Quantization of electromagnetic field

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: May 01, 2021, revised June 06, 2023)

Here we discuss the quantization of the electromagnetic (EM) field. This field consists of discrete photons. Photon are bosons with zero mass; in free space they can carry linear momentum $\hbar\boldsymbol{k}$ and energy $\hbar\omega_k = \hbar ck$; and simultaneously with sharp linear momentum, they can have a definite value $+\hbar$ or $-\hbar$ for the component $\hat{J}_z$ (helicity) of spin angular momentum along the direction of propagation. The orbital angular momentum of photon will be also discussed.

1 Maxwell equation (in cgs unit)

The electromagnetic field in free space (in vacuum) is determined by the Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{1}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \nabla \cdot \boldsymbol{E} = 0, \tag{2}$$

where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. Here we introduce the vector potential $\boldsymbol{A}$ and the scalar potential ϕ.

2. The vector potential and scalar potential

From Eq.(1), we have

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{3}$$

where $\boldsymbol{A}$ is the vector potential. Since

$$\nabla \times (\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}) = 0,$$

we get

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi, \tag{4}$$

where ϕ is the scalar potential. We have

$$\begin{aligned}\nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) \\ &= \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \\ &= \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \\ &= \frac{1}{c}\frac{\partial}{\partial t}(-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi) \\ &= -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\frac{1}{c}\frac{\partial\phi}{\partial t}\end{aligned}$$

or

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = \nabla(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}). \qquad (5)$$

We also have

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot (-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi) = -\nabla^2\phi - \frac{1}{c}\nabla \cdot \frac{\partial \boldsymbol{A}}{\partial t} = 0,$$

or

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = 0. \qquad (6)$$

3. Gauge transformation

We assume the gauge transformation,

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi, \qquad \phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t},$$

where χ is an arbitrary function of $\boldsymbol{r}$ and t. Then we get

$$\begin{aligned}\boldsymbol{E}' &= -\frac{1}{c}\frac{\partial \boldsymbol{A}'}{\partial t} - \nabla\phi' \\ &= -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{A} + \nabla\chi) - \nabla(\phi - \frac{1}{c}\frac{\partial\chi}{\partial t}) \\ &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi + (-\frac{1}{c}\frac{\partial}{\partial t}\nabla\chi + \frac{1}{c}\nabla\frac{\partial\chi}{\partial t}) \\ &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi = \boldsymbol{E}\end{aligned}$$

$$\boldsymbol{B}' = \nabla \times \boldsymbol{A}' = \nabla \times (\boldsymbol{A} + \nabla \chi) = \nabla \times \boldsymbol{A}.$$

In other words, ***E*** and ***B*** are invariant under the gauge transformation.

4. Coulomb gauge

Suppose we choose the Coulomb gauge such that

$$\nabla \cdot \boldsymbol{A} = 0,$$

or

$$\nabla \cdot \boldsymbol{A} = \nabla \cdot \boldsymbol{A}_0 - \nabla^2 \chi = 0.$$

But this is just Poisson's equation defining c in terms of the specified function, $\nabla \cdot \boldsymbol{A}_0$. Thus we prove that a gauge transformation that yields $\nabla \cdot \boldsymbol{A} = 0$ can always be carried out.

In vacuum space, the choice $\nabla \cdot \boldsymbol{A} = 0$ also leads to $\phi = 0$.

$$\nabla \cdot \boldsymbol{E} = \nabla \cdot (-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi) = -\nabla^2 \phi - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = 0$$

or

$$\nabla^2 \phi = 0$$

This is, however, simply Laplace's equation. It is well known that the only solution of this equation that is regular over all of space is $\phi = 0$. Thus we have

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}, \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \text{with } \nabla \cdot \boldsymbol{A} = 0.$$

Using these relations, we get the wave equation

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2},$$

or

$$\nabla^2 \boldsymbol{A} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2}. \qquad \phi = 0.$$

The set of solutions to the wave equation for $\boldsymbol{A}(\boldsymbol{r}, t)$ are naturally written as plane waves

$$A(r,t) = A(k)\exp[i(k\cdot r-\omega_k t)\,,$$

where the dispersion relation is given by

$$\omega_k = c|\boldsymbol{k}| = ck.$$

The Coulomb gauge condition implies that

$$\boldsymbol{k}\cdot \boldsymbol{A}(\boldsymbol{k}) = 0$$

In other words, $\boldsymbol{A}(\boldsymbol{r},t)$ is perpendicular to the propagation direction $\boldsymbol{k}$. The Coulomb gauge is frequently referred to as the transverse gauge.

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k},s}[c_{\boldsymbol{k},s}\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + c_{\boldsymbol{k},s}{}^{*}\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\,,$$

where we use the periodic boundary condition. All space is finite volume $V = L^3$ with periodic boundary conditions,

$$e^{ik_x L} = 1\,, \qquad e^{ik_y L} = 1\,, \qquad e^{ik_z L} = 1$$

or

$$\boldsymbol{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$$

where n_x, n_y, and n_z are integers, and $V = L^3$.

((**Mathematical notes added on 5/29/2023**))

When I have been teaching the plane wave representation in the Spring of 2023 (Phys.422 Quantum mechanics II) at Binghamton University, I realized that some undergraduate students have difficulty in understanding the difference in the two types of plane wave representations with two cases,

(i) the wave vector $\boldsymbol{k}$ is continuous,

(ii) the wave vector $\boldsymbol{k}$ are discrete, $\mathbf{k} = (\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y, \frac{2\pi}{L}n_y)$ where L is the side of cubic system with volume $V(=L^3)$, n_x, n_y, and n_z are integers.

(i) Plane wave representation with discrete wave vector $\mathbf{k} = (\frac{2\pi}{L}n_x, \frac{2\pi}{L}n_y, \frac{2\pi}{L}n_y)$

((**Schiff**)) **L.I. Schiff, Quantum Mechanics, 2nd edition (McGraw-Hill, 1955).**

For many applications, a representation of the potentials and fields in a complete orthonormal set of plane waves is useful. These plane waves are taken to be vector functions of **r** that are polarized perpendicular to the propagation vector, so that the condition $\nabla \cdot \mathbf{A} = 0$ is satisfied,

$$\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{r}) = \frac{1}{\sqrt{V}} \boldsymbol{\varepsilon}(\mathbf{k},\lambda) e^{i\mathbf{k}\cdot\mathbf{r}} \qquad (\lambda = 1, 2).$$

where $V = L^3$ and $\boldsymbol{\varepsilon}(\mathbf{k},\lambda)$ is the polarization vector. The vectors (***k***) are chosen as

$$k_x = \frac{2\pi}{L}n_x, \qquad k_y = \frac{2\pi}{L}n_y, \qquad k_z = \frac{2\pi}{L}n_z$$

where $\{n_x, n_y, n_z\}$ are sets of integers. So that, $u_{\mathbf{k},\lambda}$ satisfies periodic boundary conditions at the walls of a large cubic box of volume $V = L^3$ (so-called quantum box). Note that $\varepsilon(\mathbf{k},\lambda)$ are unit vectors and $\boldsymbol{\varepsilon}(\mathbf{k},\lambda = 1)$, $\boldsymbol{\varepsilon}(\mathbf{k},\lambda = 2)$, and ***k*** form a right-handed set, so that we have

$$\mathbf{k} \cdot \varepsilon(\mathbf{k},\lambda) = 0 \qquad \text{and} \qquad \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{k},\lambda) = 0.$$

It is easily verified that the orthonormality property assumes the form,

$$\int d\mathbf{r}\ \mathbf{u}_{\mathbf{k},\lambda}{}^*(\mathbf{r}) \mathbf{u}_{\mathbf{k}',\lambda'}(\mathbf{r}) = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'}.$$

In the textbook of Townsend, we find the following problem in ((**Townsend 14-2**)). Show that

$$\frac{1}{V} \int d\mathbf{r} \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}] = \delta_{\mathbf{k}',\mathbf{k}},$$

gives periodic boundary conditions.

The state vector $|\boldsymbol{k}\rangle$ for the plane wave and the orthogonality. Here we use the plane wave form in the box with volume *V*.

$$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad \langle \boldsymbol{k}|\boldsymbol{r}\rangle = \langle \boldsymbol{r}|\boldsymbol{k}\rangle^{*} = \frac{1}{\sqrt{V}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}},$$

$$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = \delta_{\boldsymbol{k},\boldsymbol{k}'} = \int d\boldsymbol{r}\langle \boldsymbol{k}'|\boldsymbol{r}\rangle\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{V}\int d\boldsymbol{r}\exp[i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}],$$

with

$$\boldsymbol{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z),$$

which are discrete values (not continuous).

