{0}}
```

```
Transpose[ψB3*].Sz.ψB3
```

```
{{0}}
```

```
Transpose[ψB4*].Sz.ψB4
```

```
{{0}}
```

21. Mathematics for spin 1/2 system

(1) Inner product between eigenkets of spin 1/2 system

The eigenkets $|+\boldsymbol{n}\rangle$ and $|-\boldsymbol{n}\rangle$ for the spin 1/2 system are obtained as

$$|+\boldsymbol{n}\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{pmatrix}, \quad \text{and} \quad |-\boldsymbol{n}\rangle = \begin{pmatrix} \sin(\frac{\theta}{2}) \\ -e^{i\phi}\cos(\frac{\theta}{2}) \end{pmatrix}.$$

Here we choose a phase factor for $|-\boldsymbol{n}\rangle$ which is a little different from the conventional notation

$$|-\boldsymbol{n}\rangle = \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ e^{i\phi}\cos(\frac{\theta}{2}) \end{pmatrix}.$$

We now calculate the inner product

$$|+\boldsymbol{a}\rangle = \begin{pmatrix} \cos(\frac{\theta_a}{2}) \\ e^{i\phi_a}\sin(\frac{\theta_a}{2}) \end{pmatrix}, \qquad |+\boldsymbol{b}\rangle = \begin{pmatrix} \cos(\frac{\theta_b}{2}) \\ e^{i\phi_b}\sin(\frac{\theta_b}{2}) \end{pmatrix},$$

$$|-\boldsymbol{a}\rangle = \begin{pmatrix} \sin(\frac{\theta_a}{2}) \\ -e^{i\phi_a}\cos(\frac{\theta_a}{2}) \end{pmatrix}, \qquad |-\boldsymbol{b}\rangle = \begin{pmatrix} \sin(\frac{\theta_b}{2}) \\ -e^{i\phi_b}\cos(\frac{\theta_b}{2}) \end{pmatrix}.$$

Then we have the scalar product

$$\begin{aligned}\langle+\boldsymbol{b}|+\boldsymbol{a}\rangle &= \begin{pmatrix} \cos(\frac{\theta_b}{2}) & e^{-i\phi_b}\sin(\frac{\theta_b}{2}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\theta_a}{2}) \\ e^{i\phi_a}\sin(\frac{\theta_a}{2}) \end{pmatrix} \\ &= \cos(\frac{\theta_a}{2})\cos(\frac{\theta_b}{2}) + \sin(\frac{\theta_a}{2})\sin(\frac{\theta_b}{2}) \qquad , \\ &= \cos(\frac{\theta_a-\theta_b}{2})\end{aligned}$$

$$\begin{aligned}\langle-\boldsymbol{b}|+\boldsymbol{a}\rangle &= \begin{pmatrix} \sin(\frac{\theta_b}{2}) & -e^{-i\phi_b}\cos(\frac{\theta_b}{2}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\theta_a}{2}) \\ e^{i\phi_a}\sin(\frac{\theta_a}{2}) \end{pmatrix} \\ &= \cos(\frac{\theta_a}{2})\sin(\frac{\theta_b}{2}) - \sin(\frac{\theta_a}{2})\cos(\frac{\theta_b}{2}) \\ &= -\sin(\frac{\theta_a-\theta_b}{2})\end{aligned}$$

$$\begin{aligned}\langle+\boldsymbol{b}|-\boldsymbol{a}\rangle &= \begin{pmatrix} \cos(\frac{\theta_b}{2}) & e^{-i\phi_b}\sin(\frac{\theta_b}{2}) \end{pmatrix}\begin{pmatrix} \sin(\frac{\theta_a}{2}) \\ -e^{i\phi_a}\cos(\frac{\theta_a}{2}) \end{pmatrix} \\ &= \sin(\frac{\theta_a}{2})\cos(\frac{\theta_b}{2}) - \cos(\frac{\theta_a}{2})\sin(\frac{\theta_b}{2}) \qquad , \\ &= \sin(\frac{\theta_a-\theta_b}{2})\end{aligned}$$

$$\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle = \begin{pmatrix} \sin(\frac{\theta_b}{2}) & -e^{-i\phi_b}\cos(\frac{\theta_b}{2}) \end{pmatrix} \begin{pmatrix} \sin(\frac{\theta_a}{2}) \\ -e^{i\phi_a}\cos(\frac{\theta_a}{2}) \end{pmatrix}$$

$$= \sin s(\frac{\theta_a}{2})\sin(\frac{\theta_b}{2}) + \cos(\frac{\theta_a}{2})\cos(\frac{\theta_b}{2})$$

$$= \cos(\frac{\theta_a - \theta_b}{2})$$

(2) Matrix elements of spin operator

$$\langle +\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle = \frac{\hbar}{2}\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$= \frac{\hbar}{2}\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}(|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}| + |-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|)|+\boldsymbol{a}\rangle$$

$$= \frac{\hbar}{2}[\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle + \langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$= \frac{\hbar}{2}[\langle +\boldsymbol{a}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle - \langle +\boldsymbol{a}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle]$$

$$= \frac{\hbar}{2}[|\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle|^2 - |\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle|^2]$$

$$= \frac{\hbar}{2}[\cos^2(\frac{\theta_a - \theta_b}{2}) - \sin^2(\frac{\theta_a - \theta_b}{2})]$$

$$= \frac{\hbar}{2}\cos(\theta_a - \theta_b)$$

where $\hat{\boldsymbol{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\sigma}}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors. $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{b}\rangle = |+\boldsymbol{b}\rangle$, and $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{b}\rangle = -|-\boldsymbol{b}\rangle$. Similarly, we have

$$
\begin{aligned}
\langle -\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle &= \frac{\hbar}{2}\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}(|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}| + |-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|)|+\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}[\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle + \langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}[\langle -\boldsymbol{a}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle - \langle -\boldsymbol{a}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{2}[\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle^{*}\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle - \langle -\boldsymbol{b}|-\boldsymbol{a}\rangle^{*}\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{2}[\sin(\frac{\theta_a-\theta_b}{2})\cos(\frac{\theta_a-\theta_b}{2}) + \cos(\frac{\theta_a-\theta_b}{2})\sin(\frac{\theta_a-\theta_b}{2})] \\
&= \frac{\hbar}{2}\sin(\theta_a-\theta_b)
\end{aligned}
$$

$$
\begin{aligned}
\langle +\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle &= \frac{\hbar}{2}\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}(|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}| + |-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|)|-\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}[\langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle + \langle +\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle \\
