Laboratory frame and center of mass frame

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The scattering experiment is actually carried out in the laboratory frame, where a beam of particles is being scattered by some fixed scattering center (target). This is called the laboratory frame. In fact, the target is always free to move. In order to compare experiment with theory, it is necessary to convert what is actually observed in the laboratory frame to what would have been observed in the center of mass frame. Since the observation of scattered flux is done with macroscopic apparatus, this is a purely classical problem. So it can be solved using classical notions. We note that the center of mass frame is favorable compared to the laboratory frame from a theoretical view-point. The separation of the system from the center of mass leads to the decrease of the degree of freedom from 6 to 3. In many cases, the scattering theory is discussed in terms of the center of mass frame.

Here the relation between the cross sections in the center of mass frame and the laboratory frame is discussed.

1. Momentum conservation and the energy conservation

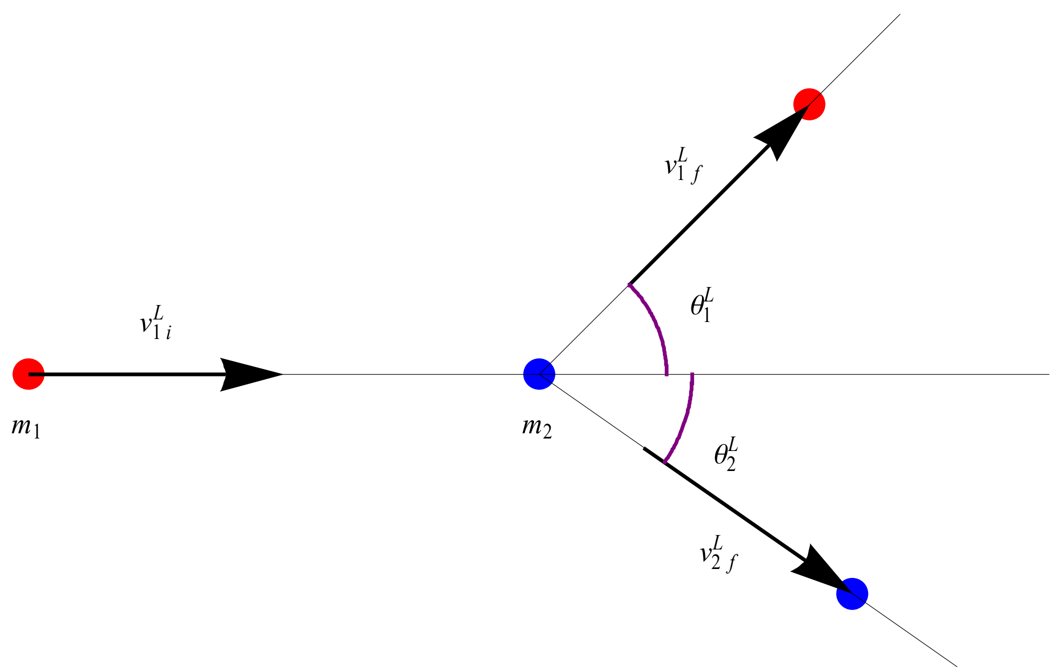


Fig. Laboratory frame. The initial momentum is $m_i \boldsymbol{v}_{1i}^L$. The scattering angle of the particle 1 with mass m_1 is θ_1^L.

In the laboratory frame, the velocities of the particles 1 and 2 before and after collision, are defined by

$$\boldsymbol{v}_{1i}^{L}, \quad \boldsymbol{v}_{2i}^{L}, \quad \boldsymbol{v}_{1f}^{L}, \quad \boldsymbol{v}_{2f}^{L},$$

where

$$\boldsymbol{v}_{2i}^{L} = 0$$

since the particle with mass m_2 is at rest before the collision. The center of mass velocity is given by

$$\boldsymbol{v}_{CM} = \frac{m_1}{m_1 + m_2}\boldsymbol{v}_{1i}^{L}$$

In the center of mass frame

$$\boldsymbol{v}_{1i}^{CM}, \quad \boldsymbol{v}_{2i}^{CM}, \quad \boldsymbol{v}_{1f}^{CM}, \quad \boldsymbol{v}_{2f}^{CM},$$

We have the relation

$$\boldsymbol{v}_{1i}^{L} = \boldsymbol{v}_{1i}^{CM} + \boldsymbol{v}_{CM}, \qquad \boldsymbol{v}_{2i}^{L} = \boldsymbol{v}_{2i}^{CM} + \boldsymbol{v}_{CM}, \tag{1a}$$

$$\boldsymbol{v}_{1f}^{L} = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}, \qquad \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM}, \tag{1b}$$

where

$$\boldsymbol{v}_{1i}^{CM} = \boldsymbol{v}_{1i}^{L} - \boldsymbol{v}_{CM} = \frac{m_2}{m_1 + m_2}\boldsymbol{v}_{1i}^{L}, \qquad \boldsymbol{v}_{2i}^{CM} = \boldsymbol{v}_{2i}^{L} - \boldsymbol{v}_{CM} = -\frac{m_1}{m_1 + m_2}\boldsymbol{v}_{1i}^{L}.$$

Note that

$$m_1\boldsymbol{v}_{1i}^{CM} + m_2\boldsymbol{v}_{2i}^{CM} = 0 \qquad \text{or} \qquad \boldsymbol{v}_{2i}^{CM} = -\frac{m_1}{m_2}\boldsymbol{v}_{1i}^{CM}$$

(i) The momentum conservation in the laboratory frame;

$$m_1 \boldsymbol{v}_{1i}^L + m_2 \boldsymbol{v}_{2i}^L = m_1 \boldsymbol{v}_{1f}^L + m_2 \boldsymbol{v}_{2f}^L \,.$$

which means that the velocity of the center of mass remains unchanged before and after the collision. Using Eqs.(1a) and (1b), we get

$$m_1 \boldsymbol{v}_{1i}^L = m_1 (\boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}) + m_2 (\boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM})$$

or

$$m_1 \boldsymbol{v}_{1f}^{CM} + m_2 \boldsymbol{v}_{2f}^{CM} = m_1 \boldsymbol{v}_{1i}^L - (m_1 + m_2) \boldsymbol{v}_{CM} = 0$$

or

$$m_1 \boldsymbol{v}_{1f}^{CM} + m_2 \boldsymbol{v}_{2f}^{CM} = 0, \qquad \text{or} \qquad \boldsymbol{v}_{2f}^{CM} = -\frac{m_1}{m_2} \boldsymbol{v}_{1f}^{CM} \qquad (2)$$

(ii) The energy conservation law;

$$\frac{1}{2} m_1 \left(v_{1i}^L\right)^2 + \frac{1}{2} m_2 \left(v_{2i}^L\right)^2 = \frac{1}{2} m_1 \left(v_{1f}^L\right)^2 + \frac{1}{2} m_2 \left(v_{2f}^L\right)^2$$

Using Eqs.(1a) and (1b), we get

$$\frac{1}{2} m_1 \left(\boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}\right)^2 + \frac{1}{2} m_2 (\boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM})^2 = \frac{1}{2} m_1 \left(\boldsymbol{v}_{1i}^L\right)^2$$

or

$$\frac{1}{2} m_1 (\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2} m_2 (\boldsymbol{v}_{2f}^{CM})^2 + (m_1 \boldsymbol{v}_{1f}^{CM} + m_2 \boldsymbol{v}_{2f}^{CM}) \cdot \boldsymbol{v}_{CM} + \frac{1}{2} (m_1 + m_2) (\boldsymbol{v}_{CM})^2 = \frac{1}{2} m_1 \left(\boldsymbol{v}_{1i}^L\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2(\boldsymbol{v}_{2f}^{CM})^2 = \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^{L}\right)^2 - \frac{1}{2}(m_1+m_2)(\boldsymbol{v}_{CM})^2$$
$$= \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^{L}\right)^2 - \frac{1}{2}(m_1+m_2)\left(\frac{m_1\boldsymbol{v}_{1i}^{L}}{m_1+m_2}\right)^2 \qquad (3)$$
$$= \frac{1}{2}\mu\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2(\boldsymbol{v}_{2f}^{CM})^2 = \frac{1}{2}\mu\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

where $m_1\boldsymbol{v}_{1f}^{CM} + m_2\boldsymbol{v}_{2f}^{CM} = 0$, μ is the reduced mass and is given by

$$\mu = \frac{m_1 m_2}{m_1+m_2}.$$

Using Eqs.(2) and (3), we get

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2\left(-\frac{m_1}{m_2}\boldsymbol{v}_{1f}^{CM}\right)^2 = \frac{1}{2}\mu\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}\frac{m_1^{\ 2}}{m_2}(\boldsymbol{v}_{1f}^{CM})^2 = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$(v_{1f}^{CM})^2 = \left(\dfrac{m_2 v_{1i}^{L}}{m_1+m_2}\right)^2$, and $\left(v_{2f}^{CM}\right)^2 = \left(\dfrac{m_1 v_{1i}^{L}}{m_1+m_2}\right)^2$

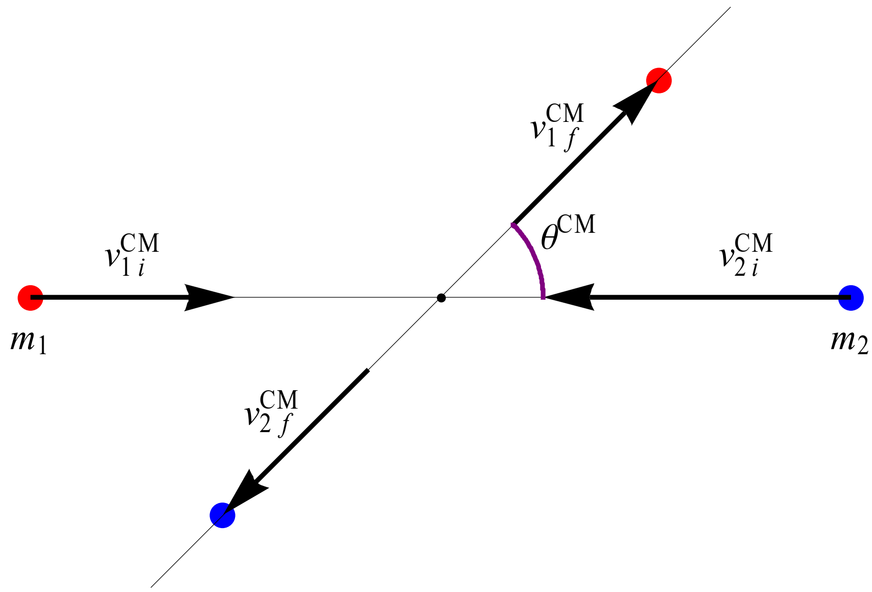


Fig. The center of mass frame. The scattering angle of the particle 1 with mass m_1 is θ^{CM}. In this plane, the z axis is in the horizontal direction and the y axis is in the vertical direction.