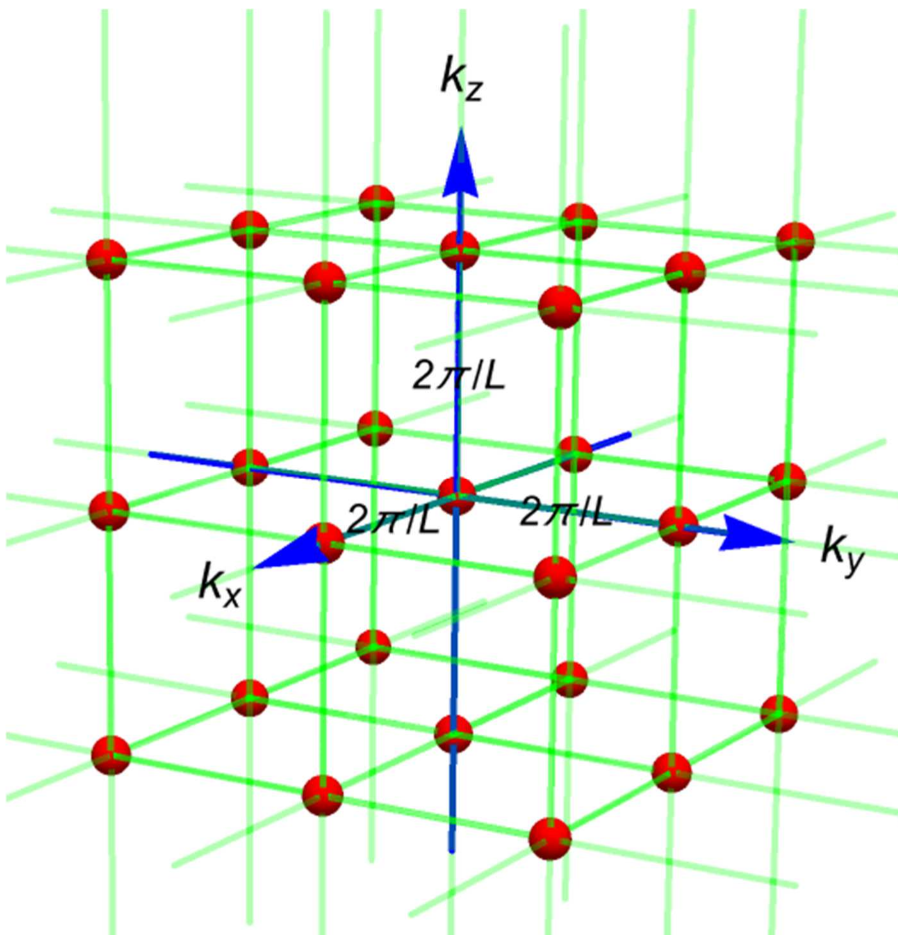

Fig. The $|\mathbf{k}\rangle$ states with $\boldsymbol{k} = (k_x, k_y, k_z) = \frac{2\pi}{L}(n_x, n_y, n_z)$ in the 3D $\boldsymbol{k}$-space. Note that there is one state per $(2\pi / L)^3$

((**Proof**))

$$\int_0^L dx \exp[i(k_x' - k_x)x] = \frac{\exp[i(k' - k)L] - 1}{i(k' - k)}$$

$$= L \exp[i\pi(n_x' - n_x)] \frac{\sin[\pi(n_x' - n_x)]}{\pi(n_x' - n_x)}$$

$$= L\delta_{n_x',n_x}$$

Similarly

$$\int_0^L dy \exp[i(k_y' - k_y)y] = L \exp[i\pi(n_y' - n_y)] \frac{\sin[\pi(n_y' - n_y)]}{\pi(n_y' - n_y)} = L\delta_{n_y',n_y} ,$$

$$\int_0^L dz \exp[i(k_z' - k_z)z] = L \exp[i\pi(n_z' - n_z)] \frac{\sin[\pi(n_z' - n_z)]}{\pi(n_z' - n_z)} = L\delta_{n_z',n_z} .$$

Thus we have

$$\int d\mathbf{r} \exp[i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{r}] = L^3 \delta_{n_x',n_x} \delta_{n_y',n_y} \delta_{n_z',n_z} = L^3 \delta_{\mathbf{k},\mathbf{k}'} .$$

We note that V $(= L^3)$ is the volume of the system. The vectors $\boldsymbol{\varepsilon}(\boldsymbol{k}, s)$ are unit vectors indicating the direction, polarization, of the vector potential for each value of $\boldsymbol{k}$. Since

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r}, t) = 0 ,$$

we have

$$\boldsymbol{k} \cdot \boldsymbol{\varepsilon}(\boldsymbol{k}, s) = 0 .$$

Thus the polarization vector is perpendicular to the direction of $\boldsymbol{k}$. For any particular $\boldsymbol{k}$, there are two linearly independent vectors [$\boldsymbol{\varepsilon}(\boldsymbol{k},1)$, $\boldsymbol{\varepsilon}(\boldsymbol{k},2)$] that satisfy this condition.

$$\boldsymbol{k} \cdot \boldsymbol{\varepsilon}(\boldsymbol{k},1) = 0 , \qquad \boldsymbol{k} \cdot \boldsymbol{\varepsilon}(\boldsymbol{k},2) = 0 .$$

(ii) Transformation function with continuous wave vector *k*: Dirac delta function

It is well known that the transformation function is given by

$$\langle \mathbf{r}|\mathbf{k}\rangle = \frac{1}{(2\pi)^{3/2}} \exp(i\mathbf{k}\cdot\mathbf{r}),$$

where $\boldsymbol{k}$ is a continuous vector. We note that

$$\begin{aligned}
\langle \mathbf{r}|\mathbf{r}'\rangle &= \int \langle \mathbf{r}|\mathbf{k}\rangle\langle \mathbf{k}|\mathbf{r}'\rangle d\mathbf{k} \\
&= \frac{1}{(2\pi)^3}\int d\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r})\exp(-i\mathbf{k}\cdot\mathbf{r}') \\
&= \frac{1}{(2\pi)^3}\int d\mathbf{k}\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')] \\
&= \delta(\mathbf{r}-\mathbf{r}')
\end{aligned}$$

Here we show that

$$\int_{-\infty}^{\infty} \exp(ikx)dk = 2\pi\delta(x), \qquad \text{(1D case))}$$

for continuous wave number k. Note that for the 3D case, we have

$$\int_{-\infty}^{\infty} \exp(ik_x x)dk_x \int_{-\infty}^{\infty} \exp(ik_y y)dk_y \int_{-\infty}^{\infty} \exp(ik_x x)dk_x = (2\pi)^3\delta(x)\delta(y)\delta(z)$$

or

$$\int d\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r}) = (2\pi)^3\delta(\mathbf{r})$$

((**Proof**))

$$\int_{-\infty}^{\infty} \exp(ikx)dk = 2\pi\delta(x).$$

We choose the finite upper limit of x as η (a large fixed value)

$$
\begin{aligned}
F(x) &= \int_{-\eta}^{\eta} \exp(ikx)dk \\
&= \int_{-\eta}^{\eta} [\cos(kx) + i\sin(kx)]dk \\
&= 2\int_{0}^{\eta} \cos(kx)dk \\
&= \frac{2\sin(\eta x)}{x}
\end{aligned}
$$

Here we show that

$$\lim_{\eta \leftrightarrow 0} F(x) = 2\pi\delta(x)\,.$$

For the finite fixed value of η, we have

$$\lim_{x \leftrightarrow 0} F(x) = \lim_{x \leftrightarrow 0} \frac{2\sin(\eta x)}{x} = 2\eta\,.$$

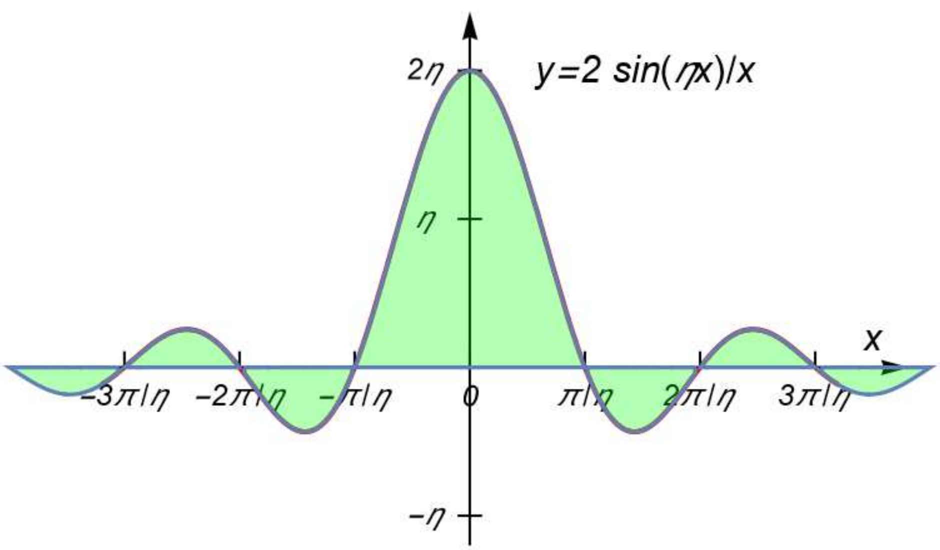

Fig. Plot of $F(x) = \frac{2\sin(\eta x)}{x}$ as a function of x where η is a fixed large value. Note that $\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{2\sin(\eta x)}{x} = 2\eta$, which diverges in the limit of $\eta \to \infty$. $\int_{-\infty}^{\infty} F(x)dx = \int_{-\infty}^{\infty} \frac{2\sin(\eta x)}{x} dx = 2\int_{-\infty}^{\infty} \frac{\sin t}{t} dt = 2\pi$.