&= \frac{\hbar}{2}[\langle +\boldsymbol{a}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle - \langle +\boldsymbol{a}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{2}[\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle^{*}\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle - \langle -\boldsymbol{b}|+\boldsymbol{a}\rangle^{*}\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle] \\
&= \frac{\hbar}{2}[\sin(\frac{\theta_a-\theta_b}{2})\cos(\frac{\theta_a-\theta_b}{2}) + \cos(\frac{\theta_a-\theta_b}{2})\sin(\frac{\theta_a-\theta_b}{2})] \\
&= \frac{\hbar}{2}\sin(\theta_a-\theta_b)
\end{aligned}
$$

$$\langle -\boldsymbol{a}|\hat{\boldsymbol{S}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle = \frac{\hbar}{2}\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{a}\rangle$$
$$= \frac{\hbar}{2}\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}(|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}| + |-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|)|-\boldsymbol{a}\rangle$$
$$= \frac{\hbar}{2}[\langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle + \langle -\boldsymbol{a}|\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle$$
$$= \frac{\hbar}{2}[\langle -\boldsymbol{a}|+\boldsymbol{b}\rangle\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle - \langle -\boldsymbol{a}|-\boldsymbol{b}\rangle\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle]$$
$$= \frac{\hbar}{2}[|\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle|^2 - |\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle|^2]$$
$$= \frac{\hbar}{2}[\sin^2(\frac{\theta_a-\theta_b}{2}) - \cos^2(\frac{\theta_a-\theta_b}{2})]$$
$$= -\frac{\hbar}{2}\cos(\theta_a-\theta_b)$$

(3) Probability

(i) $P(+\boldsymbol{a},+\boldsymbol{b}) = |\langle +\boldsymbol{a},+\boldsymbol{b}\,|\,0,0\rangle|^2$

$$\langle +\boldsymbol{a},+\boldsymbol{b}\,|\,0,0\rangle$$
$$= \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},+\boldsymbol{b}\,|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},+\boldsymbol{b}\,|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle +\boldsymbol{b}\,|-\boldsymbol{a}\rangle$$

Then we have the probability

$$P(+\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}|\langle +\boldsymbol{b}|-\boldsymbol{a}\rangle|^2 = \frac{1}{2}\sin^2\left(\frac{\theta_a-\theta_b}{2}\right).$$

(ii) $P(+\boldsymbol{a},-\boldsymbol{b}) = |\langle +\boldsymbol{a},-\boldsymbol{b}\,|\,0,0\rangle|^2$

$$\langle +\boldsymbol{a},-\boldsymbol{b}\,|\,0,0\rangle = \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},-\boldsymbol{b}\,|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle +\boldsymbol{a},-\boldsymbol{b}\,|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= \frac{1}{\sqrt{2}}\langle -\boldsymbol{b}\,|-\boldsymbol{a}\rangle$$

Then we have the probability

$$P(+\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}\left|\langle -\boldsymbol{b}|-\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\cos^2\left(\frac{\theta_a-\theta_b}{2}\right).$$

(iii) $P(-\boldsymbol{a},+\boldsymbol{b}) = \left|\langle -\boldsymbol{a},+\boldsymbol{b}|0,0\rangle\right|^2$

$$\langle -\boldsymbol{a},+\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},+\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},+\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= -\frac{1}{\sqrt{2}}\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$P(-\boldsymbol{a},+\boldsymbol{b}) = \frac{1}{2}\left|\langle +\boldsymbol{b}|+\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\cos^2\left(\frac{\theta_a-\theta_b}{2}\right).$$

(iv) $P(-\boldsymbol{a},-\boldsymbol{b}) = \left|\langle -\boldsymbol{a},-\boldsymbol{b}|0,0\rangle\right|^2$

$$\langle -\boldsymbol{a},-\boldsymbol{b}|0,0\rangle = \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},-\boldsymbol{b}|+\boldsymbol{a},-\boldsymbol{a}\rangle - \frac{1}{\sqrt{2}}\langle -\boldsymbol{a},-\boldsymbol{b}|-\boldsymbol{a},+\boldsymbol{a}\rangle]$$
$$= -\frac{1}{\sqrt{2}}\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle$$

$$P(-\boldsymbol{a},-\boldsymbol{b}) = \frac{1}{2}\left|\langle -\boldsymbol{b}|+\boldsymbol{a}\rangle\right|^2 = \frac{1}{2}\sin^2\left(\frac{\theta_a-\theta_b}{2}\right).$$

22. CHSH inequality

The **CHSH inequality** can be used in the proof of Bell's theorem, which states that certain consequences of entanglement in quantum mechanics cannot be reproduced by local hidden theories. Experimental verification of violation of the inequalities is seen as experimental confirmation that nature cannot be described by local hidden variables theories. CHSH stands for John **C**lauser, Michael **H**orne, Abner **S**himony, and Richard **H**olt, who described it in a much-cited paper published in 1969. They derived the CHSH inequality, which, as with John Bell's original inequality (Bell, 1964), is a constraint on the statistics of "coincidences" in a Bell test experiment which is necessarily true if there exist underlying local hidden variables (local realism). This constraint can, on the other hand, be infringed by quantum mechanics.
http://en.wikipedia.org/wiki/CHSH_inequality.

Clauser et al.'s 1969 derivation was oriented towards the use of "two-channel" detectors, and indeed it is for these that it is generally used, but under their method the only possible outcomes