The y and z components of v_{1f}^{CM} and v_{2f}^{CM};

$$\left(\boldsymbol{v}_{1f}^{CM}\right)_y = v_{1f}^{CM}\sin\theta^{CM} = \frac{m_2 v_{1i}^{L}}{m_1+m_2}\sin\theta^{CM},$$

$$\left(\boldsymbol{v}_{1f}^{CM}\right)_z = v_{1f}^{CM}\cos\theta^{CM} = \frac{m_2 v_{1i}^{L}}{m_1+m_2}\cos\theta^{CM},$$

$$\left(\boldsymbol{v}_{2f}^{CM}\right)_y = -v_{2f}^{CM}\sin\theta^{CM} = -\frac{m_1 v_{1i}^{L}}{m_1+m_2}\sin\theta^{CM},$$

$$\left(\boldsymbol{v}_{2f}^{CM}\right)_z = -v_{2f}^{CM}\cos\theta^{CM} = -\frac{m_1 v_{1i}^{L}}{m_1+m_2}\cos\theta^{CM}.$$

where the z axis is in the horizontal direction and the y axis is in the vertical direction.

Using Eq.(1b), we have

$$\boldsymbol{v}_{1f}^{L} = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM} , \qquad \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM} ,$$

we have

$$\left(\boldsymbol{v}_{1f}^{L}\right)_y = \left(\boldsymbol{v}_{1f}^{CM}\right)_y + \left(\boldsymbol{v}_{CM}\right)_y = v_{1f}^{CM} \sin\theta^{CM} = \frac{m_2 v_{1i}^{L}}{m_1 + m_2} \sin\theta^{CM} ,$$

$$\begin{aligned}\left(\boldsymbol{v}_{1f}^{L}\right)_z &= \left(\boldsymbol{v}_{1f}^{CM}\right)_z + \left(\boldsymbol{v}_{CM}\right)_z \\ &= v_{1f}^{CM} \cos\theta^{CM} + v_{CM} \\ &= \frac{m_2 v_{1i}^{L}}{m_1 + m_2} \cos\theta^{CM} + \frac{m_1 v_{1i}^{L}}{m_1 + m_2} \\ &= \frac{(m_1 + m_2 \cos\theta^{CM})}{m_1 + m_2} v_{1i}^{L}\end{aligned}$$

$$\left(\boldsymbol{v}_{2f}^{L}\right)_y = \left(\boldsymbol{v}_{2f}^{CM}\right)_y + \left(\boldsymbol{v}_{CM}\right)_y = -\frac{m_1 v_{1i}^{L}}{m_1 + m_2} \sin\theta^{CM} ,$$

$$\begin{aligned}\left(\boldsymbol{v}_{2f}^{L}\right)_z &= \left(\boldsymbol{v}_{2f}^{CM}\right)_z + \left(\boldsymbol{v}_{CM}\right)_z \\ &= -\frac{m_1 v_{1i}^{L}}{m_1 + m_2} \cos\theta^{CM} + \frac{m_1 v_{1i}^{L}}{m_1 + m_2} \\ &= \frac{(m_1 - m_1 \cos\theta^{CM})}{m_1 + m_2} v_{1i}^{L}\end{aligned}$$

where

$$\boldsymbol{v}_{CM} = \frac{m_1}{m_1 + m_2} \boldsymbol{v}_{1i}^{L} .$$

The scattering angle in the laboratory scheme:

$$\tan\theta_1^{L} = \frac{\left(\boldsymbol{v}_{1f}^{L}\right)_y}{\left(\boldsymbol{v}_{1f}^{L}\right)_z} = \frac{m_2 \sin\theta^{CM}}{m_1 + m_2 \cos\theta^{CM}} ,$$

$$\tan\theta_2^{L} = \frac{\left(\boldsymbol{v}_{2f}^{L}\right)_y}{\left(\boldsymbol{v}_{2f}^{L}\right)_z} = \frac{m_1 \sin\theta^{CM}}{m_1 - m_1 \cos\theta^{CM}} = \frac{\sin\theta^{CM}}{1 - \cos\theta^{CM}} .$$

Center of mass frame. $v_{1i}^{CM} = v_{1i}^{L} - v_{CM} = v_{1i}^{L} - \frac{m_1}{m_1+m_2} v_{1i}^{L} = \frac{m_2}{m_1+m_2} v_{1i}^{L}$,

$$\left|\boldsymbol{v}_{1f}^{CM}\right| = \frac{m_2\left|\boldsymbol{v}_{1i}^{L}\right|}{m_1+m_2} = \left|\boldsymbol{v}_{1i}^{CM}\right| \quad , \quad \left|\boldsymbol{v}_{2f}^{CM}\right| = \frac{m_1\left|\boldsymbol{v}_{1i}^{L}\right|}{m_1+m_2} = v_{CM} \quad . \quad \left|\boldsymbol{v}_{1i}^{L}\right| = \left|\boldsymbol{v}_{1f}^{CM}\right| + \left|\boldsymbol{v}_{2f}^{CM}\right| \quad .$$

$$\boldsymbol{v}_{2f}^{CM} = -\frac{m_1}{m_2}\boldsymbol{v}_{1f}^{CM} . \left|\boldsymbol{v}_{2f}^{CM}\right| = \frac{m_1}{m_2}\left|\boldsymbol{v}_{1f}^{CM}\right| = \frac{m_1}{m_1+m_2}\left|\boldsymbol{v}_{1i}^{L}\right| = v_{CM} .$$

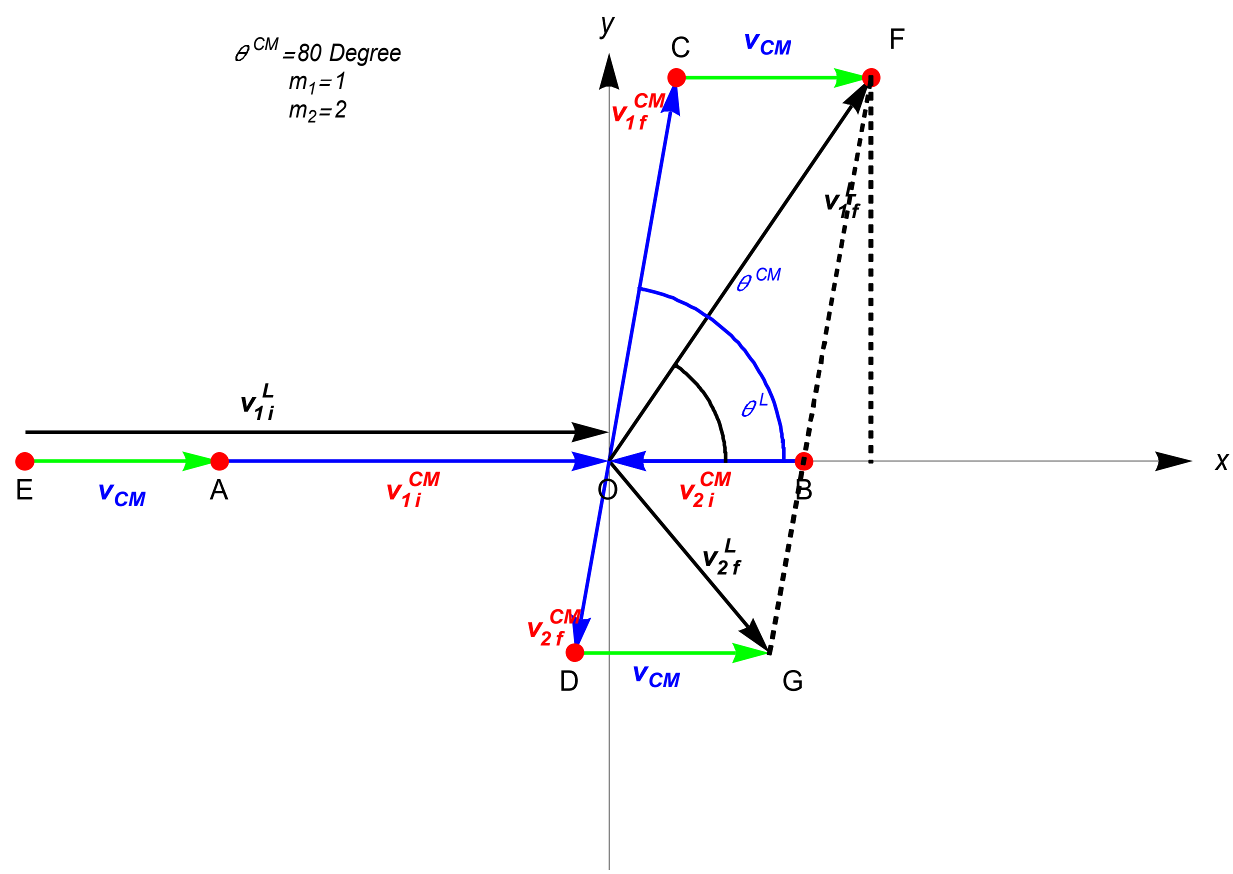
$$\boldsymbol{v}_{2i}^{CM} = -\frac{m_1}{m_2}\boldsymbol{v}_{1i}^{CM} = -\frac{m_1}{m_1+m_2}\boldsymbol{v}_{1i}^{L} = -\boldsymbol{v}_{CM} .$$

$$\boldsymbol{v}_{1f}^{L} = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM} , \; \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM} ,$$

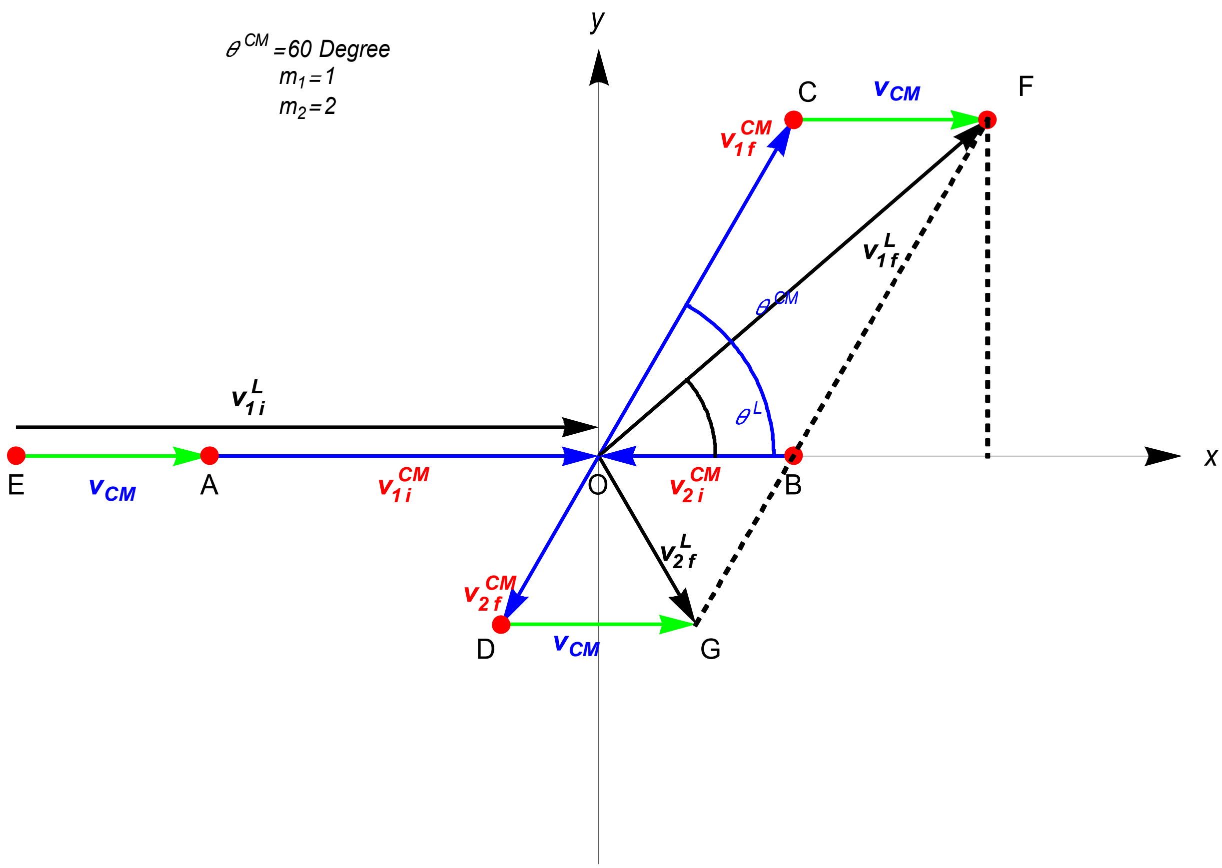
3. Example-1: $m_2/m_1 = 2$.