Now we consider the function defined by

$$F(x) = \frac{2\sin(\eta x)}{x}.$$

We define the integral by *I*,

$$I = \int_{-\infty}^{\infty} dx F(x) = \int_{-\infty}^{\infty} dx \frac{2\sin(\eta x)}{x}.$$

In order to calculate this integral, we put

$$t = \eta x, \qquad dt = \eta dx,$$

leading to

$$I = \int_{-\infty}^{\infty} dt \frac{2\sin(t)}{t} = 2\pi,$$

where we use the Mathematica. Using the Dirac delta function, we get

$$F(x) = \frac{2\sin x}{x} \to 2\pi\delta(x),$$

in the limit of $\eta \to \infty$.

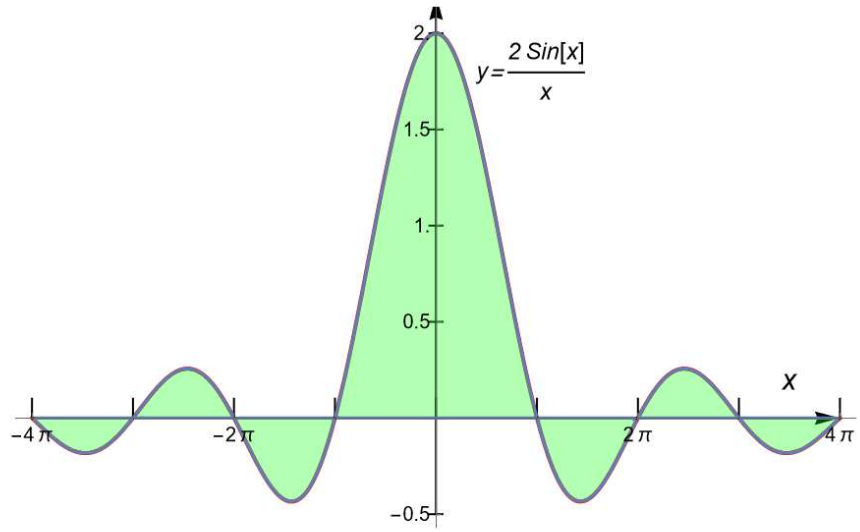

Fig. Plot of $f(x)=\dfrac{2\sin x}{x}$ as a function of *x*. $\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{\infty}\dfrac{2\sin x}{x}dx=2\pi$

From the property of the Dirac delta function, it is concluded that

$$\lim_{\eta\leftrightarrow 0} F(x)=2\pi\delta(x)\,.$$

((Further discussion from above))

Using the wave function $\langle \boldsymbol{r}|\boldsymbol{k}\rangle=\dfrac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ for the discrete wave number, as discussed above, we now calculate the energy of the electromagnetic field.

$$\begin{aligned}
H_{E\&M} &=\frac{1}{8\pi}\int d\boldsymbol{r}[\boldsymbol{E}^2(\boldsymbol{r},t)+\boldsymbol{B}^2(\boldsymbol{r},t)]\\
&=\frac{1}{8\pi}\int d\boldsymbol{r}[(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t})^2+(\nabla\times\boldsymbol{A})^2]\\
&=H_E+H_M
\end{aligned}$$

The first term is due to the electric field energy and the second term is due to the magnetic field energy. We now calculate the first term due to the electric field.

$$H_E=\frac{1}{8\pi}\int d\boldsymbol{r}\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\cdot\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\,,$$

with

$$-\frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \sum_{\boldsymbol{k},s}[-i\frac{\omega_{\boldsymbol{k}}}{c}c_{\boldsymbol{k},s}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}}{\sqrt{V}} + i\frac{\omega_{\boldsymbol{k}}}{c}c_{\boldsymbol{k},s}{}^*\boldsymbol{\varepsilon}(\boldsymbol{k},s)\frac{e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}}{\sqrt{V}}]$$

$$= \frac{-i}{c\sqrt{V}}\sum_{\boldsymbol{k},s}\omega_{\boldsymbol{k}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[c_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - c_{\boldsymbol{k},s}{}^*e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]$$

and

$$\frac{1}{c^2}\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \frac{-1}{c^2 V}\sum_{\boldsymbol{k},s}\omega_{\boldsymbol{k}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[c_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - c_{\boldsymbol{k},s}{}^*e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]$$

$$\cdot\sum_{\boldsymbol{k}',s'}\omega_{\boldsymbol{k}'}\boldsymbol{\varepsilon}(\boldsymbol{k}',s')[c_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)} - c_{\boldsymbol{k}',s'}{}^*e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}]$$

where the orthogonality relation holds as

$$\frac{1}{V}\int d\boldsymbol{r}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}e^{i\boldsymbol{k}'\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k},\boldsymbol{k}'}\,.$$

So, we get

$$\frac{1}{c^2}\int d\boldsymbol{r}[\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t}]^2 = \frac{-1}{c^2 V}\int d\boldsymbol{r}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\omega_{\boldsymbol{k}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\omega_{\boldsymbol{k}'}\boldsymbol{\varepsilon}(\boldsymbol{k}',s')[c_{\boldsymbol{k},s}c_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}c_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}$$

$$- c_{\boldsymbol{k},s}c_{\boldsymbol{k}',s'}{}^*e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)} - c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}$$

$$+ c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k}',s'}{}^*e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}]$$

$$= \frac{1}{c^2}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\omega_{\boldsymbol{k}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\omega_{\boldsymbol{k}'}\boldsymbol{\varepsilon}(\boldsymbol{k}',s')[$$

$$+ (c_{\boldsymbol{k},s}c_{\boldsymbol{k},s'}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}} - c_{\boldsymbol{k},s}c_{-\boldsymbol{k},s'}\delta_{\boldsymbol{k}',-\boldsymbol{k}}e^{-2i\omega_k t} - c_{\boldsymbol{k},s}{}^*c_{-\boldsymbol{k},s'}{}^*e^{2i\omega_k t}]$$

$$\to \frac{1}{c^2}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s')(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s'}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}}$$

$$= \frac{1}{c^2}\sum_{\boldsymbol{k},s}\sum_{s'}\omega_{\boldsymbol{k}}{}^2\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s')(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s'}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s'})$$

$$= \sum_{\boldsymbol{k},s}\frac{\omega_{\boldsymbol{k}}{}^2}{c^2}(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s})$$

where the average of the time dependent terms over the period T becomes zero. We also note that

$$\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s') = \delta_{s,s'}\,.$$

It follows that

$$H_E = \frac{1}{4\pi}\sum_{\boldsymbol{k},s}\frac{{\omega_{\boldsymbol{k}}}^2}{c^2}(c_{\boldsymbol{k},s}{c_{\boldsymbol{k},s}}^* + {c_{\boldsymbol{k},s}}^* c_{\boldsymbol{k},s}) .$$

We now calculate the second term due to the magnetic field.

$$\begin{aligned}\nabla\times\boldsymbol{A}(\boldsymbol{r},t) &= \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k},s}\{c_{\boldsymbol{k},s}\nabla\times[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}] + {c_{\boldsymbol{k},s}}^*\nabla\times[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\}\\ &= \frac{1}{\sqrt{V}}\sum_{\boldsymbol{k},s} i\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)[c_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - {c_{\boldsymbol{k},s}}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\end{aligned}$$

where

$$\nabla\times[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}] = i\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} .$$