were +1 and −1. In order to adapt to real situations, which at the time meant the use of polarized light and single-channel polarizers, they had to interpret '−' as meaning "non-detection in the '+' channel", i.e. either '−' or nothing. They did not in the original article discuss how the two-channel inequality could be applied in real experiments with real imperfect detectors, though it was later proved (Bell, 1971) that the inequality itself was equally valid. The occurrence of zero outcomes, though, means it is no longer so obvious how the values of E are to be estimated from the experimental data. The mathematical formalism of quantum mechanics predicts a maximum value for S of (Tsirelson's bound),[4] which is greater than 2, and CHSH violations are therefore predicted by the theory of quantum mechanics.
https://en.wikipedia.org/wiki/CHSH_inequality

The usual form of the CHSH inequality is given by

$$\begin{aligned} S &= E(\boldsymbol{a},\boldsymbol{b}) - E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}) + E(\boldsymbol{a}',\boldsymbol{b}') \\ &= \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle - \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b}')\rangle + \langle \sigma_1(\boldsymbol{a}')\sigma_2(\boldsymbol{b})\rangle + \langle \sigma_1(\boldsymbol{a}')\sigma_2(\boldsymbol{b}')\rangle \\ &= \langle \sigma_1(\boldsymbol{a})[\sigma_2(\boldsymbol{b}) - \sigma_2(\boldsymbol{b}')]\rangle + \langle \sigma_1(\boldsymbol{a}')[\sigma_2(\boldsymbol{b}) + \sigma_2(\boldsymbol{b}')]\rangle \end{aligned}$$

where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are detector settings on side A, $\boldsymbol{b}$ and $\boldsymbol{b}'$ on side B, the four combinations being tested in separate sub-experiments. The terms $E(\boldsymbol{a},\boldsymbol{b})$ etc. are the quantum correlations of the particle pairs, where the quantum correlation is defined to be the expectation value of the product of the "outcomes" of the experiment, i.e. the statistical average of $A(a)\cdot B(b)$, where A and B are the separate outcomes, using the coding +1 for the '+' channel and −1 for the '−' channel.

$$\begin{aligned} E(\boldsymbol{a},\boldsymbol{b}) &= \langle \sigma_1(\boldsymbol{a})\sigma_2(\boldsymbol{b})\rangle \\ &= \int d^n \boldsymbol{v} \rho(\boldsymbol{v}) \sigma_1(\boldsymbol{v},\boldsymbol{a})\sigma_2(\boldsymbol{v},\boldsymbol{b}) \\ &= -\boldsymbol{a}\cdot\boldsymbol{b} \end{aligned}$$

The CHSH inequality is predicted as

$$-2 \le S \le 2,$$

from the local hidden theories. On the other hand, the mathematical formalism of quantum mechanics predicts a maximum value for S of $2\sqrt{2}$, which is greater than 2, and CHSH violations are therefore predicted by the theory of quantum mechanics.

Here we consider the operator defined by

$$\hat{Q} = \hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')] + \hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')].$$

We calculate the square of this operator based on the quantum mechanics.

$$
\begin{aligned}
\hat{Q}^2 &= \{\hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')] + \hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')]\} \\
&\quad \cdot \{\hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')] + \hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')]\} \\
&= \hat{\sigma}_1{}^2(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')]^2 + \hat{\sigma}_1{}^2(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')]^2 \\
&\quad + \hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')]\hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')] \\
&\quad + \hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b}) + \hat{\sigma}_2(\boldsymbol{b}')]\hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b}) - \hat{\sigma}_2(\boldsymbol{b}')]
\end{aligned}
$$

Here we use the formula

$$
(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{a})(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b})\hat{1} + i\hat{\boldsymbol{\sigma}} \cdot (\boldsymbol{a} \times \boldsymbol{b}) .
$$

So, we have

$$
(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{a})(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{a}\hat{1} = \hat{1} .
$$

Then, we get

$$
\begin{aligned}
\hat{Q}^2 &= [2\hat{1} - \hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') - \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] + [2\hat{1} + \hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') + \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\
&\quad + \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') - \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\
&\quad + \hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_1(\boldsymbol{a})[-\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') + \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\
&= 4\hat{1} + \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') - \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\
&\quad - \hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') + \hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})]
\end{aligned}
$$

or

$$
\hat{Q}^2 = 4\hat{1} + [\hat{\sigma}_1(\boldsymbol{a}), \hat{\sigma}_1(\boldsymbol{a}')][\hat{\sigma}_2(\boldsymbol{b}), \hat{\sigma}_2(\boldsymbol{b}')] .
$$

Since

$$
\begin{aligned}
(\hat{\boldsymbol{\sigma}} \cdot \mathbf{a}), (\hat{\boldsymbol{\sigma}} \cdot \mathbf{a}') &= (\hat{\boldsymbol{\sigma}} \cdot \mathbf{a})(\hat{\boldsymbol{\sigma}} \cdot \mathbf{a}') - (\hat{\boldsymbol{\sigma}} \cdot \mathbf{a}')(\hat{\boldsymbol{\sigma}} \cdot \mathbf{a}) \\
&= (\mathbf{a} \cdot \mathbf{a}')\hat{1} + i\hat{\boldsymbol{\sigma}} \cdot (\mathbf{a} \times \mathbf{a}') - [(\mathbf{a}' \cdot \mathbf{a})\hat{1} + i\hat{\boldsymbol{\sigma}} \cdot (\mathbf{a}' \times \mathbf{a})] \\
&= 2i\hat{\boldsymbol{\sigma}} \cdot (\mathbf{a} \times \mathbf{a}')
\end{aligned}
$$

we get

$$\hat{Q}^2 = 4\hat{1} - 4[\hat{\boldsymbol{\sigma}}_1 \cdot (\boldsymbol{a} \times \boldsymbol{a}')][\hat{\boldsymbol{\sigma}}_2 \cdot (\boldsymbol{b} \times \boldsymbol{b}')].$$

The operator $\hat{\boldsymbol{\sigma}}_1 \cdot (\boldsymbol{a} \times \boldsymbol{a}')$ has eigenvalues $\pm|\boldsymbol{a} \times \boldsymbol{a}'| = \pm\sin\theta_{\boldsymbol{a},\boldsymbol{a}'}$, since