Here we assume that $m_2/m_1 = 2$.

(a) $\theta^{CM} = 80°$.

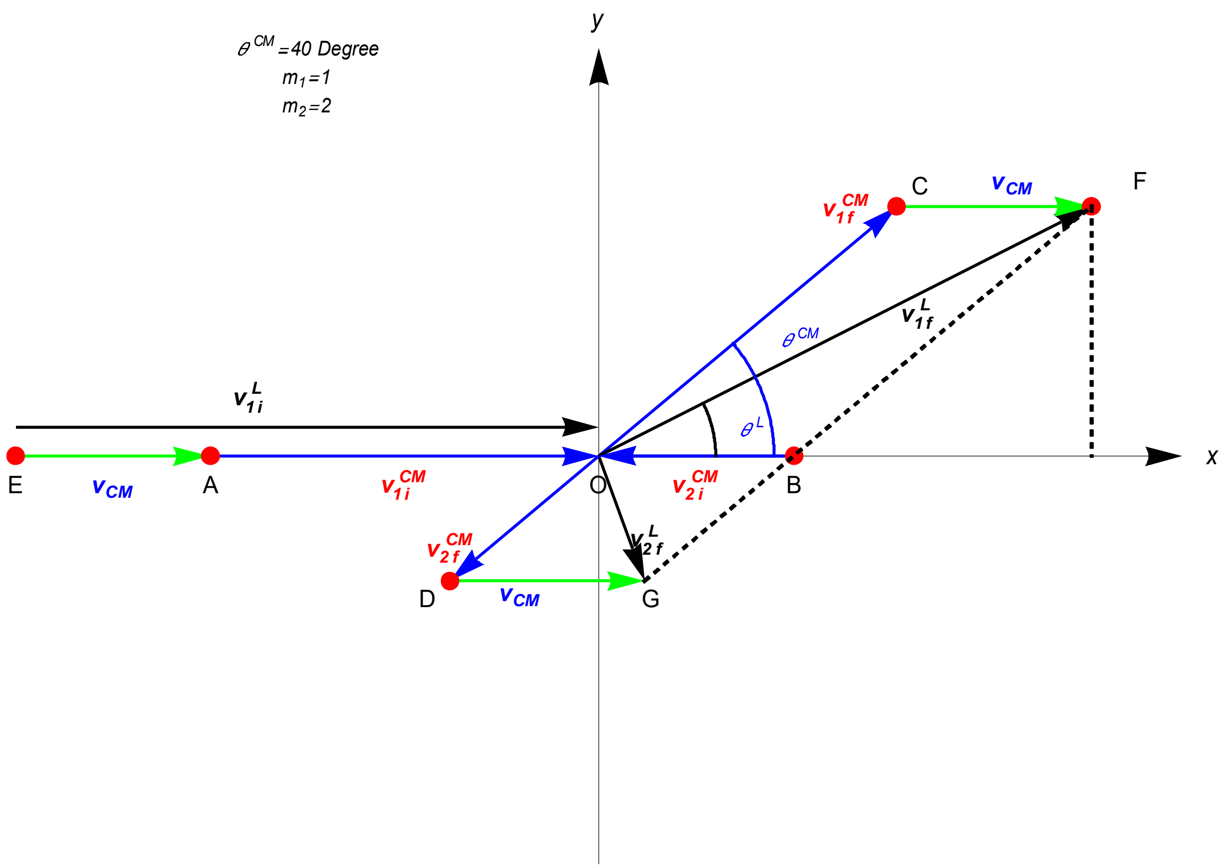


(b) $\theta^{CM} = 60°$.



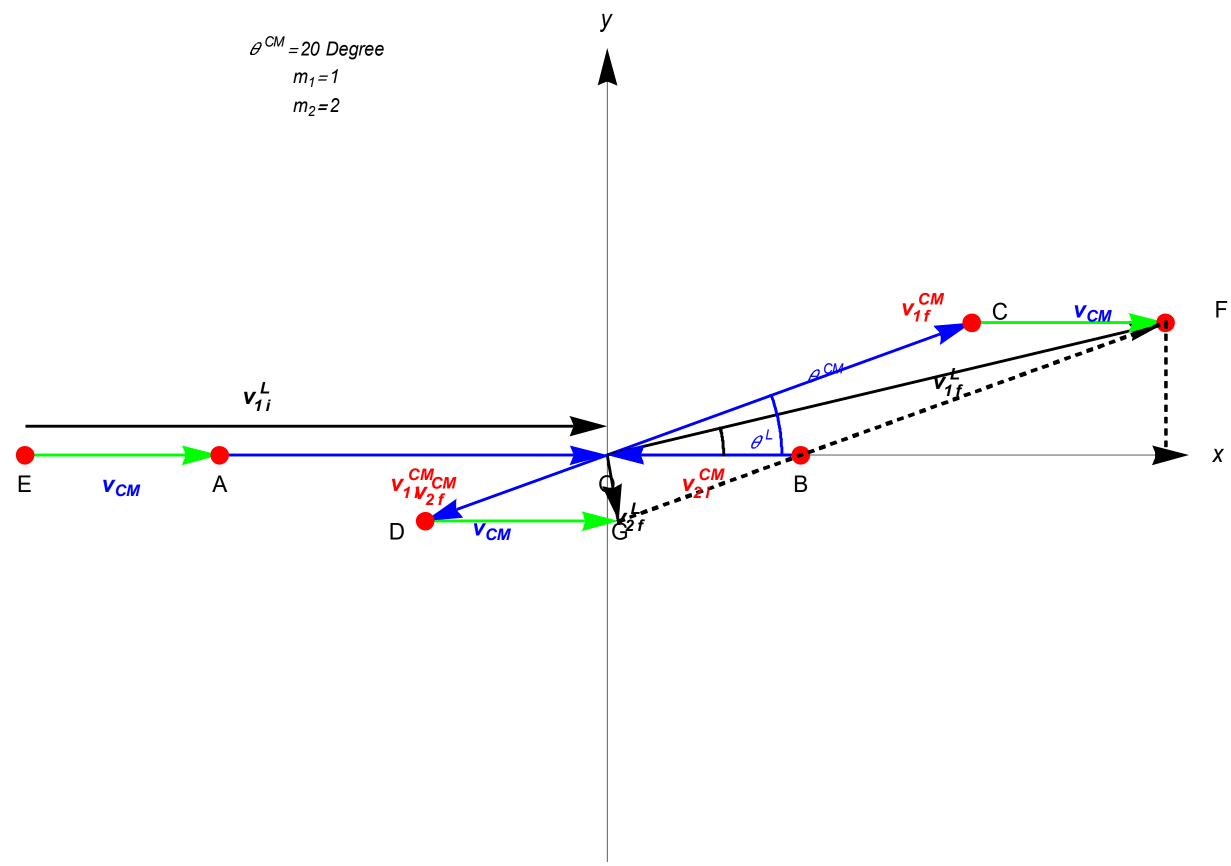
θ^{CM} =60 Degree
m_1=1
m_2=2
y
x
C
F
E
A
O
B
D
G
v_{CM}
v_{1f}^{CM}
v_{1f}^{L}
θ^{CM}
θ^{L}
v_{1i}^{L}
v_{1i}^{CM}
v_{2i}^{CM}
v_{2f}^{L}
v_{2f}^{CM}

(c) $\theta^{CM} = 40°$



y
θ^CM =40 Degree
m1=1
m2=2
C
F
v1f CM
vCM
v1f L
θ CM
v1i L
θ L
x
E
vCM
A
v1i CM
O
v2i CM
B
v2f CM
v2f L
D
vCM
G

(d) $\theta^{CM} = 20°$



3. Example-2: Identical particles ($m_2 = m_1$)

Here we consider the case of $m_2 = m_1$ (identical particles)

$$\left|\boldsymbol{v}_{1f}^{CM}\right| = \frac{\left|\boldsymbol{v}_{1i}^{L}\right|}{2}, \quad \left|\boldsymbol{v}_{2f}^{CM}\right| = \frac{\left|\boldsymbol{v}_{1i}^{L}\right|}{2} = v_{CM}. \quad \left|\boldsymbol{v}_{1i}^{L}\right| = \left|\boldsymbol{v}_{1f}^{CM}\right| + \left|\boldsymbol{v}_{2f}^{CM}\right|. \quad \boldsymbol{v}_{2f}^{CM} = -\boldsymbol{v}_{1f}^{CM}. \quad \boldsymbol{v}_{2i}^{CM} = -\boldsymbol{v}_{1i}^{CM}.$$

$$\boldsymbol{v}_{1f}^{L} = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}, \quad \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM}$$

In the laboratory scheme;

$$\overrightarrow{EO} = \boldsymbol{v}_{1i}^{L}, \qquad \overrightarrow{OF} = \boldsymbol{v}_{1f}^{L}, \qquad \overrightarrow{OG} = \boldsymbol{v}_{2f}^{L}.$$

$$\boldsymbol{v}_{1i}^{L} = \boldsymbol{v}_{1i}^{CM} + \boldsymbol{v}^{CM}, \qquad \boldsymbol{v}_{1f}^{L} = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}^{CM}, \qquad \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}^{CM}$$

In the center of mass scheme:

$\overrightarrow{AO} = \boldsymbol{v}_{1i}^{CM}$, $\overrightarrow{BO} = \boldsymbol{v}_{2i}^{L}$, $\overrightarrow{OC} = \boldsymbol{v}_{1f}^{CM}$, $\overrightarrow{OD} = \boldsymbol{v}_{2f}^{CM}$

$\boldsymbol{v}^{CM}$: center of mass velocity

$$\overrightarrow{EA} = \overrightarrow{CF} = \overrightarrow{DG} = \boldsymbol{v}^{CM}$$

((**Elastic scattering**))

Here we show some examples for the laboratory and center of mass schemes for the elastic scattering of two particles with the same mass, where $\boldsymbol{v}_{1i}^{L}$ and θ^{CM} are given and $\boldsymbol{v}_{2i}^{L} = 0$. We note that the points A, B, C, and D are on the same circle with the radius ($|\boldsymbol{v}_{CM}| = |\boldsymbol{v}_{1i}^{L}|/2$) centered at the origin O. The vectors $\boldsymbol{v}_{1f}^{L}$ and $\boldsymbol{v}_{2f}^{L}$ are denoted by the vectors $\overline{\mathrm{OF}}$ and $\overline{\mathrm{OG}}$, respectively. The angle $\angle\mathrm{FOG} = 90°$. Because of the momentum conservation ($\boldsymbol{v}_{1f}^{L} + \boldsymbol{v}_{2f}^{L} = \boldsymbol{v}_{1i}^{L}$), we have $\overrightarrow{\mathrm{OF}} + \overrightarrow{\mathrm{OG}} = \overrightarrow{\mathrm{OH}}$. The point H is on the same circle with the radius ($|\boldsymbol{v}_{1i}^{L}|$) centered at the origin. Since $\angle\mathrm{FOG} = 90°$ (which is independent of θ^{CM}), the points O, E, and F lie on the same circle with the radius $|\boldsymbol{v}_{CM}|$.