Then we get

$$\begin{aligned}\int d\boldsymbol{r}[\nabla\times\boldsymbol{A}(\boldsymbol{r},t)]^2 &= -\frac{1}{V}\int d\boldsymbol{r}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')]\\ &\quad [c_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - {c_{\boldsymbol{k},s}}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}][c_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)} - {c_{\boldsymbol{k}',s'}}^* e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)}]\\ &= -\frac{1}{V}\int d\boldsymbol{r}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')]\\ &\quad [c_{\boldsymbol{k},s}c_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)} - c_{\boldsymbol{k},s}{c_{\boldsymbol{k}',s'}}^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)}\\ &\quad -{c_{\boldsymbol{k},s}}^* c_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)} + {c_{\boldsymbol{k},s}}^*{c_{\boldsymbol{k}',s'}}^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'} t)}]\\ &= -\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')]\\ &\quad [-(c_{\boldsymbol{k},s}{c_{\boldsymbol{k},s'}}^* + {c_{\boldsymbol{k},s}}^* c_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}} + (c_{\boldsymbol{k},s}c_{-\boldsymbol{k},s'}e^{-2i\omega_k t} + {c_{\boldsymbol{k},s}}^*{c_{-\boldsymbol{k},s'}}^* e^{2i\omega_k t})\delta_{\boldsymbol{k}',-\boldsymbol{k}}]\\ &\to \sum_{\boldsymbol{k},s}\sum_{s'}\{[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)]\cdot[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s')]\}(c_{\boldsymbol{k},s}{c_{\boldsymbol{k},s'}}^* + {c_{\boldsymbol{k},s}}^* c_{\boldsymbol{k},s'})\end{aligned}$$

where the average of the time dependent terms over the period T becomes zero. We also note that

$$\begin{aligned}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)]\cdot[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s')] &= k^2[\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s')] - [\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s')]\cdot[\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)]\\ &= k^2\delta_{s,s'}\end{aligned}$$

where

$$\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s')=0\,,\qquad\qquad \boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)=0\,.$$

Then we have

$$\begin{aligned}\int d\boldsymbol{r}[\nabla\times\boldsymbol{A}(\boldsymbol{r},t)]^2 &= \sum_{\boldsymbol{k},s}\sum_{s'}k^2\delta_{s,s'}(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s'}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s'})\\ &= \sum_{\boldsymbol{k},s}k^2(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s})\\ &= \sum_{\boldsymbol{k},s}\frac{\omega_k{}^2}{c^2}(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s})\end{aligned}$$

It follows that

$$H_B=\frac{1}{8\pi}\sum_{\boldsymbol{k},s}\frac{\omega_{\boldsymbol{k}}{}^2}{c^2}(c_{\boldsymbol{k},s}c_{\boldsymbol{k},s}{}^* + c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s})$$

The total energy is given by

$$\begin{aligned}H &= H_E + H_B\\ &= \frac{1}{4\pi}\sum_{\boldsymbol{k},s}\frac{\omega_{\boldsymbol{k}}{}^2}{c^2}(c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s} + c_{\boldsymbol{k},s}c_{\boldsymbol{k},s}{}^*)\,.\\ &= \frac{1}{2\pi}\sum_{\boldsymbol{k},s}\frac{\omega_{\boldsymbol{k}}{}^2}{c^2}c_{\boldsymbol{k},s}{}^*c_{\boldsymbol{k},s}\end{aligned}$$

We introduce $q_{\boldsymbol{k},s}$ and $p_{\boldsymbol{k},s}$ such that

$$q_{\boldsymbol{k},s}=\frac{1}{c\sqrt{4\pi}}(c_{\boldsymbol{k},s}+c_{\boldsymbol{k},s}{}^*)\,,\qquad p_{\boldsymbol{k},s}=\frac{-i\omega_{\boldsymbol{k}}}{c\sqrt{4\pi}}(c_{\boldsymbol{k},s}-c_{\boldsymbol{k},s}{}^*)\,,$$

or

$$c_{\boldsymbol{k},s}=c\sqrt{\pi}(q_{\boldsymbol{k},s}+\frac{i}{\omega_k}p_{\boldsymbol{k},s})\,,\qquad c_{\boldsymbol{k},s}{}^*=c\sqrt{\pi}(q_{\boldsymbol{k},s}-\frac{i}{\omega_k}p_{\boldsymbol{k},s})\,.$$

Then we get

$$H = \sum_{\boldsymbol{k},s} \left(\frac{p_{\boldsymbol{k},s}{}^2}{2} + \frac{1}{2} \omega_{\boldsymbol{k}}{}^2 q_{\boldsymbol{k},s}{}^2 \right).$$

Thus we see that formally the electromagnetic field can be considered as a collection of independent harmonic oscillators.

5. Quantizing the radiation field

The Hamiltonian of the 3D isotropic simple harmonics is described by

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3$$

with

$$\hat{H}_i = \frac{1}{2m} \hat{p}_i{}^2 + \frac{1}{2} m \omega_0{}^2 \hat{x}_i{}^2,$$

where

$$[\hat{H}, \hat{H}_i] = 0$$

The eigenvectors of the Hamiltonian $\hat{H}$ are also eigenvectors of $\hat{H}_x$, $\hat{H}_y$, and $\hat{H}_z$ .Let us introduce three pairs of creation and annihilation operators.

$$\hat{a}_i = \frac{\beta}{\sqrt{2}} (\hat{x}_i + \frac{i}{m\omega_0} \hat{p}_i), \qquad \hat{a}_i^+ = \frac{\beta}{\sqrt{2}} (\hat{x}_i - \frac{i}{m\omega_0} \hat{p}_i),$$

$$[\hat{a}_i, \hat{a}_i^+] = \hat{1},$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}. \qquad \frac{\beta}{\sqrt{2}} = \sqrt{\frac{m\omega_0}{2\hbar}}$$

Note that

$$\hat{x}_i = \frac{1}{\sqrt{2}\beta}(\hat{a}_i + \hat{a}_i^+), \qquad \hat{p}_i = \frac{1}{\sqrt{2}\beta}\frac{m\omega}{i}(\hat{a}_i - \hat{a}_i^+)$$

The Hamiltonian can be expressed by

$$\hat{H} = \sum_{i=1}^{3} \hbar\omega_0 (\hat{a}_i{}^+ a_i + \frac{1}{2}\hat{1}).$$

We now discuss the quantum mechanics of the electromagnetic field. We assume that the variables $q_{\boldsymbol{k},s}$ and $p_{\boldsymbol{k},s}$ should be operators obeying the commutation relations

$$[\hat{q}_{\boldsymbol{k},s}, \hat{p}_{\boldsymbol{k}',s'}] = i\hbar\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}\hat{1}.$$

Here we define the creation and annihilation operators as

$$\hat{a}_{\boldsymbol{k},s} = \sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}}(\hat{q}_{\boldsymbol{k},s} + \frac{i}{\omega_k}\hat{p}_{\boldsymbol{k},s}), \qquad \hat{a}^+{}_{\boldsymbol{k},s} = \sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}}(\hat{q}_{\boldsymbol{k},s} - \frac{i}{\omega_k}\hat{p}_{\boldsymbol{k},s}),$$

or

$$\hat{q}_{\boldsymbol{k},s} = \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}}}(\hat{a}_{\boldsymbol{k},s} + \hat{a}^+{}_{\boldsymbol{k},s}), \qquad \hat{p}_{\boldsymbol{k},s} = -i\sqrt{\frac{\hbar\omega_{\boldsymbol{k}}}{2}}(\hat{a}_{\boldsymbol{k},s} - \hat{a}^+{}_{\boldsymbol{k},s}).$$

The commutation relation:

$$[\hat{a}_{\boldsymbol{k},s}, \hat{a}^+{}_{\boldsymbol{k}',s'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}.$$

A comparison of these equations with $c_{\boldsymbol{k},s}$ and $c_{\boldsymbol{k},s}{}^*$,

$$\frac{1}{c\sqrt{4\pi}}c_{\boldsymbol{k},s} \to \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}}}\hat{a}_{\boldsymbol{k},s}, \qquad \frac{1}{c\sqrt{4\pi}}c_{\boldsymbol{k},s}{}^* \to \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}}}\hat{a}^+{}_{\boldsymbol{k},s},$$

or

$$c_{\boldsymbol{k},s} \to c\sqrt{\frac{2\pi\hbar}{\omega_{\boldsymbol{k}}}}\hat{a}_{\boldsymbol{k},s}, \qquad c_{\boldsymbol{k},s}{}^{*} \to c\sqrt{\frac{2\pi\hbar}{\omega_{\boldsymbol{k}}}}\hat{a}^{+}{}_{\boldsymbol{k},s}.$$

where

$$\frac{c_{\boldsymbol{k},s}}{c\sqrt{\pi}}\sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}} \to \hat{a}_{\boldsymbol{k},s} = \sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}}(\hat{q}_{\boldsymbol{k},s} + \frac{i}{\omega_k}\hat{p}_{\boldsymbol{k},s}),$$

$$\frac{c_{\boldsymbol{k},s}{}^{*}}{c\sqrt{\pi}}\sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}} \to \hat{a}_{\boldsymbol{k},s}{}^{+} = \sqrt{\frac{\omega_{\boldsymbol{k}}}{2\hbar}}(\hat{q}_{\boldsymbol{k},s} - \frac{i}{\omega_k}\hat{p}_{\boldsymbol{k},s})$$

((**Second quantization**))

The quantization of the radiation can be achieved by writing the electromagnetic field in terms of creation and annihilation operators, by analogy with the harmonic oscillator. This process which is called *second quantization*, leads to the replacement of the various *fields* (such as the vector potential, the electric field, and the magnetic field by operator quantities, which in turn are expressed in terms of creation and annihilation operators.