$$\hat{\boldsymbol{\sigma}}_1 \cdot (\boldsymbol{a} \times \boldsymbol{a}') = |\boldsymbol{a} \times \boldsymbol{a}'|[\hat{\boldsymbol{\sigma}}_1 \cdot \frac{\boldsymbol{a} \times \boldsymbol{a}'}{|\boldsymbol{a} \times \boldsymbol{a}'|}],$$

where $\frac{\boldsymbol{a} \times \boldsymbol{a}'}{|\boldsymbol{a} \times \boldsymbol{a}'|}$ is the unit vector. The operator $\hat{\boldsymbol{\sigma}}_1 \cdot (\boldsymbol{b} \times \boldsymbol{b}')$ has eigenvalues $\pm|\boldsymbol{b} \times \boldsymbol{b}'| = \pm\sin\theta_{\boldsymbol{b},\boldsymbol{b}'}$, since

$$\hat{\boldsymbol{\sigma}}_2 \cdot (\boldsymbol{b} \times \boldsymbol{b}') = |\boldsymbol{b} \times \boldsymbol{b}'|[\hat{\boldsymbol{\sigma}}_2 \cdot \frac{\boldsymbol{b} \times \boldsymbol{b}'}{|\boldsymbol{b} \times \boldsymbol{b}'|}],$$

where $\frac{\boldsymbol{b} \times \boldsymbol{b}'}{|\boldsymbol{b} \times \boldsymbol{b}'|}$ is the unit vector.

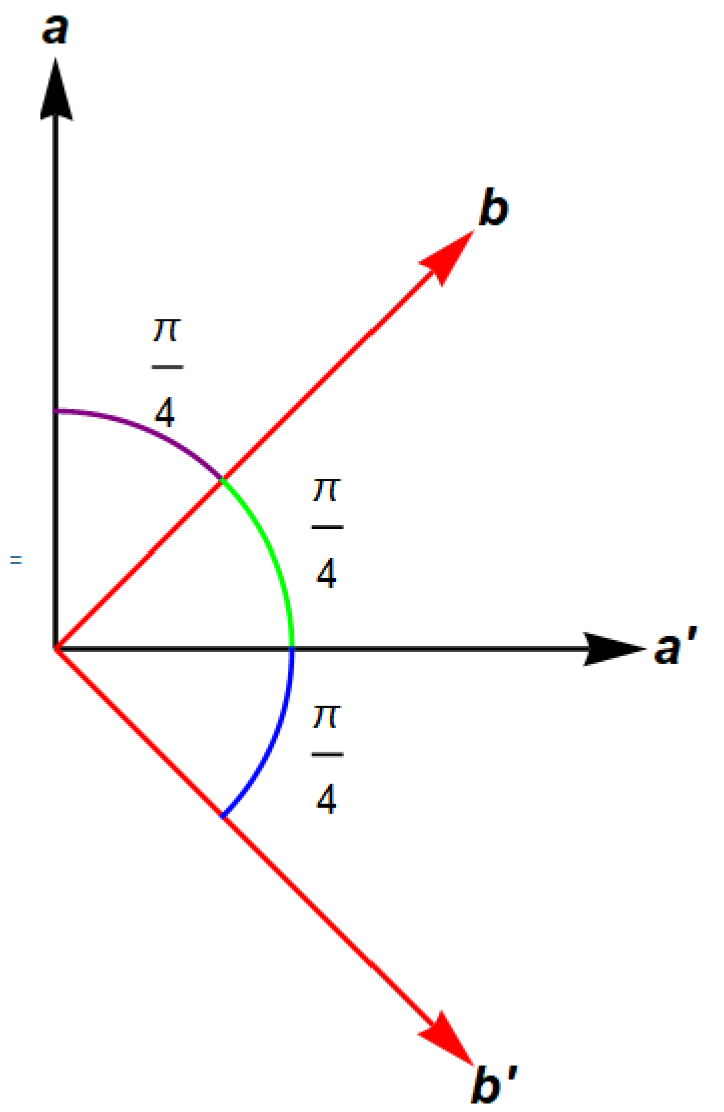


Fig. ***a*** and ***a′*** are detector settings on side A (Alice), ***b*** and ***b′*** on side B (Bob), The four combinations are tested in separate sub-experiments.

Suppose that the four vectors $\boldsymbol{a}$, $\boldsymbol{a}'$, $\boldsymbol{b}$ and $\boldsymbol{b}'$ are in the same plane and each of them makes an angle of $\frac{\pi}{4}$ with the preceding vector.

$$\boldsymbol{a}\times\boldsymbol{a}' = \boldsymbol{e}_z \sin\theta_{aa'} \qquad \boldsymbol{b}\times\boldsymbol{b}' = \boldsymbol{e}_z \sin\theta_{bb'}$$

$$\hat{Q}^2 = 4\hat{1} - 4\sin\theta_{aa'}\sin\theta_{bb'}(\hat{\sigma}_{1z}\otimes\hat{\sigma}_{2z})$$

When $\theta_{aa'} = \theta_{bb'} = \dfrac{\pi}{4}$,

$$\hat{Q}^2 = 4\hat{1} - 4(\hat{\sigma}_{1z} \otimes \hat{\sigma}_{2z}) = 8\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$\hat{Q}^2 \left| +z, +z \right\rangle = 0$$

$\left| +z, +z \right\rangle$ is the eigenket of $\hat{\boldsymbol{Q}}^2$ with the eigenvalue 0 (minimum value)

$$\hat{Q}^2 \left| +z, -z \right\rangle = 8 \left| +z, -z \right\rangle$$

$\left| +z, -z \right\rangle$ is the eigenket of $\hat{Q}^2$ with the eigenvalue 8 (maximum value)

$$\hat{Q}^2 \left| -z, +z \right\rangle = 8 \left| -z, +z \right\rangle$$

$\left| -z, +z \right\rangle$ is the eigenket of $\hat{Q}^2$ with the eigenvalue 8 (maximum value)

$$\hat{Q}^2 \left| -z, -z \right\rangle = 0$$

$\left| -z, -z \right\rangle$ is the eigenket of $\hat{Q}^2$ with the eigenvalue 0 (minimum value)

Note that

$$\left| +z, +z \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \left| +z, -z \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|-z,+z\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad |-z,-z\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Thus, the average value of the operator $\hat{Q}^2$ over the state vector $|\psi\rangle$ is given by

$$\langle Q^2\rangle \le 8 \qquad \text{or} \qquad \sqrt{\langle Q^2\rangle} \le 2\sqrt{2}.$$

where the equality holds for $|\psi\rangle = |+z,-z\rangle$ or $|-z,+z\rangle$

((**Note**)) **Classical case (local hidden theory)**

In the local hidden theory where the commutation relations hold,

$$[\hat{\sigma}_1(\boldsymbol{a}),\hat{\sigma}_1(\boldsymbol{a}')] = 0, \qquad [\hat{\sigma}_2(\boldsymbol{b}),\hat{\sigma}_2(\boldsymbol{b}')] = 0.$$

(which can be regarded as the classical case), we get

$$\begin{aligned}\hat{Q}^2 &= 4\hat{1} + \hat{\sigma}_1(\boldsymbol{a})\hat{\sigma}_1(\boldsymbol{a}')[\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') - \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\ &\quad -\hat{\sigma}_1(\boldsymbol{a}')\hat{\sigma}_1(\boldsymbol{a})[\hat{\sigma}_2(\boldsymbol{b})\hat{\sigma}_2(\boldsymbol{b}') - \hat{\sigma}_2(\boldsymbol{b}')\hat{\sigma}_2(\boldsymbol{b})] \\ &= 4\hat{1}\end{aligned}$$

Thus we have

$$\langle Q^2\rangle = 4, \qquad \text{or} \qquad \sqrt{\langle Q^2\rangle} = 2.$$

The value of 2 is called the Tsirelson bound for the CHSH inequality. The Tsirelson bound is named after B.S. Tsirelson (or Boris Cirel'son).

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23. Derivation of CHSH inequality by Mathematica

Here we show a proof of the CHSH inequality by using the KroneckerProduct (Mathematica). To this end, we use the following notations for the 2D vectors in the *x*-*z* plane with the magnitudes of unity,

$\mathbf{a} = (\sin\alpha, 0, \cos\alpha)$, $\mathbf{a}' = (\sin\beta, 0, \cos\beta)$

$\mathbf{b} = (\sin\gamma, 0, \cos\gamma)$, $\mathbf{b}' = (\sin\delta, 0, \cos\delta)$

We now calculate the KroneckerProduct

$$\hat{Q} = (\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a}) \otimes [(\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b}) - (\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b}')] + (\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a}') \otimes [(\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b}) + (\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b}')]$$

Using the Mathematica we get

$$\hat{Q}^2 = \begin{pmatrix} 4 & 0 & 0 & 4\sin(\alpha-\beta)\sin(\gamma-\delta) \\ 0 & 4 & -4\sin(\alpha-\beta)\sin(\gamma-\delta) & 0 \\ 0 & -4\sin(\alpha-\beta)\sin(\gamma-\delta) & 4 & 0 \\ 4\sin(\alpha-\beta)\sin(\gamma-\delta) & 0 & 0 & 4 \end{pmatrix}$$

$$\langle \Phi_B | \hat{Q}^2 | \Phi_B \rangle = 4 + 4\sin(\alpha-\beta)\sin(\gamma-\delta) \le 8$$

where $|\Phi_B\rangle$ is the Bell tate,

$$|\Phi_B\rangle = \frac{1}{\sqrt{2}}(|+z,-z\rangle - |-z,+z\rangle)$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

((**Mathematica**))

CHSH (Clauser-Horne-Shimoney-Holt) inequality: proof

The CHSH inequality can be used in the proof of Bell's theorem, which states that certain consequences of entanglement in quantum mechanics can not be reproduced by local hidden variable theories.