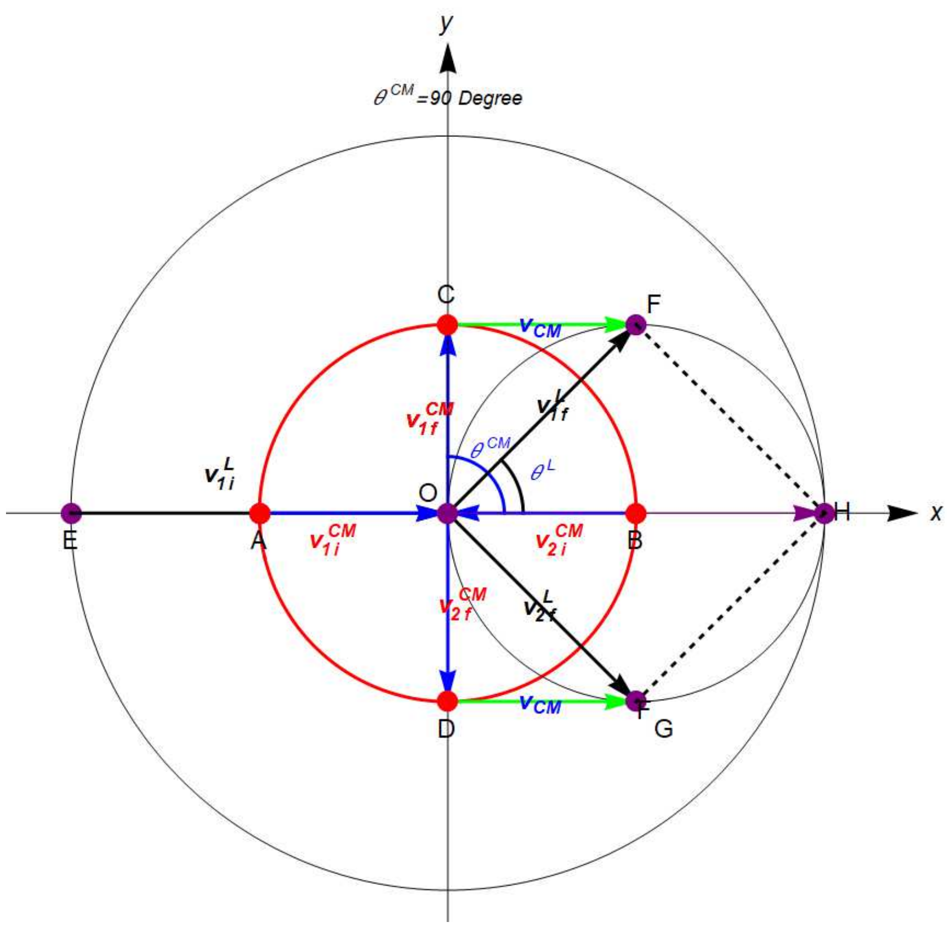


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes. $\theta^{CM} = 90°$. $\theta^{L} = 45°$.

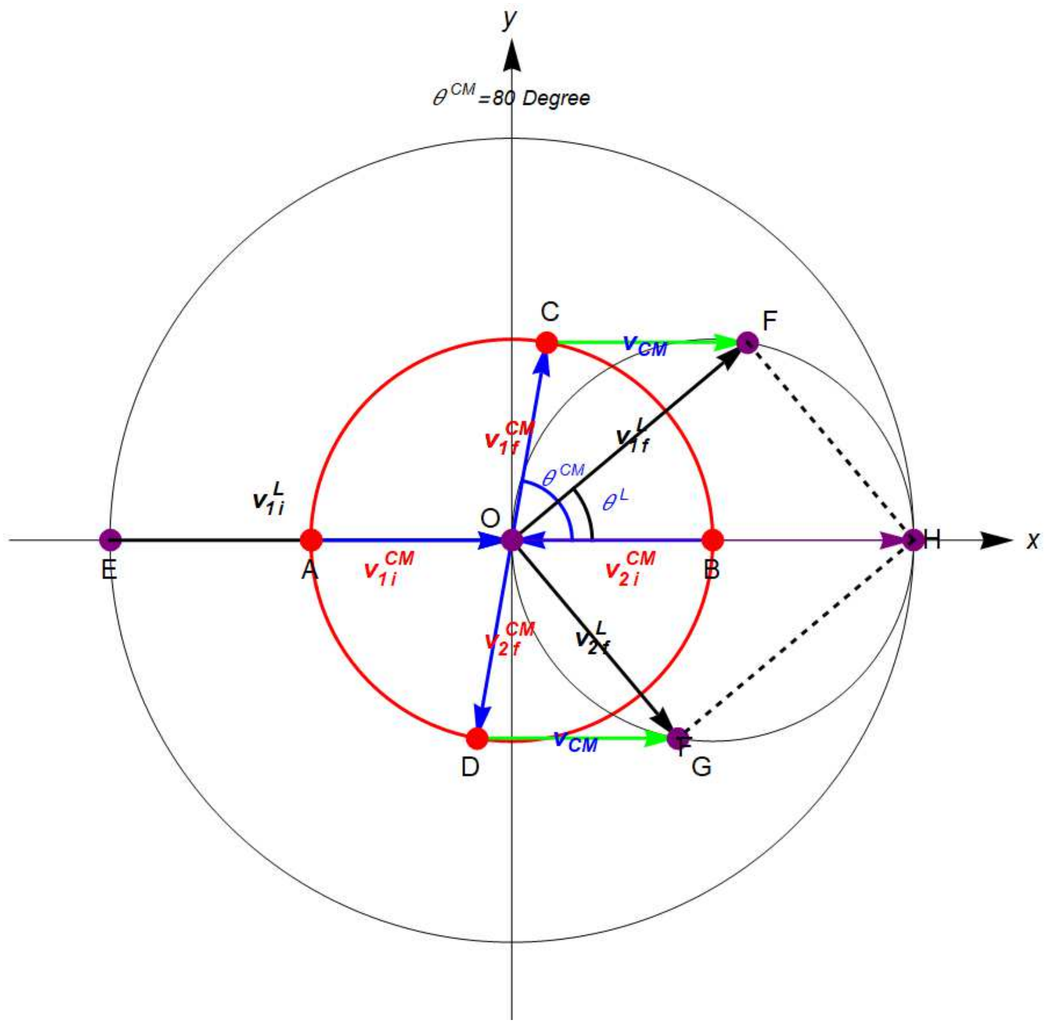


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes. $\theta^{CM} = 80°$. $\theta^{L} = 40°$.

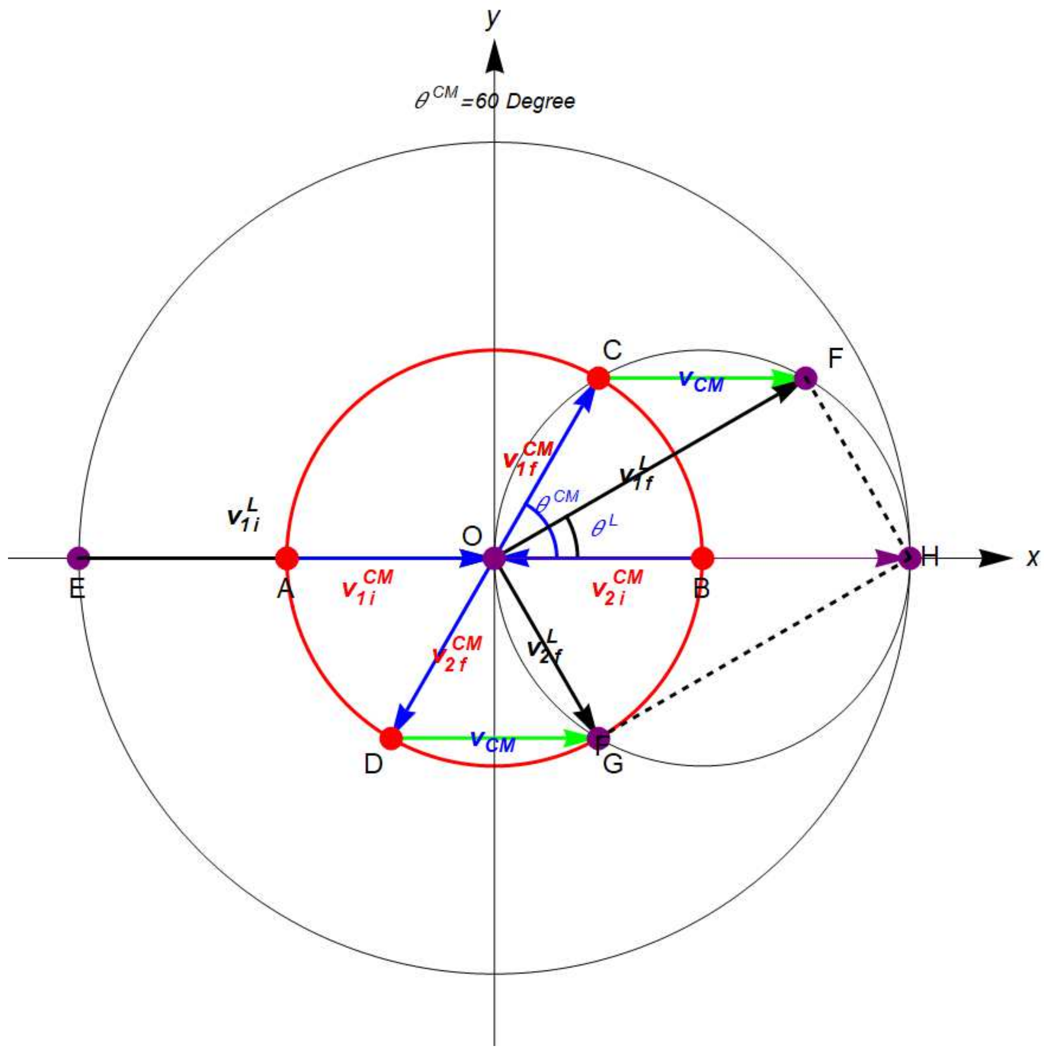


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes. $\theta^{CM} = 60°$. . $\theta^{L} = 30°$