The Hamiltonian $\hat{H}$ is

$$\begin{aligned}\hat{H} &= \sum_{\boldsymbol{k},s}(\frac{\hat{p}_{\boldsymbol{k},s}{}^{2}}{2} + \frac{1}{2}\omega_{\boldsymbol{k}}{}^{2}\hat{q}_{\boldsymbol{k},s}{}^{2}) \\ &= \frac{1}{2}\sum_{\boldsymbol{k},s}\hbar\omega_{\boldsymbol{k}}(\hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s} + \hat{a}_{\boldsymbol{k},s}\hat{a}^{+}{}_{\boldsymbol{k},s}) \\ &= \sum_{\boldsymbol{k},s}\hbar\omega_{\boldsymbol{k}}(\hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s} + \frac{1}{2}\hat{1})\end{aligned}$$

The vector potential operator $\hat{\boldsymbol{A}}$ can be obtained as

$$\begin{aligned}\hat{\boldsymbol{A}}(\boldsymbol{r},t) &= \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\hat{a}_{\boldsymbol{k},s}\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + \hat{a}^{+}{}_{\boldsymbol{k},s}\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}] \\ &= \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + \hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}] \\ &= \hat{\boldsymbol{A}}^{(+)}(\boldsymbol{r},t) + \hat{\boldsymbol{A}}^{(-)}(\boldsymbol{r},t)\end{aligned}$$

with

$$\hat{\boldsymbol{A}}^{(+)}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},s} \sqrt{\frac{2\pi\hbar c}{\omega_{\mathbf{k}} V}} \hat{a}_{\boldsymbol{k},s} \boldsymbol{\varepsilon}(\boldsymbol{k},s) e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)},$$

$$\hat{\mathbf{A}}^{(-)}(\mathbf{r},t) = \sum_{\mathbf{k},s} \sqrt{\frac{2\pi\hbar c}{\omega_{\mathbf{k}} V}} \hat{a}^{+}_{\mathbf{k},s} \boldsymbol{\varepsilon}(\mathbf{k},s) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_{\mathbf{k}}t)},$$

and

$$\hat{\boldsymbol{A}}^{(-)}(\boldsymbol{r},t) = [\hat{\boldsymbol{A}}^{(+)}(\boldsymbol{r},t)]^{+}.$$

The electric field is evaluated as

$$\begin{aligned}
\hat{\boldsymbol{E}}(\boldsymbol{r},t) &= -\frac{1}{c}\frac{\partial}{\partial t}\hat{\boldsymbol{A}}(\boldsymbol{r},t) \\
&= i\sum_{\boldsymbol{k},s} \sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}} \boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s} e^{i(\mathbf{k}\cdot\boldsymbol{r}-\omega_{\mathbf{k}}t)} - \hat{a}^{+}{}_{\boldsymbol{k},s} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&= \hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r},t) + \hat{\boldsymbol{E}}^{(-)}(\boldsymbol{r},t)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r},t) &= i\sum_{\boldsymbol{k},s} \sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}} [\boldsymbol{\varepsilon}(\boldsymbol{k},s)\hat{a}_{\boldsymbol{k},s} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&= i\sum_{\boldsymbol{k}} \sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}} [\boldsymbol{\varepsilon}(\boldsymbol{k},1)\hat{a}_{\boldsymbol{k},1} + \boldsymbol{\varepsilon}(\boldsymbol{k},2)\hat{a}_{\boldsymbol{k},2}] e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}
\end{aligned},$$

and the Hermitian conjugate of $\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r},t)$ is given by

$$\hat{\boldsymbol{E}}^{(-)}(\boldsymbol{r},t) = -i\sum_{\boldsymbol{k},s} \sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}} [\boldsymbol{\varepsilon}(\boldsymbol{k},s)\hat{a}^{+}{}_{\boldsymbol{k},s} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}].$$

The magnetic field is evaluated as

$$
\begin{aligned}
\hat{\boldsymbol{B}}(\boldsymbol{r},t) &= \nabla\times\hat{\boldsymbol{A}}(\boldsymbol{r},t) \\
&= \sum_{\boldsymbol{k},s} c\sqrt{\frac{2\pi\hbar}{\omega_{\boldsymbol{k}}V}}[\hat{a}_{\boldsymbol{k},s}\nabla\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} + \hat{a}^{+}{}_{\boldsymbol{k},s}\nabla\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&= i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&= i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}[\frac{\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)}{k}][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\boldsymbol{B}}^{(+)}(\boldsymbol{r},t) &= i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)\hat{a}_{\boldsymbol{k},s}]e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} \\
&= i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}[\frac{\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)}{k}\hat{a}_{\boldsymbol{k},s}]e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} \\
&= i\sum_{\boldsymbol{k}}\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\boldsymbol{k}}}}\boldsymbol{k}\times[\boldsymbol{\varepsilon}(\boldsymbol{k},1)\hat{a}_{\boldsymbol{k},1} + \boldsymbol{\varepsilon}(\boldsymbol{k},2)\hat{a}_{\boldsymbol{k},2}]e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}
\end{aligned}
$$

and the Hermitian conjugate of $\hat{\boldsymbol{B}}^{(+)}(\boldsymbol{r},t)$ is given by

$$
\begin{aligned}
\hat{\boldsymbol{B}}^{(-)}(\boldsymbol{r},t) &= -i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)\hat{a}^{+}{}_{\boldsymbol{k},s}]e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} \\
&= -i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}[\frac{\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)}{k}\hat{a}^{+}{}_{\boldsymbol{k},s}]e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}
\end{aligned}
$$

Note that

$$
\nabla\times[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] = \nabla e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s) = ie^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)]
$$

where we use the formula

$$
\nabla\times(f\boldsymbol{F}) = \nabla f\times\boldsymbol{F} \ .
$$

f is a scalar function and $\boldsymbol{F}$ is an arbitrary vector which is independent of $\boldsymbol{r}$.

We also have the Fourier transforms of the above vector functions

$$\boldsymbol{\varepsilon}(\boldsymbol{k},1)\hat{a}_{\boldsymbol{k},1} + \boldsymbol{\varepsilon}(\boldsymbol{k},2)\hat{a}_{\boldsymbol{k},2} = -i\frac{1}{\sqrt{2\pi\hbar\omega_{\boldsymbol{k}}V}}\int d\boldsymbol{r}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\hat{\boldsymbol{E}}^{(+)}(\boldsymbol{r},t),$$

and

$$\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},1)\hat{a}_{\boldsymbol{k},1} + \boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},2)\hat{a}_{\boldsymbol{k},2} = -i\sqrt{\frac{\omega_{\boldsymbol{k}}}{2\pi\hbar c^2 V}}\int d\boldsymbol{r}\hat{\boldsymbol{B}}^{(+)}(\boldsymbol{r},t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}.$$

Note that the direction of the circular polarization is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{k},+) = -\frac{1}{\sqrt{2}}[\boldsymbol{\varepsilon}(\boldsymbol{k},1) + i\boldsymbol{\varepsilon}(\boldsymbol{k},2)], \qquad \boldsymbol{\varepsilon}(\boldsymbol{k},-) = \frac{1}{\sqrt{2}}[\boldsymbol{\varepsilon}(\boldsymbol{k},1) - i\boldsymbol{\varepsilon}(\boldsymbol{k},2)]$$

6. Momentum operator for photon

The momentum density (erg s/cm^4) of an electromagnetic field is the Poynting vector

$$\hat{\boldsymbol{S}}(\boldsymbol{r},t) = \frac{c}{4\pi}\hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{B}}(\boldsymbol{r},t), \qquad [\text{erg}/(\text{cm}^2\ \text{s})]$$

divided by c^2.

Note that the energy dispersion of photon is given by $\varepsilon = cp$. Suppose that the total energy is E_{tot} and that the total momentum is P_{tot} in the system with the volume V.

$$E_{tot} = cP_{tot}.$$

The energy density u and the momentum density are defined by

$$u = \frac{E_{tot}}{V}, \qquad \tilde{P} = \frac{P_{tot}}{V}$$

Thus, we have

$$u = c\tilde{P} = \frac{S}{c}$$

leading to the relation

$$\tilde{P} = \frac{\boldsymbol{S}}{c^2}$$

where $\boldsymbol{S}$ is the Poynting vector.