```
Clear["Global`*"];
σx = PauliMatrix[1];
σy = PauliMatrix[2];
σz = PauliMatrix[3];
A1 = {Sin[α], 0, Cos[α]};
A2 = {Sin[β], 0, Cos[β]};
B1 = {Sin[γ], 0, Cos[γ]};
B2 = {Sin[δ], 0, Cos[δ]};
X[A1_] := σx A1⟦1⟧ + σy A1⟦2⟧ + σz A1⟦3⟧;
```

```
X[A1] // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\alpha] & \text{Sin}[\alpha] \\ \text{Sin}[\alpha] & -\text{Cos}[\alpha] \end{pmatrix}$$

```
Q1 = KroneckerProduct[X[A1], X[B1] - X[B2]] +
    KroneckerProduct[X[A2], X[B1] + X[B2]] //
   FullSimplify;
```

```
Q11 = Q1.Q1 // FullSimplify
```

```
{{4, 0, 0, 4 Sin[α - β] Sin[γ - δ]},
 {0, 4, -4 Sin[α - β] Sin[γ - δ], 0},
 {0, -4 Sin[α - β] Sin[γ - δ], 4, 0},
 {4 Sin[α - β] Sin[γ - δ], 0, 0, 4}}
```

The average of Q.Q over the B ell's state

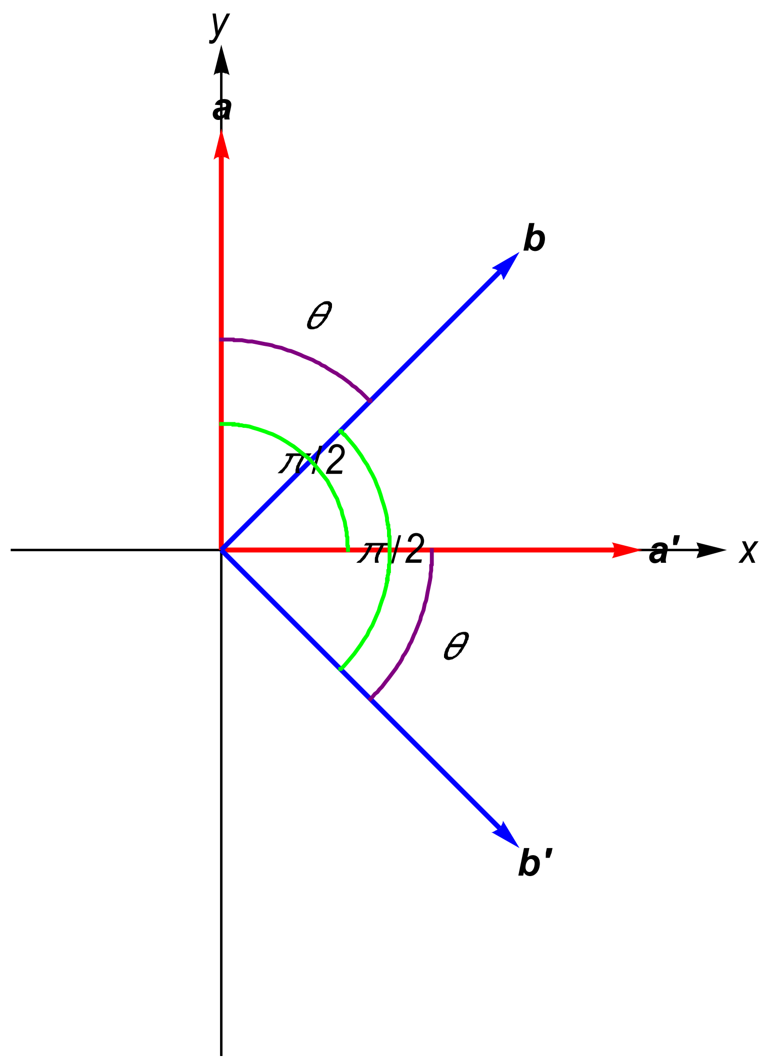
$$\psi 1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix};$$

```
Transpose[ψ1].Q11.ψ1 // Simplify
```

```
{{4 + 4 Sin[α - β] Sin[γ - δ]}}
```

24. Evaluation of *S*

Here we evaluate the value of *S*. We choose the directions of the unit vectors ***a***, ***b***, ***a***', and, ***b***' as shown in Fig.



$\boldsymbol{a} = z$, $\boldsymbol{a}' = x$. The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is θ.

$$E(\boldsymbol{a},\boldsymbol{b}) = E(\boldsymbol{a}',\boldsymbol{b}') = -\cos\theta$$

$$E(\boldsymbol{a}',\boldsymbol{b}) = -\cos(\frac{\pi}{2}-\theta) = -\sin\theta$$

$$E(\boldsymbol{a},\boldsymbol{b}') = -\cos(\frac{\pi}{2}+\theta) = \sin\theta$$

leading to

$$\begin{aligned} S &= E(\boldsymbol{a},\boldsymbol{b}) - E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}) + E(\boldsymbol{a}',\boldsymbol{b}') \\ &= -\cos\theta - \sin\theta - \sin\theta - \cos\theta \\ &= -2(\sin\theta + \cos\theta) \\ &= -2\sqrt{2}\sin(\theta + \frac{\pi}{4}) \end{aligned}$$

We make a plot of S as a function of θ. The value of $|S|$ becomes larger than 2 for $0 < \theta < \pi/2$ and $\pi < \theta < 3\pi/2$.

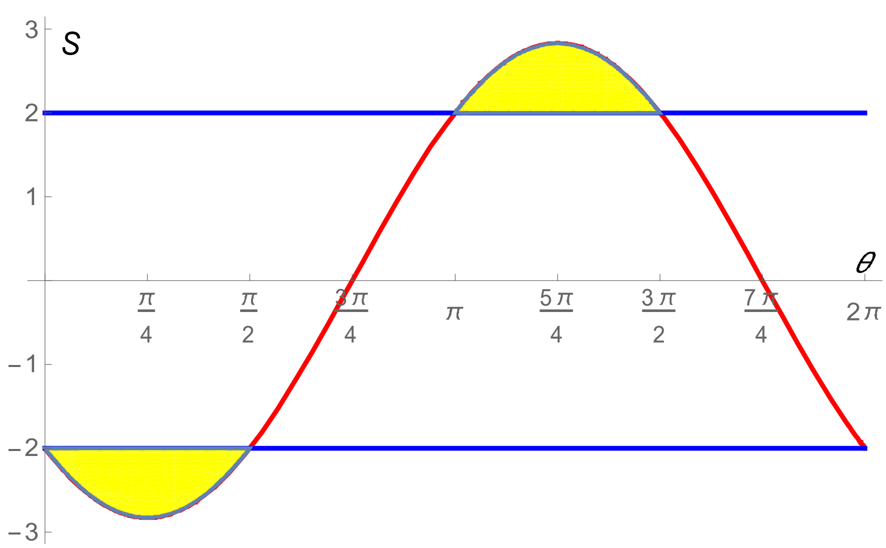
When $\theta = \pi/4$, we have

$$E(\boldsymbol{a},\boldsymbol{b}) = E(\boldsymbol{a}',\boldsymbol{b}) = E(\boldsymbol{a}',\boldsymbol{b}') = -\frac{1}{\sqrt{2}},$$

$$E(\boldsymbol{a}',\boldsymbol{b}') = -\cos(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}.$$

Then we have

$$\begin{aligned} S &= E(\boldsymbol{a},\boldsymbol{b}) - E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}) + E(\boldsymbol{a}',\boldsymbol{b}') \\ &= -2\sqrt{2} \end{aligned}$$



((**Note**))

Suppose that $\theta_{ab} = \theta_{ba'} = \theta_{a'b'} = \theta$, and $\theta_{ab'} = 3\theta$. We find the maximum and the minimum of S for $0 \le \theta \le 2\pi$ When the maximum is larger than $S = 2$, this yields the greatest conflict between a quantum mechanical calculation of S and the Bell's inequality ($|S| \le 2$).

$$S = E(\boldsymbol{a},\boldsymbol{b}) - E(\boldsymbol{a},\boldsymbol{b}') + E(\boldsymbol{a}',\boldsymbol{b}) + E(\boldsymbol{a}',\boldsymbol{b}')$$
$$= -\cos\theta_{ab} + \cos\theta_{ab'} - \cos\theta_{a'b} - \cos\theta_{a'b'}$$
$$= -3\cos\theta + \cos(3\theta)$$
$$= f(\theta)$$

where

$$\theta_{ab} = \theta_{ba'} = \theta_{a'b'} = \theta, \text{ and } \theta_{ab'} = 3\theta .$$

We make a plot of $f(\theta)$ as a function of θ. It is found that $f(\theta)$ has a maxim $2\sqrt{2}$ which is larger than 2 predicted from the hidden local theory.