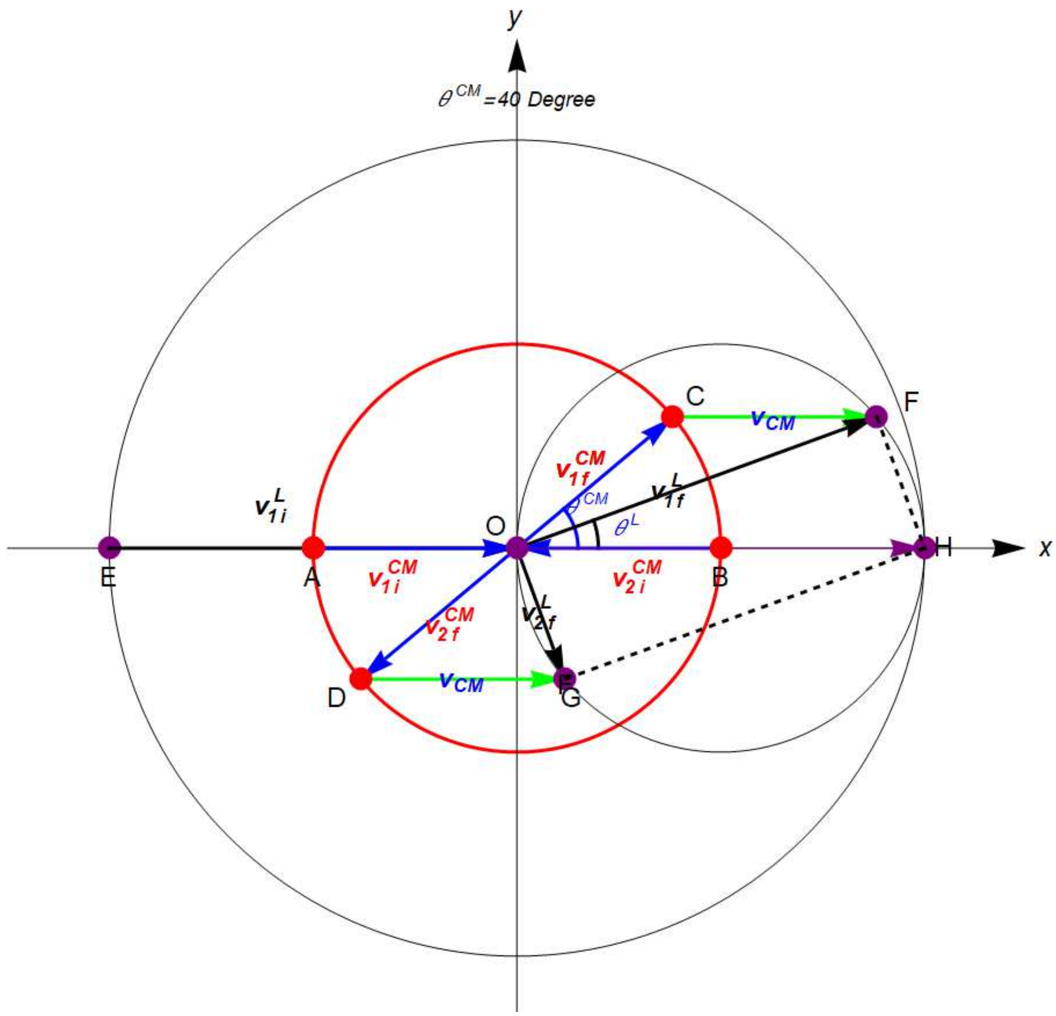


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes. $\theta^{CM} = 40°$. $\theta^{L} = 20°$.

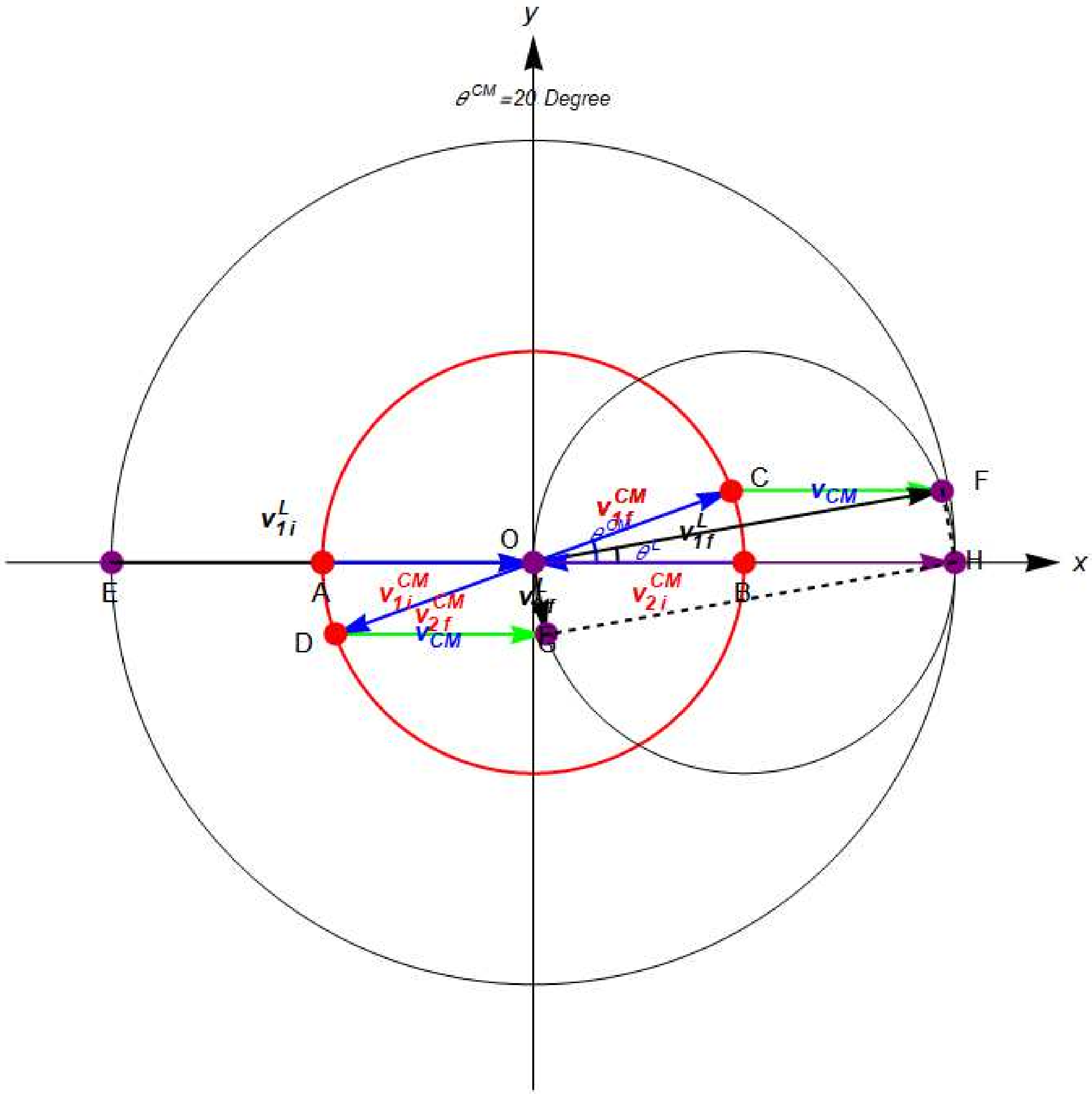


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes. $\theta^{CM} = 20°$. $\theta^{L} = 10°$.

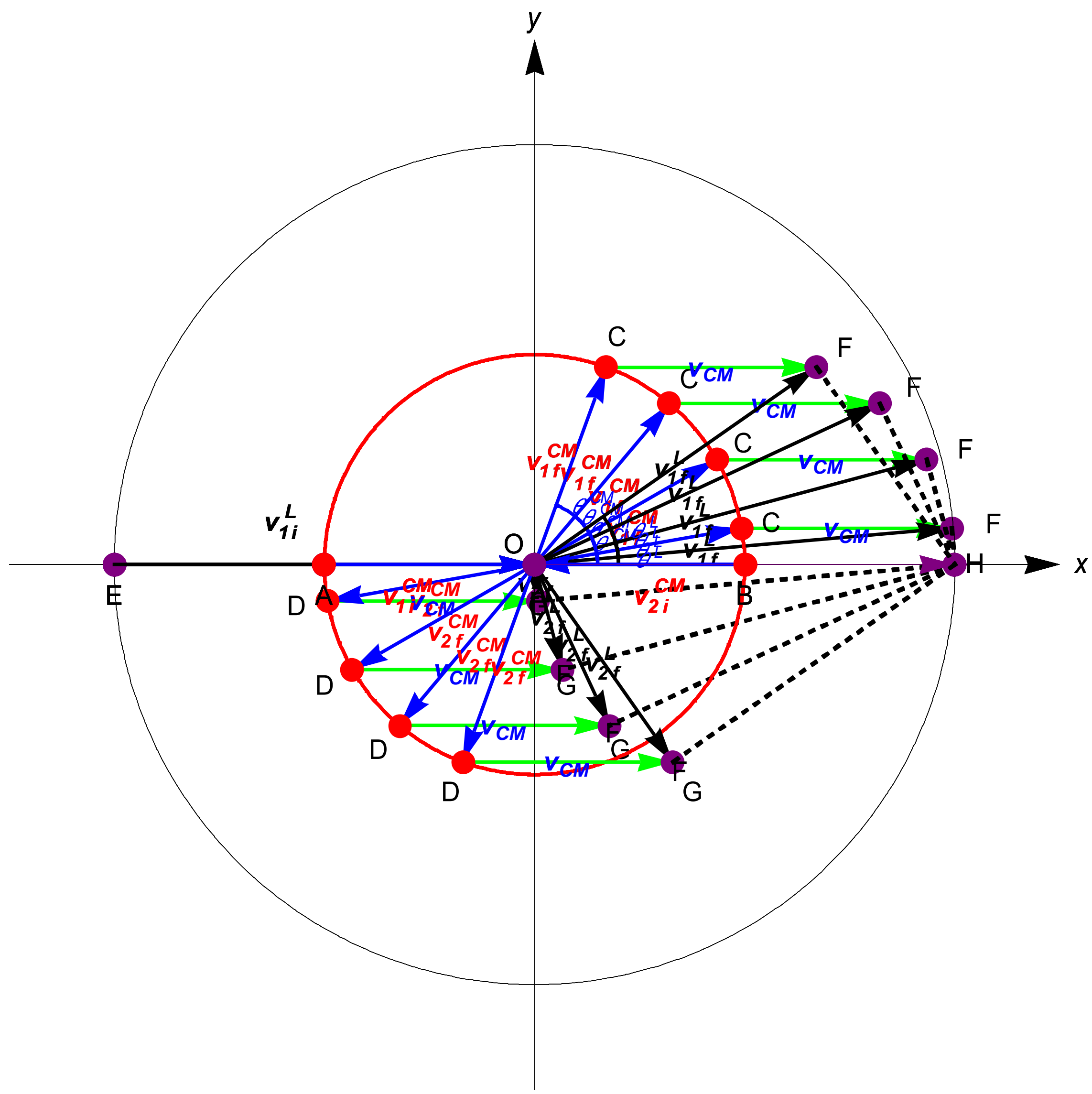


Fig. Scattering of two identical particles (the same mass) in the laboratory and center of mass schemes, θ^{CM} is changed as a parameter. $\theta^{L} = \theta^{CM}/2$.