The total momentum in the field can be written as

$$\hat{\boldsymbol{P}}_{EM} = \frac{1}{c^2}\int \hat{\boldsymbol{S}}(\boldsymbol{r},t)d\boldsymbol{r} = \frac{1}{4\pi c}\int \hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{B}}(\boldsymbol{r},t)d\boldsymbol{r}, \qquad \text{(erg s/cm)}$$

where

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = -\frac{1}{c}\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}[\frac{\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)}{k}][i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

Then we have

$$\begin{aligned}
\hat{\boldsymbol{P}}_{EM} &= \frac{1}{4\pi c}\int d\boldsymbol{r}\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&\times\sum_{\boldsymbol{k}',s'}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}'}}{V}}[\frac{\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')}{k'}][i\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - i\hat{a}_{\boldsymbol{k}',s'}{}^{+}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= \frac{1}{4\pi c}\frac{2\pi\hbar}{V}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k},s'}\frac{\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}}{k'}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times[\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')] \\
&\int d\boldsymbol{r}[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}_{\boldsymbol{k},s}{}^{+}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}][i\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - i\hat{a}_{\boldsymbol{k}',s'}{}^{+}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}]
\end{aligned}$$

Here we note that

$$\begin{aligned}
&\frac{1}{V}\int d\boldsymbol{r}[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}_{\boldsymbol{k},s}{}^{+}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}][i\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - i\hat{a}_{\boldsymbol{k}',s'}{}^{+}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= (\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'}{}^{+} + \hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}} - (\hat{a}_{\boldsymbol{k},s}\hat{a}_{-\boldsymbol{k},s'}e^{-i2\omega_{\boldsymbol{k}}t} + \hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{-\boldsymbol{k},s'}{}^{+}e^{2i\omega_{\boldsymbol{k}}t})\delta_{\boldsymbol{k}',-\boldsymbol{k}} \\
&\to (\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'}{}^{+} + \hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}}
\end{aligned}$$

Note that the time average of the time dependent term over a period T becomes zero. Then we get

$$\hat{\boldsymbol{P}}_{EM} = \frac{\hbar}{2c}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k},s'}\frac{\sqrt{\omega_k\omega_{k'}}}{k'}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times[\boldsymbol{k}'\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')](\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'} + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'})\delta_{\boldsymbol{k}',\boldsymbol{k}}$$

$$= \frac{\hbar}{2}\sum_{\boldsymbol{k},s}\sum_{s'}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s')](\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'} + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'})$$

Here we have

$$\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s')] = [\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s')]\boldsymbol{k} - [\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{k}]\boldsymbol{\varepsilon}(\boldsymbol{k},s')]$$
$$= \delta_{s,s'}\boldsymbol{k}$$

where

$$\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s') = \delta_{s,s'}, \qquad \boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{k} = 0$$

Then we get

$$\hat{\boldsymbol{P}}_{EM} = \frac{\hbar}{2}\sum_{\boldsymbol{k},s}\sum_{s'}\boldsymbol{k}\delta_{s,s'}(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'} + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s'})$$
$$= \frac{\hbar}{2}\sum_{\boldsymbol{k},s}\boldsymbol{k}(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k},s} + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s})$$
$$= \sum_{\boldsymbol{k},s}\hbar\boldsymbol{k}(\hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s} + \frac{1}{2}\hat{1})$$

or

$$\hat{\boldsymbol{P}}_{EM} = \sum_{\boldsymbol{k},s}\hbar\boldsymbol{k}\hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}.$$

Note that there is no term 1/2 in the summand of the last expression in the above equation. The reason is that

$$\sum_{\boldsymbol{k},s}\hbar\boldsymbol{k} = 0,$$

since for every $\boldsymbol{k}$ in the sum there is $-\boldsymbol{k}$ to cancel it. We note that

$$\hat{\boldsymbol{P}}_{EM}|0\rangle = 0.$$

Applying the momentum operator to the state $\left|1_{\boldsymbol{k},s}\right\rangle$ (the photon number state, Fock state)

$$\hat{\boldsymbol{P}}_{EM}\left|1_{\boldsymbol{k},s}\right\rangle = \hbar\boldsymbol{k}\hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}\left|1_{\boldsymbol{k},s}\right\rangle = \hbar\boldsymbol{k}\left|1_{\boldsymbol{k},s}\right\rangle$$

Note that

$$\hat{H}\left|1_{\boldsymbol{k},s}\right\rangle = \hbar\omega_{\boldsymbol{k},s}(\hat{n}_{\boldsymbol{k},s} + \frac{1}{2}\hat{1})\left|1_{\boldsymbol{k},s}\right\rangle = \frac{3}{2}\hbar\omega_{\boldsymbol{k},s}\left|1_{\boldsymbol{k},s}\right\rangle.$$

So the state $\left|1_{\boldsymbol{k},s}\right\rangle$ is the eigenket of both $\hat{\boldsymbol{P}}_{EM}$ and $\hat{H}$.

7. Angular momentum

The angular momentum $\hat{\boldsymbol{J}}$ of the radiation is another constant of the motion for a free field. It is proportional to the integral over volume *V* of the moment of the Poynting vector about the coordinate origin:

$$\hat{\boldsymbol{J}}_{EM} = \int \boldsymbol{r}\times\frac{[\hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{B}}(\boldsymbol{r},t)]}{4\pi c}d\boldsymbol{r},$$

where the pointing vector $\hat{\boldsymbol{S}}(r,t)$ is defined as

$$\hat{\boldsymbol{S}}(r,t) = \frac{1}{4\pi c}[\hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{B}}(\boldsymbol{r},t)].$$

The *i*-th component of $\hat{\boldsymbol{J}}_{EM}$ is given by

$$\begin{aligned}
\left(\hat{\boldsymbol{J}}_{EM}\right)_i &= \frac{1}{4\pi c}\int d\boldsymbol{r}\varepsilon_{ijk}x_j[\hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{B}}(\boldsymbol{r},t)]_k \\
&= \frac{1}{4\pi c}\int d\boldsymbol{r}\varepsilon_{ijk}x_j\varepsilon_{klm}\hat{E}_l\hat{B}_m \\
&= \frac{1}{4\pi c}\int d\boldsymbol{r}\varepsilon_{ijk}x_j\varepsilon_{klm}\hat{E}_l\varepsilon_{mnp}\frac{\partial\hat{A}_p}{\partial x_n} \\
&= \frac{1}{4\pi c}\varepsilon_{ijk}\varepsilon_{klm}\varepsilon_{mnp}\int d\boldsymbol{r}x_j\hat{E}_l\frac{\partial\hat{A}_p}{\partial x_n}
\end{aligned}$$

where ε_{ijk} is the Levi-Civita symbol and we use the simplified notation for the summation such that

$$\sum_{l,m} \varepsilon_{klm} \hat{E}_l \hat{B}_m = \varepsilon_{klm} \hat{E}_l \hat{B}_m$$

Using the property of the Levi-Civita symbol,

$$\varepsilon_{klm}\varepsilon_{mnp} = \varepsilon_{mkl}\varepsilon_{mnp} = \delta_{k,n}\delta_{l,p} - \delta_{k,p}\delta_{l,n}$$

we get

$$\begin{aligned}
\left(\hat{\boldsymbol{J}}_{EM}\right)_i &= \frac{1}{4\pi c}\varepsilon_{ijk}\int d\boldsymbol{r}(\delta_{k,n}\delta_{l,p} - \delta_{k,p}\delta_{l,n})x_j \hat{E}_l \frac{\partial \hat{A}_p}{\partial x_n} \\
&= \frac{1}{4\pi c}\varepsilon_{ijk}\int d\boldsymbol{r}(\delta_{k,n}\delta_{l,p}x_j \hat{E}_l \frac{\partial \hat{A}_p}{\partial x_n} - \delta_{k,p}\delta_{l,n}x_j \hat{E}_l \frac{\partial \hat{A}_p}{\partial x_n}) \\
&= \frac{1}{4\pi c}\varepsilon_{ijk}\int d\boldsymbol{r}(x_j \hat{E}_l \frac{\partial \hat{A}_l}{\partial x_k} - x_j \hat{E}_l \frac{\partial \hat{A}_k}{\partial x_l})
\end{aligned}$$

On integrating by parts in the last term, we have

$$\left(\hat{\boldsymbol{J}}_{EM}\right)_i = \frac{1}{4\pi c}\varepsilon_{ijk}\int d\boldsymbol{r}[x_j \hat{E}_l \frac{\partial \hat{A}_l}{\partial x_k} + \frac{\partial}{\partial x_l}(x_j \hat{E}_l)\hat{A}_k].$$

We note that

$$\frac{\partial}{\partial x_l}(x_j \hat{E}_l) = \frac{\partial x_j}{\partial x_l}\hat{E}_l + x_j \frac{\partial}{\partial x_l}\hat{E}_l = \delta_{j,l}\hat{E}_l + x_j \frac{\partial \hat{E}_l}{\partial x_l} = \hat{E}_j$$

where

$$\frac{\partial \hat{E}_l}{\partial x_l} = \nabla \cdot \hat{\boldsymbol{E}} = 0$$