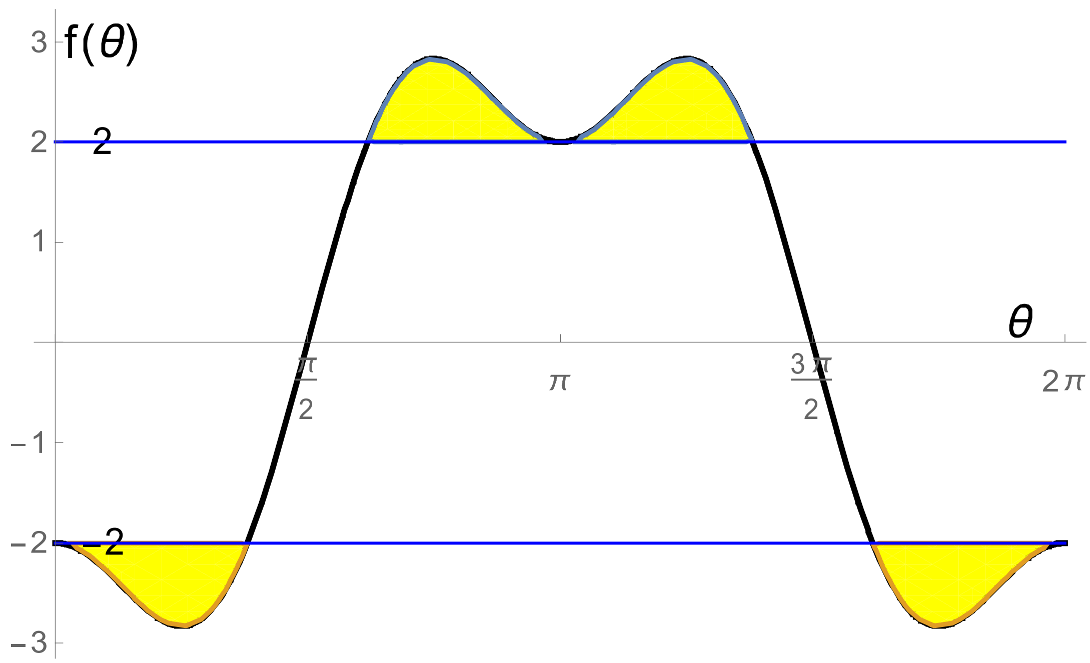


Fig. Plot of $S = f(\theta)$ vs θ. The yellow region where $|S| > 2$, violating the Bell's inequality.

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APPENDIX
Definitions of key words in quantum entanglement from the Wikipedia

Action at a distance

In physics, action at a distance is the nonlocal interaction of objects that are separated in space. This term was used most often in the context of early theories of gravity and electromagnetism to describe how an object responds to the influence of distant objects. More generally "action at a distance" describes the failure of early atomistic and mechanistic theories which sought to reduce all physical interaction to collision. The exploration and resolution of this problematic phenomenon led to significant developments in physics, from the concept of a field, to descriptions of quantum entanglement and the mediator particles of the standard model.

Spooky action at a distance [spukhafte Fernwirkung (in german)]

Entanglement arises naturally when two particles are created at the same point and instant in space, for example. Entangled particles can become widely separated in space. But even so, the mathematics implies that a measurement on one immediately influences the other, regardless of the distance between them. Einstein and co-authors pointed out that according to special relativity, this was impossible and therefore, quantum mechanics must be wrong, or at least incomplete. Einstein famously called it spooky action at a distance. The basic idea here is to think about the transfer of information. Entanglement allows one particle to instantaneously influence another but not in a way that allows classical information to travel faster than light. This resolved the paradox with special relativity but left much of the mystery intact.

Wave function collapse

In quantum mechanics, wave function collapse is said to occur when a wave function—initially in a superposition of several eigenstates—appears to reduce to a single eigenstate due to interaction with the external world; this is called an "observation". It is the essence of measurement in quantum mechanics and connects the wave function with classical observables like position and momentum.

Separability

Different particles or systems that occupy different regions in space have an independent reality.

Locality

An action involving one of these particles or systems cannot influence a particle or system in another part of space unless something travels the distance between them, a process limited by the speed of light.

Nonlocality

In physics, nonlocality or action at a distance is the direct interaction of two objects that are separated in space with no intermediate agency or mechanism. Regarding the unexplained nature of gravity, Isaac Newton (1642-1727) considered action-at-a-distance "so great an Absurdity that I believe no Man who has in philosophical Matters a competent Faculty of thinking can ever fall into it". Quantum nonlocality refers to what Einstein called the "spooky action at a distance" of quantum entanglement.

(Local) hidden variable theory

Historically, in physics, hidden variable theories were espoused by some physicists who argued that the state of a physical system, as formulated by quantum mechanics, does not give a complete description for the system; i.e., that quantum mechanics is ultimately incomplete, and that a complete theory would provide descriptive categories to account for all observable behavior and thus avoid any indeterminism. The existence of indeterminacy for some measurements is a characteristic of prevalent interpretations of quantum mechanics; moreover, bounds for indeterminacy can be expressed in a quantitative form by the Heisenberg uncertainty principle.

Principle of locality

In physics, the principle of locality states that an object is influenced directly only by its immediate surroundings. Experiments have shown that quantum mechanically entangled particles must either violate the principle of locality or engage in superluminal communication.

Local realism

Local realism is a significant feature of classical mechanics, of general relativity, and of electrodynamics; but quantum mechanics largely rejects this principle due to the theory of distant quantum entanglements, an interpretation rejected by Einstein in the EPR paradox but subsequently proven by Bell's inequalities. Any theory, such as quantum mechanics, that violates Bell's inequalities must abandon *either* locality *or* realism; but some physicists dispute that experiments have demonstrated Bell's violations, on the grounds that the sub-class of inhomogeneous Bell inequalities has not been tested or due to experimental limitations in the tests. Different interpretations of quantum mechanics violate different parts of local realism and/or counterfactual definiteness.

Qubit

In quantum computing, a qubit or quantum bit is a unit of quantum information—the quantum analogue of the classical bit. A qubit is a two-state quantum-mechanical system, such as the polarization of a single photon: here the two states are vertical polarization and horizontal polarization. In a classical system, a bit would have to be in one state or the other, but quantum mechanics allows the qubit to be in a superposition of both states at the same time, a property which is fundamental to quantum computing.

Quantum entanglement

Quantum entanglement is a physical phenomenon that occurs when pairs (or groups) of particles are generated or interact in ways such that the quantum state of each member must subsequently be described relative to each other. Quantum entanglement is a product of quantum superposition. However, the state of each member is indefinite in terms of physical properties such as position, momentum, spin, polarization, etc. in a manner distinct from the intrinsic uncertainty of quantum superposition. When a measurement is made on one member of an entangled pair and the outcome is thus known (e.g., clockwise spin), the other member of the pair is at any subsequent time always found (when measured) to have taken the appropriately correlated value (e.g., counterclockwise spin). There is thus a correlation between the results of measurements performed on entangled pairs, and this correlation is observed even though the entangled pair may be separated by arbitrarily large distances. Repeated experiments have verified that this works even when the measurements are performed more quickly than light could travel between the sites of measurement: there is no light speed or slower influence that can pass between the entangled particles. Recent experiments have measured entangled particles within less than one part in 10,000 of the light travel time between them; according to the formalism of quantum theory, the effect of measurement happens instantly.

This behavior is consistent with quantum theory, and has been demonstrated experimentally with photons, electrons, molecules the size of buckyballs, and even small diamonds. It is an area of extremely active research by the physics community. However, there is some heated debate about whether a possible classical underlying mechanism could explain entanglement. The difference in opinion derives from espousal of various interpretations of quantum mechanics.

Research into quantum entanglement was initiated by a 1935 paper by Albert Einstein, Boris Podolsky, and Nathan Rosen describing the EPR paradox and several papers by Erwin Schrödinger shortly thereafter. Although these first studies focused on the counterintuitive properties of entanglement, with the aim of criticizing quantum mechanics, eventually entanglement was verified experimentally, and recognized as a valid, fundamental feature of quantum mechanics. The focus of the research has now changed to its utilization as a resource for communication and computation.