4. Differential cross sections in the laboratory frame and the center of mass frame

The relation between the cross sections in the center of mass frame and the laboratory frame can be obtained from the fact that the same particles which go into the solid angle $d\Omega^{CM}$ at θ^{CM} in the center of mass frame go into the solid angle $d\Omega^{L}$ at θ^{L} in the laboratory frame. Therefore, we have

$$\sigma_L(\theta_1^L)\sin\theta_1^L d\theta_1^L = \sigma_{CM}(\theta^{CM})\sin\theta^{CM} d\theta^{CM} ,$$

or

$$\sigma_L(\theta_1^L) = \frac{\sin\theta^{CM} d\theta^{CM}}{\sin\theta_1^L d\theta_1^L}\sigma_{CM}(\theta^{CM}) .$$

Using the relation

$$\tan\theta_1^L = \frac{m_2 \sin\theta^{CM}}{m_1 + m_2\cos\theta^{CM}} ,$$

we have

$$\sigma_L(\theta_1^L) = \frac{(1+\gamma^2+2\gamma\cos\theta^{CM})^{3/2}}{\left|1+\gamma\cos\theta^{CM}\right|}\sigma_{CM}(\theta^{CM}),$$

where

$$\gamma = \frac{m_1}{m_2} .$$

The kinetic energy of the laboratory frame is

$$K_L = \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^L\right)^2 .$$

In the center of mass frame, it is

$$K_{CM} = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(\boldsymbol{v}_{1i}^L\right)^2 .$$

Then we have

$$K_{CM} = \frac{m_2}{m_1+m_2}K_L .$$

((**Mathematica**))

```
Clear["Global`"]; θL = ArcTan[m2 Sin[θ] / (m1 + m2 Cos[θ])];

f1 = 1/D[θL, θ] Sin[θ]/Sin[θL] // Simplify[#, (m1 + m2 Cos[θ]) > 0] & ;

f2 = f1 /. {m1 → γ m2} // Simplify[#, m2 > 0] &
```

$$\frac{\left(1+\gamma^2+2\gamma\,\mathrm{Cos}[\theta]\right)^{3/2}}{1+\gamma\,\mathrm{Cos}[\theta]}$$

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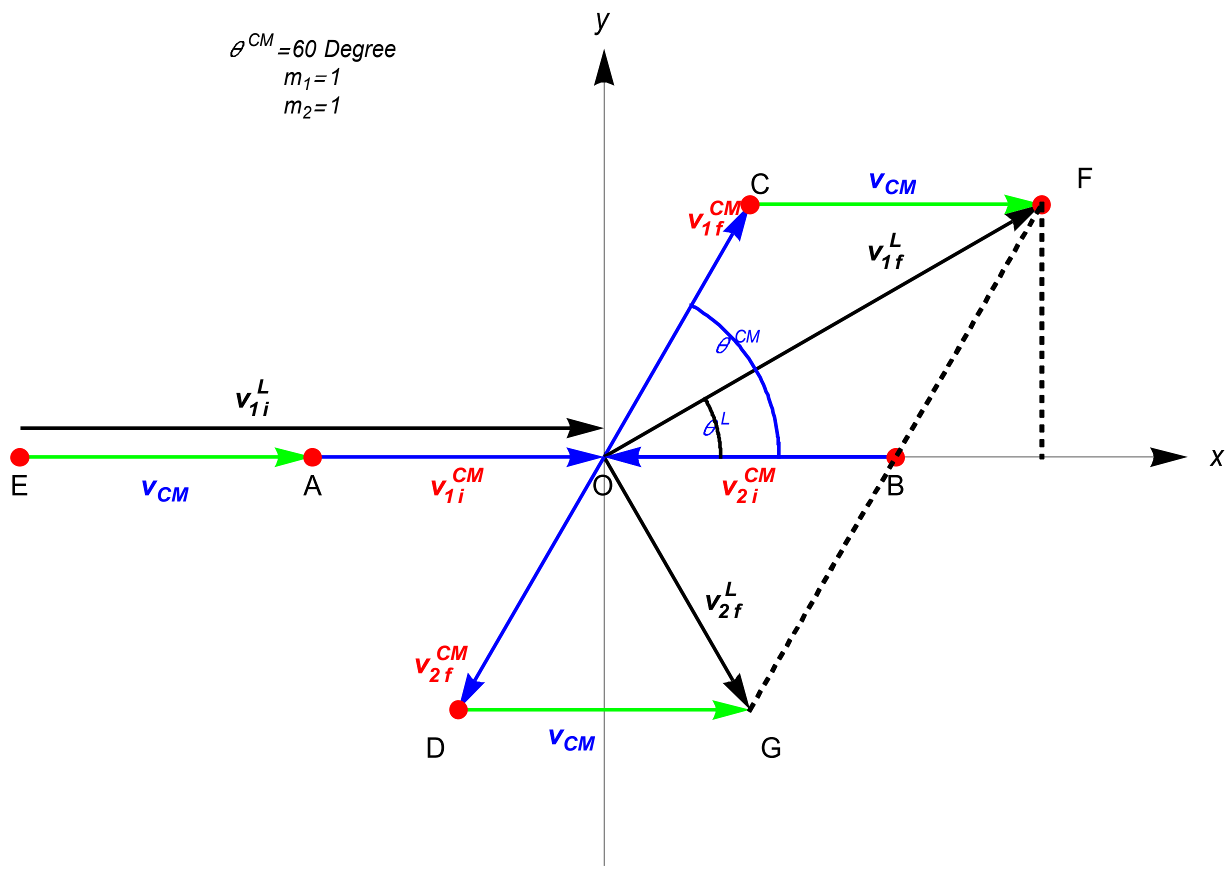
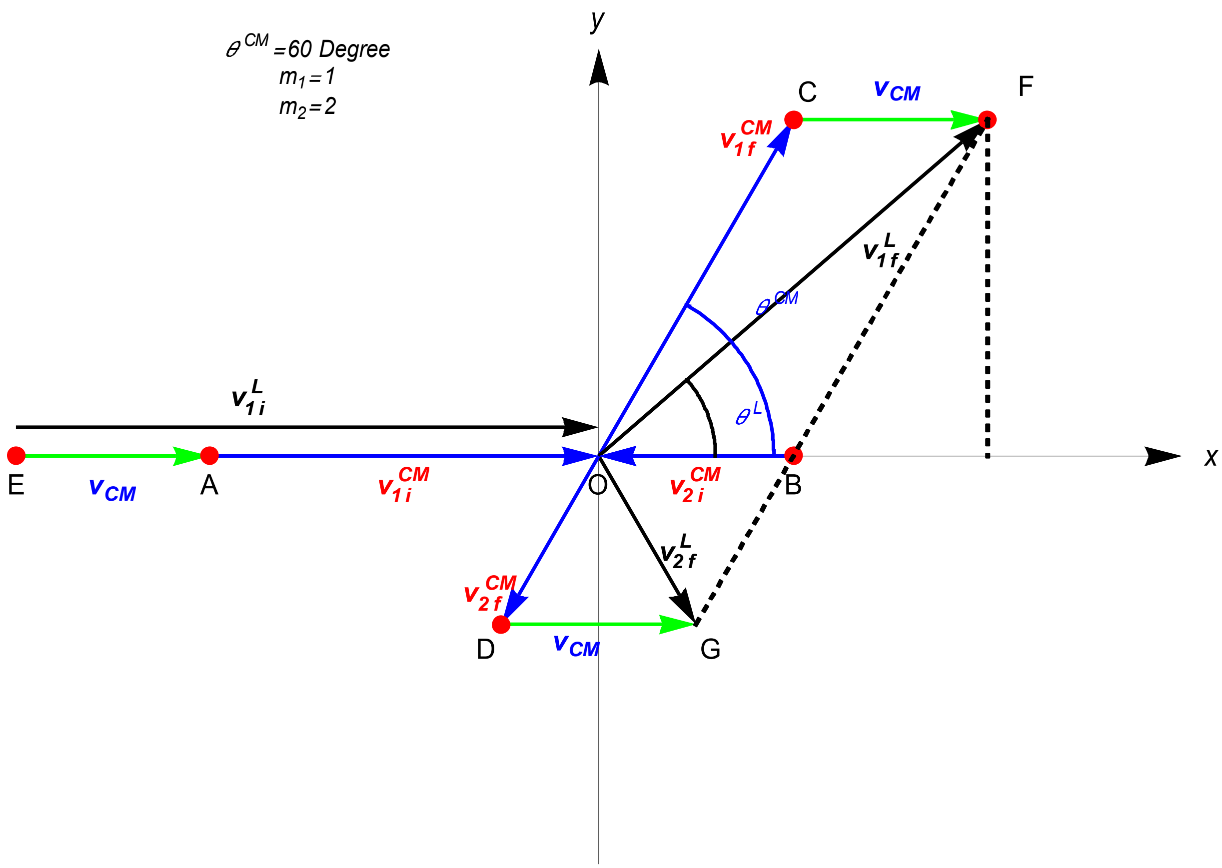
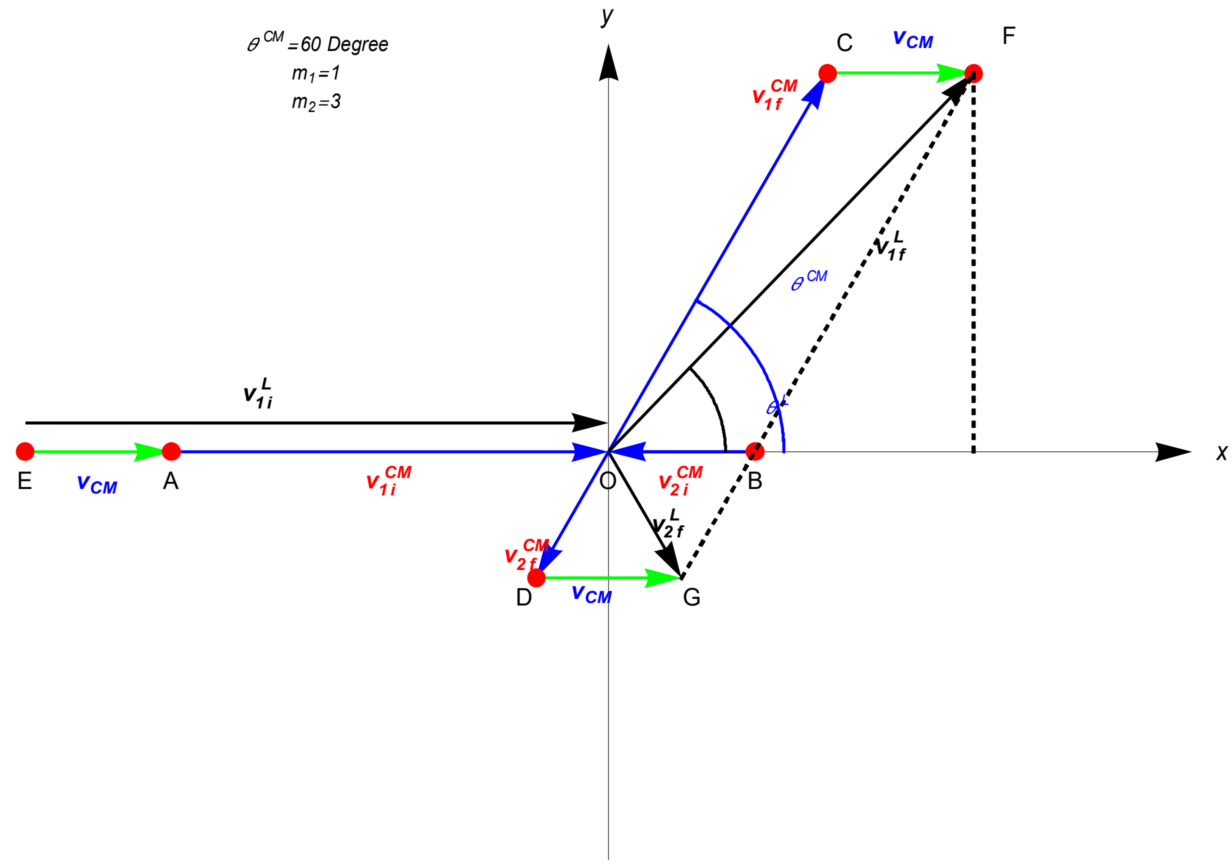
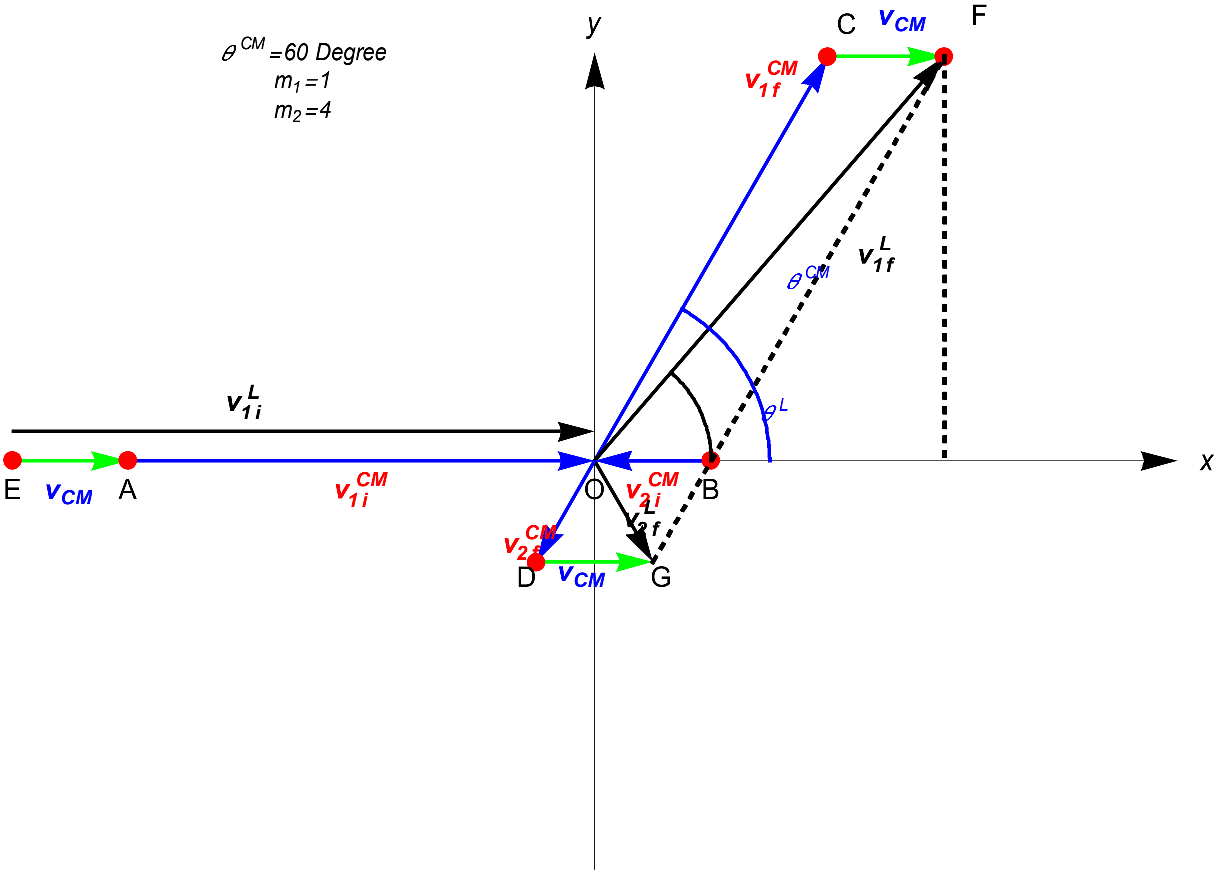
APPENDIX-I: Examples.