Note that

$$\begin{aligned}\nabla\cdot\hat{\boldsymbol{E}}(\boldsymbol{r},t) &= \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\{i\hat{a}_{\boldsymbol{k},s}\nabla\cdot[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]-i\hat{a}^{+}{}_{\boldsymbol{k},s}\nabla\cdot[\boldsymbol{\varepsilon}(\boldsymbol{k},s)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\} \\ &= -\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{k}\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}+\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}] \\ &= 0\end{aligned}$$

The total angular momentum can be rewritten as

$$\left(\hat{\boldsymbol{J}}_{EM}\right)_i = \frac{1}{4\pi c}\varepsilon_{ijk}\int d\boldsymbol{r}[\hat{E}_l x_j\frac{\partial\hat{A}_l}{\partial x_k}+\hat{E}_j\hat{A}_k]$$

The first term can be recognized as the orbital angular momentum of the field. The orbital angular momentum operator $\hat{L}_i$ is given by

$$\hat{L}_i = -i\hbar\varepsilon_{ijk}x_j\frac{\partial}{\partial x_k}$$

Then we get

$$\begin{aligned}\left(\hat{J}_{EM}\right)_i &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}[\hat{E}_l(-i\hbar x_j\frac{\partial}{\partial x_k})\hat{A}_l - i\hbar\varepsilon_{ijk}\hat{E}_j\hat{A}_k] \\ &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}(\hat{E}_l\hat{L}_i\hat{A}_l - i\hbar\varepsilon_{ijk}\hat{E}_j\hat{A}_k) \\ &= (\hat{\boldsymbol{L}}_{EM})_i + \left(\hat{\boldsymbol{S}}_{EM}\right)_i\end{aligned}$$

The first term is the orbital angular momentum and the second term is the spin angular momentum of photon.

(a) Spin angular momentum

$$\begin{aligned}\left(\hat{\boldsymbol{S}}_{EM}\right)_i &= \frac{1}{4\pi c}\int d\boldsymbol{r}\varepsilon_{ijk}\hat{E}_j\hat{A}_k \\ &= \frac{1}{4\pi c}\int d\boldsymbol{r}(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{A}})_i\end{aligned}$$

or

$$\hat{\boldsymbol{S}}_{EM} = \frac{1}{4\pi c}\int d\boldsymbol{r}(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{A}})$$

((**Note**))

If one uses a definition like (coordinate x conjugate momentum), integrated over space, where the coordinate is the vector potential, and its conjugate momentum is the electric field.

We now calculate the integral $\int d\boldsymbol{r}(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{A}})$.

$$\begin{aligned}
\hat{\boldsymbol{E}}(\boldsymbol{r},t)\times\hat{\boldsymbol{A}}(\boldsymbol{r},t) &= \sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[i\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - i\hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}] \\
&\quad\times\sum_{\boldsymbol{k}',s'}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}'}V}}\boldsymbol{\varepsilon}(\boldsymbol{k}',s')[\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} + \hat{a}^{+}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= \frac{2\pi i c\hbar}{V}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\sqrt{\frac{\omega_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}'}}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s') \\
&\quad [\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}][\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} + \hat{a}^{+}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{V}\int d\boldsymbol{r}[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}][\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} + \hat{a}^{+}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= \frac{1}{V}\int dr[\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} + \hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} \\
&- \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= \hat{a}_{\boldsymbol{k},s}\hat{a}_{-\boldsymbol{k}',s'}\delta_{\boldsymbol{k}',-\boldsymbol{k}}e^{-2i\omega_{\boldsymbol{k}}t} + \hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'}\delta_{\boldsymbol{k}',\boldsymbol{k}} - \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}\delta_{\boldsymbol{k}',\boldsymbol{k}} - \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}{}^{+}\delta_{\boldsymbol{k}',-\boldsymbol{k}}e^{2i\omega_{\boldsymbol{k}}t} \\
&\to (\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'} - \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'})\delta_{\boldsymbol{k}',\boldsymbol{k}}
\end{aligned}$$

since the time average of the time dependent terms over a period time T is zero. Then we get

$$
\begin{aligned}
\hat{\boldsymbol{S}}_{EM} &= \frac{1}{4\pi c}\int d\boldsymbol{r}(\hat{\boldsymbol{E}}\times\hat{\boldsymbol{A}}) \\
&= \frac{2\pi i c\hbar}{4\pi c}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\sqrt{\frac{\omega_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}'}}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times\boldsymbol{\varepsilon}(\boldsymbol{k}',s')(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}',s'}-\hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'})\delta_{\boldsymbol{k}',\boldsymbol{k}} \\
&= \frac{i\hbar}{2}\sum_{\boldsymbol{k}}\sum_{s,s'}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\times\boldsymbol{\varepsilon}(\boldsymbol{k},s')(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k},s}-\hat{a}_{\boldsymbol{k},s}{}^{+}\hat{a}_{\boldsymbol{k},s'}) \\
&= \frac{i\hbar}{2}\sum_{\boldsymbol{k}}[\boldsymbol{\varepsilon}(\boldsymbol{k},1)\times\boldsymbol{\varepsilon}(\boldsymbol{k},2)(\hat{a}_{\boldsymbol{k},1}\hat{a}^{+}_{\boldsymbol{k},2}-\hat{a}^{+}_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2}) \\
&\quad +\boldsymbol{\varepsilon}(\boldsymbol{k},2)\times\boldsymbol{\varepsilon}(\boldsymbol{k},1)(\hat{a}_{\boldsymbol{k},2}\hat{a}^{+}_{\boldsymbol{k},1}-\hat{a}^{+}_{\boldsymbol{k},2}\hat{a}_{\boldsymbol{k},1})] \\
&= \frac{i\hbar}{2}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k}[(\hat{a}_{\boldsymbol{k},1}\hat{a}^{+}_{\boldsymbol{k},2}-\hat{a}^{+}_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2})-(\hat{a}_{\boldsymbol{k},2}\hat{a}^{+}_{\boldsymbol{k},1}-\hat{a}^{+}_{\boldsymbol{k},2}\hat{a}_{\boldsymbol{k},1})] \\
&= i\hbar\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k}(\hat{a}_{\boldsymbol{k},1}\hat{a}^{+}_{\boldsymbol{k},2}-\hat{a}^{+}_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2})
\end{aligned}
$$

where

$$\boldsymbol{\varepsilon}(\boldsymbol{k},1)\times\boldsymbol{\varepsilon}(\boldsymbol{k},1)=0\,,\qquad \boldsymbol{\varepsilon}(\boldsymbol{k},2)\times\boldsymbol{\varepsilon}(\boldsymbol{k},2)=0\,,$$

$$\boldsymbol{\varepsilon}(\boldsymbol{k},1)\times\boldsymbol{\varepsilon}(\boldsymbol{k},2)=\frac{\boldsymbol{k}}{k}\,,\qquad \boldsymbol{\varepsilon}(\boldsymbol{k},2)\times\boldsymbol{\varepsilon}(\boldsymbol{k},1)=-\frac{\boldsymbol{k}}{k}\,,$$

and the commutation relation

$$[\hat{a}_{\boldsymbol{k},s},\hat{a}^{+}_{\boldsymbol{k}',s'}]=\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}$$

In summary we have

$$\hat{\boldsymbol{S}}_{EM}=i\hbar\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k}(\hat{a}_{\boldsymbol{k},1}\hat{a}^{+}_{\boldsymbol{k},2}-\hat{a}^{+}_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2})\,.$$

(b) Orbital angular momentum

The orbital angular momentum is given by

$$\hat{\boldsymbol{L}}_{EM}=\frac{i}{4\pi c\hbar}\int d\boldsymbol{r}E_l\boldsymbol{L}A_l=\frac{1}{4\pi c}\int d\boldsymbol{r}E_l(\boldsymbol{r}\times\nabla)A_l\,.$$

We show that the component of $\boldsymbol{L}_{EM}$ along the direction $\boldsymbol{k}$ of a photon vanishes, so calling these pieces as orbital and spin angular momentum makes some sense.

$$(\boldsymbol{r}\times\nabla)A_l=\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)](\boldsymbol{r}\times i\boldsymbol{k})[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]$$

where $\boldsymbol{e}_l$ is the unit vector along the ***l***-direction. Now we combine with the same component of the electric field, E_l,

$$E_l(\boldsymbol{r}\times\nabla)A_l=i\sum_{\boldsymbol{k}',s'}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}'}}{V}}[\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s')][\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}-\hat{a}^{+}{}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}]$$
$$\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\boldsymbol{k}}V}}[\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)](\boldsymbol{r}\times i\boldsymbol{k})[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]$$