SG measurements

$$\left|\Phi^{+}\right\rangle = \frac{1}{\sqrt{2}}[\left|++\right\rangle + \left|--\right\rangle],$$

$$\left|\Phi^{-}\right\rangle = \frac{1}{\sqrt{2}}[\left|++\right\rangle - \left|--\right\rangle],$$

$$\left|\Psi^{+}\right\rangle = \frac{1}{\sqrt{2}}[\left|+-\right\rangle + \left|-+\right\rangle],$$

$$\left|\psi^{-}\right\rangle = \frac{1}{\sqrt{2}}[\left|+-\right\rangle - \left|-+\right\rangle].$$

where these ket vectors are orthogonal to each other.

$$\hat{\boldsymbol{S}}_1 \cdot \boldsymbol{n}_1 = \frac{\hbar}{2}(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1), \qquad \hat{\boldsymbol{S}}_2 \cdot \boldsymbol{n}_2 = \frac{\hbar}{2}(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2).$$

(a)

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^+\right\rangle = \cos(\theta_1 - \theta_2)\left|\Phi^+\right\rangle - \sin(\theta_1 - \theta_2)\left|\psi^-\right\rangle,$$

leading to

$$\left\langle\Phi^+\right|(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^+\right\rangle = \cos(\theta_1 - \theta_2)\,.$$

(b)

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^-\right\rangle = \cos(\theta_1 + \theta_2)\left|\Phi^-\right\rangle + \sin(\theta_1 + \theta_2)\left|\psi^+\right\rangle,$$

leading to

$$\left\langle\psi^+\right|(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^+\right\rangle = -\cos(\theta_1 + \theta_2)\,.$$

(c)

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^-\right\rangle = -\sin(\theta_1 - \theta_2)\left|\Phi^+\right\rangle - \cos(\theta_1 - \theta_2)\left|\psi^-\right\rangle,$$

leading to

$$\left\langle\psi^-\right|(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^-\right\rangle = -\cos(\theta_1 - \theta_2)\,.$$

(i) When $\theta_1 - \theta_2 = 0$,

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^+\right\rangle = \left|\Phi^+\right\rangle,$$

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^-\right\rangle = -\left|\psi^-\right\rangle.$$

(ii) When $\theta_1 - \theta_2 = \dfrac{\pi}{2}$,

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^+\right\rangle = -\left|\psi^-\right\rangle,$$

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^-\right\rangle = -\left|\Phi^+\right\rangle.$$

(iii) When $\theta_1 + \theta_2 = 0$

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^-\right\rangle = \left|\Phi^-\right\rangle,$$

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^+\right\rangle = -\left|\psi^+\right\rangle.$$

(iv) When $\theta_1 + \theta_2 = \dfrac{\pi}{2}$,

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\Phi^-\right\rangle = \left|\psi^+\right\rangle,$$

$$(\hat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n}_1)(\hat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n}_2)\left|\psi^+\right\rangle = \left|\Phi^-\right\rangle.$$

APPENDIX-B

Therefore, all three components of spin correspond to "elements of reality," as defined by EPR, because a definite value will be predictable with certainty, for any one of them, if we measure the corresponding spin component of the *other* particle. This claim, however, is incompatible with quantum mechanics, which asserts that at most *one* spin component of each particle may be definite.

We consider the two spin particles, far apart from each other, in a singlet state

$$\left|\Phi^-\right\rangle_{12} = \frac{1}{\sqrt{2}}[\left|+z\right\rangle_1\left|-z\right\rangle_2 - \left|-z\right\rangle_1\left|+z\right\rangle_2].$$

We know that measurements of $\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x}$, if performed, shall yield *opposite* values,

$$(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2x})\left|\Phi^-\right\rangle_{12} = m_{1x}m_{2x}\left|\Phi^-\right\rangle_{12} = -\left|\Phi^-\right\rangle_{12}. \qquad \text{(quantum mechanics)}$$

Sincee $m_{1x}m_{2x} = -1$ and $m_{1x}{}^2 = m_{2x}{}^2 = 1$, we have

$$m_{1x} = -m_{2x},$$

Similarly, the measurements of $\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2y}$, if performed, shall yield *opposite* values,

$$(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2y})\left|\Phi^-\right\rangle_{12} = m_{1y}m_{2y}\left|\Phi^-\right\rangle_{12} = -\left|\Phi^-\right\rangle_{12}. \qquad \text{(quantum mechanics)}$$

Since $m_{1y}m_{2y} = -1$ and $m_{1y}{}^2 = m_{2y}{}^2 = 1$, we have

$$m_{1y} = -m_{2y}.$$

Furthermore, since $\hat{\sigma}_{1x}$ and $\hat{\sigma}_{2y}$ commute, and both correspond to elements of reality, their product $\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y}$ also corresponds to an element of reality. The numerical value assigned to the product $\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y}$ is the product of the individual numerical values, $m_{1x}\, m_{2y}$,such that

$$(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y})\left|\Phi^-\right\rangle_{12} = m_{1x}m_{2y}\left|\Phi^-\right\rangle_{12}. \qquad \text{(EPR)}$$

Likewise, the numerical value of $\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x}$ is the product $m_{1y}\, m_{2x}$, such that

$$(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x})\left|\Phi^-\right\rangle_{12} = m_{1y}m_{2x}\left|\Phi^-\right\rangle_{12}.$$

These two products must be *equal,*

$$m_{1x}m_{2y} = (-m_{2x})(-m_{1y}) = m_{1y}m_{2x},$$

because $m_{1x} = -\, m_{2x}$ and $m_{1y} = -\, m_{2y}$.

The quantum theory asserts that these products have *opposite* values, because the singlet state satisfies

$$(\hat{\sigma}_{1x} \otimes \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \otimes \hat{\sigma}_{2x})\left|\Phi^-\right\rangle_{12} = 0. \qquad \text{(quantum mechanics)}$$

From the EPR, element of reality, on the other hand, we should obtain the relation

$$m_{1x}m_{2y} + m_{1y}m_{2x} = 0,$$

which is inconsistent with the value

$$m_{1x}m_{2y} + m_{1y}m_{2x} = 2m_{1x}m_{2y} \neq 0 .$$

which can be derived from the above result. Then we are thus forced to the conclusion that the definition of elements of reality is incompatible with quantum theory.

APPENDIX-C

(1) Quantum Entanglement Documentary- Atomic Physics and Reality

http://www.youtube.com/watch?v=BFvJOZ51tmc

where J. Bell, D. Bohm, A. Aspect, J. Wheeler talked about physics.