θ^CM =60 Degree
m1=1
m2=1
y
x
C
F
vCM
v1f CM
v1f L
θ CM
θ L
v1i L
E
A
O
B
vCM
v1i CM
v2i CM
v2f L
v2f CM
D
G
vCM



θ^{CM} =60 Degree
m_1=1
m_2=2
y
x
C
F
E
A
O
B
D
G
v_{CM}
v_{1f}^{CM}
v_{1f}^{L}
θ^{CM}
θ^{L}
v_{1i}^{L}
v_{1i}^{CM}
v_{2i}^{CM}
v_{2f}^{L}
v_{2f}^{CM}


θ^CM = 60 Degree
m1 = 1
m2 = 3
y
x
C
F
vCM
v1f^CM
v1f^L
θ^CM
θ^L
v1i^L
E
vCM
A
v1i^CM
O
v2i^CM
B
v2f^L
v2f^CM
D
vCM
G

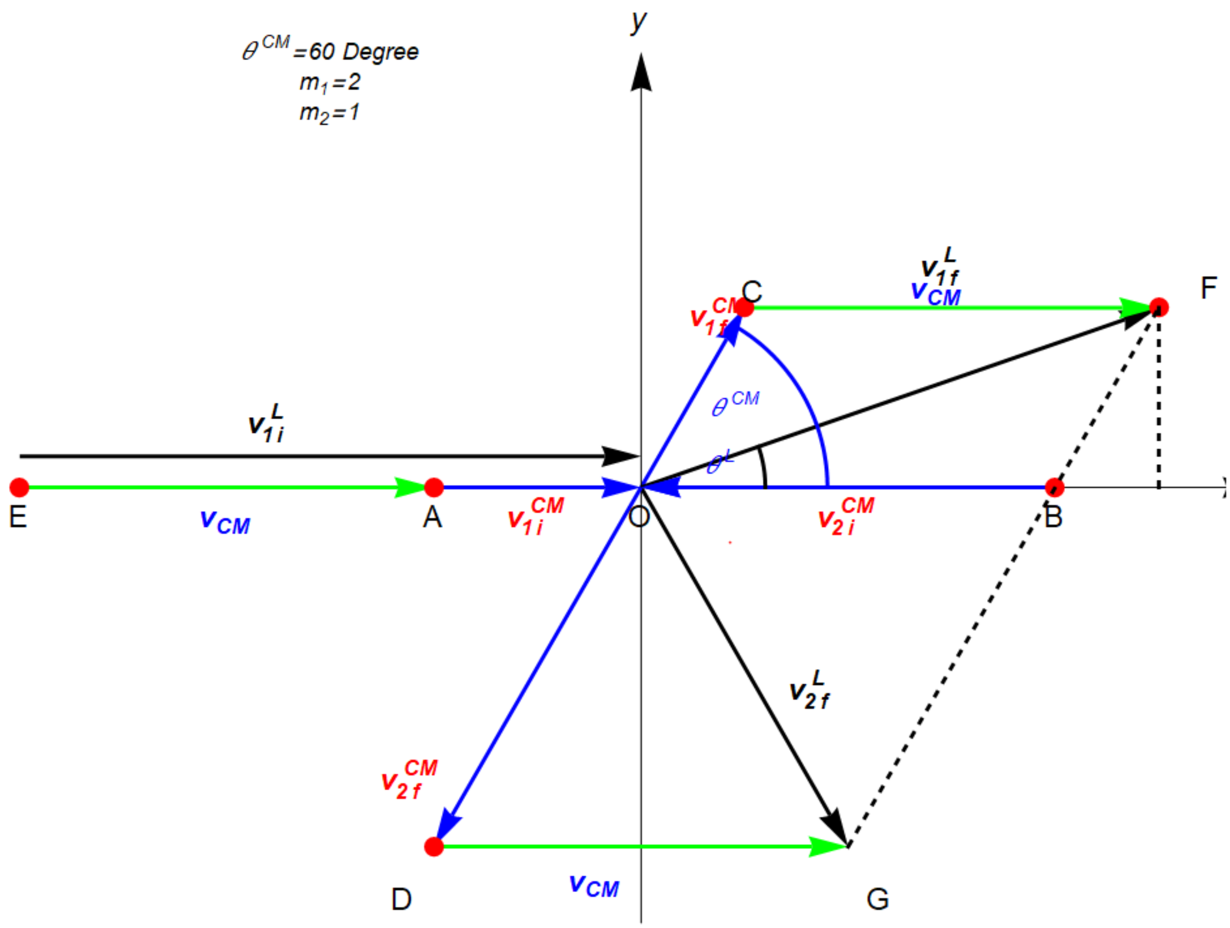


APPENDIX-II Mathematica

```
Clear["Global`*"];
CMSCHEME[m1_, m2_, ϕ1_] :=
 Module[{O1, E11, δ1, E12, X1, M1, Y1, AV, BV, S1, B1, C1, χ0, VCM, D1, G1,
   F1, g1, g2, h0, h1, h2, h3}, O1 = {0, 0};
  ϕ = ϕ1;
  VCM = {m1/(m1 + m2), 0};
  AV = m1/(m1 + m2);
  BV = m2/(m1 + m2); E11 = {-1, 0};
  E12 = E11 + VCM;
  R1[θ1_] := {Cos[θ1], Sin[θ1]};
  S1 = R1[ϕ];
  C1 = BV S1;
  D1 = -AV S1;
  G1 = D1 + VCM;
  B1 = VCM;
  F1 = C1 + VCM;
  δ1 = {0, 0.05};
  M1 = BV S1 + VCM; χ0 = ArcTan[M1[[2]]/M1[[1]]];
```

```
h1 = Graphics[{Arrowheads[0.03], Black, Thick, Arrow[{E11 + δ1, δ1}], Blue,
   Arrow[{O1, C1}] , Thin, Black, Arrow[{{0, -0.7}, {0, 0.7}}],
   Arrow[{{-1, 0}, {1, 0}}], Thick, Arrow[{O1, G1}], Arrow[{O1, F1}],
   Green, Thick, Arrow[{C1, F1}], Arrow[{D1, G1}], Arrow[{E11, E12}],
   Blue, Arrow[{O1, D1}], Arrow[{B1, O1}], Arrow[{E12, O1}], Blue,
   Thick, Arrow[{B1, O1}], Red, PointSize[0.015], Point[D1], Point[E12],
   Point[E11], Point[C1], Point[F1], Point[B1], Dashed, Black,
   Line[{G1, F1}], Line[{F1, {F1[[1]], 0}}]}];
h2 = Graphics[{Text[Style["v^CM_1i", Red, 15, Italic, Bold], {-0.5 BV, -0.05}],
   Text[Style["v^CM_2i", Red, 15, Italic, Bold], {0.5 AV, -0.05}],
   Text[Style["v^L_1i", Black, 15, Italic, Bold], {-0.6, 0.1}],
   Text[Style["v^L_1f", Black, 15, Italic, Bold], 0.6 R1[0.6 ϕ]],
   Text[Style["v^CM_1f", Red, 15, Italic, Bold], 0.9 BV R1[ϕ + 5 °]],
   Text[Style["v^CM_2f", Red, 15, Italic, Bold], -0.9 AV R1[ϕ - 8 °]],
   Text[Style["v^L_2f", Black, 15, Italic, Bold], 0.5 (-AV R1[ϕ + 20 °] + VCM)],
   Text[Style["θ^CM=" <> ToString[ϕ], Black, 12, Italic], {-0.5, 0.7}],
   Text[Style["m1=" <> ToString[m1], Black, 12, Italic], {-0.5, 0.65}],
   Text[Style["m2=" <> ToString[m2], Black, 12, Italic], {-0.5, 0.60}]}];
h3 = Graphics[{Text[Style["E", Black, 15], E11 + {0, -0.05}],
   Text[Style["A", Black, 15], {-BV, -0.05}],
   Text[Style["O", Black, 15], {0, -0.05}],
   Text[Style["B", Black, 15], VCM + {0.0, -0.05}],
   Text[Style["C", Black, 15], 1.08 BV S1],
   Text[Style["D", Black, 15], -1.15 AV S1],
   Text[Style["F", Black, 15], 1.1 (BV S1 + VCM)],
   Text[Style["G", Black, 15], 1.15 (-AV S1 + VCM)],
   Text[Style["vCM", Blue, Italic, Bold, 15], 1.1 (BV S1 + 0.4 VCM)],
   Text[Style["vCM", Blue, Italic, Bold, 15], 1.1 (- AV S1 + 0.4 VCM)],
   Text[Style["vCM", Blue, Italic, Bold, 15], (E11 + VCM / 2) + {0, -0.05}],
   Text[Style["θ^L", Blue, Italic, 12], 0.4 BV R1[ϕ / 4]],
   Text[Style["θ^CM", Blue, Italic, 12], 0.6 BV R1[(ϕ / 2) + 10 °]],
   Text[Style["x", Black, Italic, 15], {1.05, 0}],
   Text[Style["y", Black, Italic, 15], {0, 0.75}]}];
g1 = ParametricPlot[0.2 R1[θ], {θ, 0, χ0}, PlotStyle → {Black, Thick}];
g2 = ParametricPlot[0.3 R1[θ], {θ, 0, ϕ}, PlotStyle → {Blue, Thick}];
h0 = Show[ h1, h2, h3, g1, g2, PlotRange → All];
```

```
CMSCHEME[2, 1, 60 °]
```



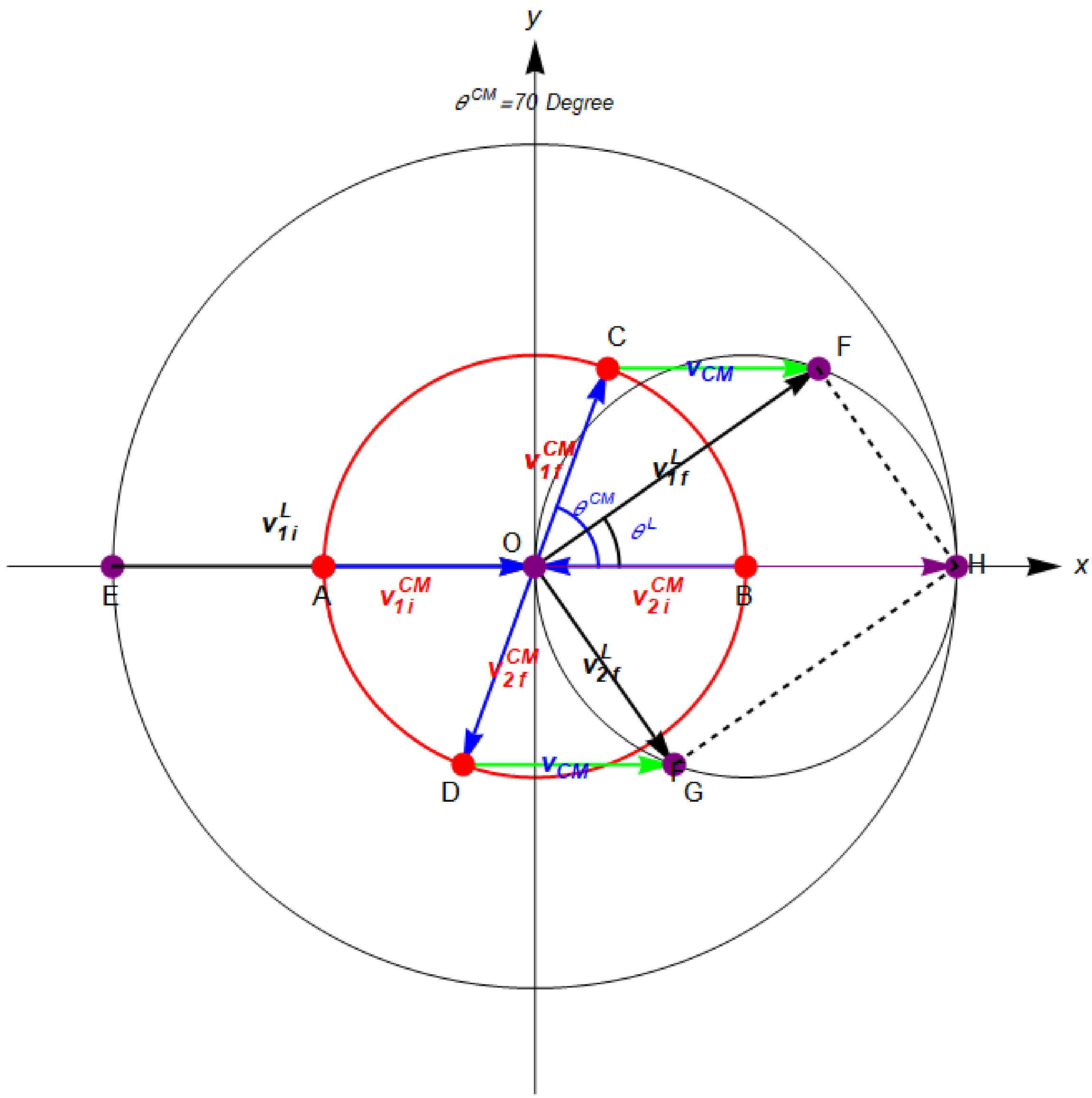
APPENDIX-III Mathemtica (identical mass)

Identical mass; $m_1 = m_2$

```
Clear["Global`*"];
CMSCHEME[ϕ_] := Module[{O1, X1, Y1, S1, g1, g2, g3, g4, h1, h2, h3},
  O1 = {0, 0};
  X1 = {1, 0}; Y1 = {0, 1};
  R1[θ1_] := {Cos[θ1], Sin[θ1]};
  S1 = R1[ϕ];
  g1 = ParametricPlot[R1[θ], {θ, 0, 2 π}, PlotStyle → {Red, Thick},
    Axes → False];
  g2 = ParametricPlot[2 R1[θ], {θ, 0, 2 π}, PlotStyle → {Black, Thin}];
  g3 = ParametricPlot[0.4 R1[θ], {θ, 0, ϕ / 2}, PlotStyle → {Black, Thick}];
  g4 = ParametricPlot[0.3 R1[θ], {θ, 0, ϕ}, PlotStyle → {Blue, Thick}];
  g5 = ParametricPlot[{1, 0} + R1[θ], {θ, 0, 2 π}, PlotStyle → {Black, Thin}];
  h1 = Graphics[{Arrowheads[0.03], Black, Thick, Arrow[{-X1, O1}],
     Blue, Arrow[{O1, R1[ϕ]}] , Thin, Black, Arrow[{{0, -2.5}, {0, 2.5}}],
     Arrow[{{-2.5, 0}, {2.5, 0}}], Green, Thick, Arrow[{S1, S1 + X1}],
     Arrow[{-S1, - S1 + X1}], Black, Arrow[{O1, S1 + X1}], Arrow[{O1, -S1 + X1}],
     Blue, Arrow[{X1, O1}], Blue, Thick, Arrow[{-X1, O1}] ,
     Arrow[{O1, -S1}], Purple, PointSize[0.02], Point[-2 X1], Point[O1],
     Point[2 X1], Point[S1 + X1], Point[-X1], Point[-S1 + X1], Thin,
     Arrow[{O1, 2 X1}], Black, Thick, Line[{-2 X1, -X1}], Black,
     Dashed, Line[{S1 + X1, 2 X1}], Line[{2 X1, -S1 + X1}],
     Text[Style["v1f^CM", Black, 15, Italic, Bold], 0.5 R1[ϕ + 10 °]],
     Red, PointSize[0.02], Point[-X1], Point[X1], Point[S1], Point[-S1]}];
```

```
h2 = Graphics[{Text[Style["v^CM_1 i", Red, 15, Italic, Bold], {-0.6, -0.15}],
   Text[Style["v^CM_2 i", Red, 15, Italic, Bold], {0.6, -0.15}],
   Text[Style["v^L_1 i", Black, 15, Italic, Bold], {-1.2, 0.2}],
   Text[Style["v^L_1 f", Black, 15, Italic, Bold], 0.8 R1[ϕ / 2]],
   Text[Style["v^CM_1 f", Red, 15, Italic, Bold], 0.5 R1[ϕ + 10 °]],
   Text[Style["v^CM_2 f", Red, 15, Italic, Bold], -0.5 R1[ϕ + 10 °]],
   Text[Style["v^L_2 f", Black, 15, Italic, Bold], 0.5 (-S1 + X1)],
   Text[Style["θ^CM=" <> ToString[ϕ], Black, 12, Italic], {0, 2.2}]}];
h3 = Graphics[{Text[Style["E", Black, 15], {-2, -0.15}],
   Text[Style["A", Black, 15], {-1, -0.15}],
   Text[Style["O", Black, 15], {-0.10, 0.10}],
   Text[Style["B", Black, 15], {1, -0.15}],
   Text[Style["C", Black, 15], 1.15 S1],
   Text[Style["D", Black, 15], -1.15 S1],
   Text[Style["F", Black, 15], 1.05 (-S1 + X1)] ,
   Text[Style["F", Black, 15], 1.1 (S1  + X1)],
   Text[Style["H", Black, 15], 1.05 (2 X1)],
   Text[Style["G", Black, 15], 1.15 (-S1 + X1)],
   Text[Style["v_CM", Blue, Italic, Bold, 15], S1 + X1 / 2],
   Text[Style["v_CM", Blue, Italic, Bold, 15], - S1 + X1 / 2],
   Text[Style["θ^L", Blue, Italic, 12], 0.55 R1[ϕ / 4]],
   Text[Style["θ^CM", Blue, Italic, 12], 0.4 R1[(ϕ / 2) + 10 °]],
   Text[Style["x", Black, Italic, 15], {2.6, 0}],
   Text[Style["y", Black, Italic, 15], {0, 2.6}]}];
```

```
CMSCHEME[70 °]
```



y
θ^{CM} =70 Degree
C
F
v_{CM}
v_{1f}^{CM}
v_{1f}^{L}
θ^{CM}
θ^{L}
v_{1i}^{L}
O
E
A
v_{1i}^{CM}
v_{2i}^{CM}
B
H
x
v_{2f}^{CM}
v_{2f}^{L}
D
v_{CM}
G