(2) Fabric of the Cosmos-Quantum Leap (Brian Greene)

http://www.youtube.com/watch?v=NbIcg0XsbFQ

APPENDIX-D Formula

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+\boldsymbol{a}\rangle = \begin{pmatrix} \cos\left(\frac{\theta_a}{2}\right) \\ \sin\left(\frac{\theta_a}{2}\right) \end{pmatrix}, \qquad |-\boldsymbol{a}\rangle = \begin{pmatrix} -\sin\left(\frac{\theta_a}{2}\right) \\ \cos\left(\frac{\theta_a}{2}\right) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}(|+\boldsymbol{a},-\boldsymbol{a}\rangle - |-\boldsymbol{a},+\boldsymbol{a}\rangle) = \frac{1}{\sqrt{2}}(|+z,-z\rangle - |-z,+z\rangle)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\langle 0,0|(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{a})(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{b})|0,0\rangle = -\cos(\theta_a - \theta_b)$$

$$\langle +\boldsymbol{a}|+\boldsymbol{b}\rangle = \cos\left(\frac{\theta_a - \theta_b}{2}\right), \qquad \langle +\boldsymbol{a}|-\boldsymbol{b}\rangle = \sin\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$\langle -\boldsymbol{a}|+\boldsymbol{b}\rangle = -\sin\left(\frac{\theta_a - \theta_b}{2}\right), \qquad \langle -\boldsymbol{a}|-\boldsymbol{b}\rangle = \cos\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$\langle +\boldsymbol{a}, +\boldsymbol{b} | 0,0\rangle = -\frac{1}{\sqrt{2}}\sin\left(\frac{\theta_a - \theta_b}{2}\right), \qquad \langle +\boldsymbol{a}, -\boldsymbol{b} | 0,0\rangle = \frac{1}{\sqrt{2}}\cos\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$\langle -\boldsymbol{a}, +\boldsymbol{b} | 0,0\rangle = -\frac{1}{\sqrt{2}}\cos\left(\frac{\theta_a - \theta_b}{2}\right), \qquad \langle -\boldsymbol{a}, -\boldsymbol{b} | 0,0\rangle = -\frac{1}{\sqrt{2}}\sin\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$P_{++}(\boldsymbol{a},\boldsymbol{b}) = \left|\langle +\boldsymbol{a}, +\boldsymbol{b} | 0,0\rangle\right| = \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$P_{+-}(\boldsymbol{a},\boldsymbol{b}) = \left|\langle +\boldsymbol{a}, +\boldsymbol{b} | 0,0\rangle\right| = \frac{1}{2}\cos^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$P_{-+}(\boldsymbol{a},\boldsymbol{b}) = \left|\langle -\boldsymbol{a}, +\boldsymbol{b} | 0,0\rangle\right| = \frac{1}{2}\cos^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$P_{--}(\boldsymbol{a},\boldsymbol{b}) = \left|\langle -\boldsymbol{a}, -\boldsymbol{b} | 0,0\rangle\right| = \frac{1}{2}\sin^2\left(\frac{\theta_a - \theta_b}{2}\right)$$

$$\begin{aligned} E(\boldsymbol{a},\boldsymbol{b}) &= P_{++}(\boldsymbol{a},\boldsymbol{b}) + P_{--}(\boldsymbol{a},\boldsymbol{b}) - P_{+-}(\boldsymbol{a},\boldsymbol{b}) - P_{-+}(\boldsymbol{a},\boldsymbol{b}) \\ &= -\cos(\theta_a - \theta_b) \end{aligned}$$

APPENDIX-E

$$\frac{1}{\sqrt{2}}[|+\mathbf{a}\rangle_1 \otimes |-\mathbf{a}\rangle_1 - |+\mathbf{a}\rangle_1 \otimes |+\mathbf{a}\rangle_1] = \hat{U}_a \otimes \hat{U}_a \frac{1}{\sqrt{2}}[|+z\rangle_1 \otimes |-z\rangle_2 - |+z\rangle_1 \otimes |+z\rangle_2]$$

$$= e^{i\phi}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= e^{i\phi}|0,0\rangle$$

where

$$\hat{U}_a = \begin{pmatrix} \cos\frac{\theta_a}{2} & -\sin\frac{\theta_a}{2} \\ e^{i\phi_a}\sin\frac{\theta_a}{2} & e^{i\phi_a}\cos\frac{\theta_a}{2} \end{pmatrix}$$

((**Mathematica**))

```
Clear["Global`*"];
Ua = ( Cos[θ/2]                -Sin[θ/2]
       Exp[ⅈ ϕ] Sin[θ/2]   Exp[ⅈ ϕ] Cos[θ/2] );
H11 = KroneckerProduct[Ua, Ua] // Simplify;
ψ0 = 1/√2 ( 0
            1
           -1
            0 );
H11.ψ0 // Simplify
```

$$\left\{\{0\}, \left\{\frac{e^{i\phi}}{\sqrt{2}}\right\}, \left\{-\frac{e^{i\phi}}{\sqrt{2}}\right\}, \{0\}\right\}$$

APPENDIX-F

$$E(\mathbf{a},\mathbf{b}) = \langle 0,0|(\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a})(\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b})|0,0\rangle$$

$$\begin{aligned}(\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b})|0,0\rangle &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\sigma}}_2 \cdot \mathbf{b})[|+\mathbf{b}\rangle \otimes |-\mathbf{b}\rangle - |-\mathbf{b}\rangle \otimes |+\mathbf{b}\rangle] \\ &= -\frac{1}{\sqrt{2}}[|+\mathbf{b}\rangle \otimes |-\mathbf{b}\rangle + |-\mathbf{b}\rangle \otimes |+\mathbf{b}\rangle]\end{aligned}$$

$$\begin{aligned}(\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a})|0,0\rangle &= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a})[|+\mathbf{a}\rangle \otimes |-\mathbf{a}\rangle - |-\mathbf{a}\rangle \otimes |+\mathbf{a}\rangle] \\ &= \frac{1}{\sqrt{2}}[|+\mathbf{a}\rangle \otimes |-\mathbf{a}\rangle + |-\mathbf{a}\rangle \otimes |+\mathbf{a}\rangle]\end{aligned}$$

or

$$\langle 0,0|(\hat{\boldsymbol{\sigma}}_1 \cdot \mathbf{a}) = \frac{1}{\sqrt{2}}[\langle +\mathbf{a}| \otimes \langle -\mathbf{a}| + \langle -\mathbf{a}| \otimes \langle +\mathbf{a}|]$$

$$
\begin{aligned}
E(\mathbf{a},\mathbf{b}) &= \langle 0,0|(\hat{\boldsymbol{\sigma}}_1\cdot\mathbf{a})(\hat{\boldsymbol{\sigma}}_2\cdot\mathbf{b})|0,0\rangle \\
&= -\frac{1}{2}[\langle +\mathbf{a}|\otimes\langle -\mathbf{a}| + \langle -\mathbf{a}|\otimes\langle +\mathbf{a}|][|+\mathbf{b}\rangle\otimes|-\mathbf{b}\rangle + |-\mathbf{b}\rangle\otimes|+\mathbf{b}\rangle] \\
&= -\frac{\hbar^2}{2}[\langle +\mathbf{a}|\otimes\langle -\mathbf{a}| + \langle -\mathbf{a}|\otimes\langle +\mathbf{a}|][|+\mathbf{b}\rangle\otimes|-\mathbf{b}\rangle + |-\mathbf{b}\rangle\otimes|+\mathbf{b}\rangle] \\
&= -[\langle +\mathbf{a}|+\mathbf{b}\rangle\langle -\mathbf{a}|-\mathbf{b}\rangle + \langle +\mathbf{a}|-\mathbf{b}\rangle\langle -\mathbf{a}|+\mathbf{b}\rangle] \\
&= -[\cos^2(\frac{\theta_{ab}}{2}) - \sin^2(\frac{\theta_{ab}}{2})] \\
&= -\cos(\theta_{ab}) \\
&= -\mathbf{a}\cdot\mathbf{b}
\end{aligned}
$$

((**Note**))

$$
\begin{aligned}
&P(+\mathbf{a},+\mathbf{b}) + P(-\mathbf{a},-\mathbf{b}) - P(+\mathbf{a},-\mathbf{b}) - P(-\mathbf{a},+\mathbf{b}) \\
&= -[\cos^2\left(\frac{\theta_{ab}}{2}\right) - \sin^2\left(\frac{\theta_{ab}}{2}\right)] \\
&= -\cos\theta_{ab} \\
&= -\mathbf{a}\cdot\mathbf{b}
\end{aligned}
$$