Then the orbital angular momentum is

$$\boldsymbol{L}_{EM}=-\frac{i\hbar}{2V}\sum_{\boldsymbol{k}',s'}\sum_{\boldsymbol{k},s}\sqrt{\frac{\omega_{\boldsymbol{k}'}}{\omega_{\boldsymbol{k}}}}[\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s')][\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)]$$
$$\int d\boldsymbol{r}\boldsymbol{r}[\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}-\hat{a}^{+}{}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\times i\boldsymbol{k}$$
$$=-\frac{i\hbar}{2V}\sum_{\boldsymbol{k}',s'}\sum_{\boldsymbol{k},s}\sqrt{\frac{\omega_{\boldsymbol{k}'}}{\omega_{\boldsymbol{k}}}}[\boldsymbol{\varepsilon}(\boldsymbol{k}',s')\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)]$$
$$\int d\boldsymbol{r}\boldsymbol{r}[\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}-\hat{a}^{+}{}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\times i\boldsymbol{k}$$

Note that

$$[\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s')][\boldsymbol{e}_l\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)]=\boldsymbol{\varepsilon}(\boldsymbol{k}',s')\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s).$$

and

$$
\begin{aligned}
I &= \boldsymbol{r}[\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)} - \hat{a}^{+}{}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\times i\boldsymbol{k} \\
&= \boldsymbol{r}[\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} \\
&\quad - \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{k'}t)}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\times i\boldsymbol{k} \\
&= V\boldsymbol{r}[\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle\langle\boldsymbol{r}|\boldsymbol{k}\rangle e^{-i(\omega_{k'}+\omega_k)t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}'\rangle e^{-i(\omega_{k'}-\omega_k)t} \\
&\quad - \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}\rangle e^{i(\omega_{k'}-\omega_k)t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{k}|\boldsymbol{r}\rangle e^{i(\omega_{k'}+\omega_k)t}]\times i\boldsymbol{k} \\
&= V[\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle e^{-i(\omega_{k'}+\omega_k)t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle^{*}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle e^{-i(\omega_{k'}-\omega_k)t} \\
&\quad - \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle e^{i(\omega_{k'}-\omega_k)t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle^{*}e^{i(\omega_{k'}+\omega_k)t}]\times i\boldsymbol{k} \\
&= V[\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle e^{-i(\omega_{k'}+\omega_k)t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle^{*}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle e^{-i(\omega_{k'}-\omega_k)t} \\
&\quad - \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle e^{i(\omega_{k'}-\omega_k)t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle^{*}e^{i(\omega_{k'}+\omega_k)t}]\times i\boldsymbol{k}
\end{aligned}
$$

where we use the notation

$$\langle\boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{\sqrt{V}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad \langle\boldsymbol{k}|\boldsymbol{r}\rangle = \langle\boldsymbol{r}|\boldsymbol{k}\rangle^{*} = \frac{1}{\sqrt{V}}e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\langle\boldsymbol{k}|\hat{\boldsymbol{r}}|\psi\rangle = i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{k}|\psi\rangle, \qquad \langle\boldsymbol{k}|\hat{\boldsymbol{k}}|\psi\rangle = \boldsymbol{k}\langle\boldsymbol{k}|\psi\rangle$$

for arbitrary $|\psi\rangle$. We also have

$$\langle\boldsymbol{k}|\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle = i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{k}|\boldsymbol{r}\rangle,$$

$$\langle\boldsymbol{r}|\hat{\boldsymbol{r}}|\boldsymbol{k}\rangle = \langle\boldsymbol{r}|\hat{\boldsymbol{r}}^{+}|\boldsymbol{k}\rangle = \langle\boldsymbol{k}|\hat{\boldsymbol{r}}|\boldsymbol{r}\rangle^{*} = -i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{r}|\boldsymbol{k}\rangle$$

Thus we have

$$
\begin{aligned}
I &= \frac{1}{V}[\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}\langle\boldsymbol{r}|\boldsymbol{k}'\rangle[-i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{r}|\boldsymbol{k}\rangle]e^{-i(\omega_{k'}+\omega_k)t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}[-i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}'\rangle]^{*}e^{-i(\omega_{k'}-\omega_k)t} \\
&\quad - \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}[-i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{k}'|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}\rangle]e^{i(\omega_{k'}-\omega_k)t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}[-i\frac{\partial}{\partial\boldsymbol{k}}\langle\boldsymbol{k}|\boldsymbol{r}\rangle\langle\boldsymbol{r}|\boldsymbol{k}'\rangle]^{*}e^{i(\omega_{k'}+\omega_k)t}]\times i\boldsymbol{k}
\end{aligned}
$$

We take the integral of I over the whole space of $\boldsymbol{r}$, noting that

$$\int d\boldsymbol{r}\langle \boldsymbol{r}|\boldsymbol{k}'\rangle\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \int d\boldsymbol{r}e^{i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}} = V\delta_{\boldsymbol{k}',-\boldsymbol{k}}$$

$$\int d\boldsymbol{r}\langle \boldsymbol{k}|\boldsymbol{r}\rangle\langle \boldsymbol{r}|\boldsymbol{k}'\rangle = V\langle \boldsymbol{k}|\boldsymbol{k}'\rangle = V\delta_{\boldsymbol{k}',\boldsymbol{k}}$$

Then we have

$$\int I d\boldsymbol{r} = [\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})e^{-i(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}})t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})^{*}e^{-i(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}})t}$$
$$- \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})e^{i(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}})t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})^{*}e^{i(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}})t}]\times i\boldsymbol{k}$$

or

$$\int I d\boldsymbol{r} = [\hat{a}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})e^{-i(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}})t} - \hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})^{*}e^{-i(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}})t}$$
$$- \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})e^{i(\omega_{\boldsymbol{k}'}-\omega_{\boldsymbol{k}})t} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s}(-i\frac{\partial}{\partial\boldsymbol{k}}\delta_{\boldsymbol{k}',\boldsymbol{k}})^{*}e^{i(\omega_{\boldsymbol{k}'}+\omega_{\boldsymbol{k}})t}]\times i\boldsymbol{k}$$

The average of this integral over a period T leads to

$$\int I d\boldsymbol{r} = (\hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s})(i\frac{\partial}{\partial\boldsymbol{k}}\times i\boldsymbol{k})$$

$$\boldsymbol{L}_{EM} = -\frac{i\hbar}{2}\sum_{\boldsymbol{k}',s'}\sum_{\boldsymbol{k},s}\sqrt{\frac{\omega_{\boldsymbol{k}'}}{\omega_{\boldsymbol{k}}}}\boldsymbol{\varepsilon}(\boldsymbol{k}',s')\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)\delta_{\boldsymbol{k}',\boldsymbol{k}}(\hat{a}_{\boldsymbol{k}',s'}\hat{a}^{+}{}_{\boldsymbol{k},s} + \hat{a}^{+}{}_{\boldsymbol{k}',s'}\hat{a}_{\boldsymbol{k},s})(i\frac{\partial}{\partial\boldsymbol{k}}\times i\boldsymbol{k})$$
$$= -\frac{i\hbar}{2}\sum_{,s'}\sum_{\boldsymbol{k},s}\boldsymbol{\varepsilon}(\boldsymbol{k},s')\cdot\boldsymbol{\varepsilon}(\boldsymbol{k},s)(\hat{a}_{\boldsymbol{k},s'}\hat{a}^{+}{}_{\boldsymbol{k},s} + \hat{a}^{+}{}_{\boldsymbol{k},s'}\hat{a}_{\boldsymbol{k},s})(i\frac{\partial}{\partial\boldsymbol{k}}\times i\boldsymbol{k})$$

Since

$$\boldsymbol{\varepsilon}(\boldsymbol{k},s')\boldsymbol{\varepsilon}(\boldsymbol{k},s) = \delta_{s,s'}$$

we get

$$\begin{aligned}
\boldsymbol{L}_{EM} &= \frac{1}{2}\sum_{\boldsymbol{k},s}(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}{}_{\boldsymbol{k},s} + \hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s})\langle \boldsymbol{k}|\hat{\boldsymbol{L}}|\boldsymbol{k}\rangle \\
&= \sum_{\boldsymbol{k},s}(\hat{a}^{+}{}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s} + \frac{1}{2})\langle \boldsymbol{k}|\hat{\boldsymbol{L}}|\boldsymbol{k}\rangle \\
&= \sum_{\boldsymbol{k},s}(\hat{N}_{\boldsymbol{k},s} + \frac{1}{2})[i\hbar\frac{\partial}{\partial \boldsymbol{k}}\times \boldsymbol{k}] \\
&= \sum_{\boldsymbol{k},s}(\hat{N}_{\boldsymbol{k},s} + \frac{1}{2})(\boldsymbol{r}\times \boldsymbol{p})
\end{aligned}$$

where

$$\begin{aligned}
&\langle \boldsymbol{k}|\hat{\boldsymbol{L}}|\boldsymbol{k}\rangle = \langle \boldsymbol{k}|\hat{\boldsymbol{r}}\times\hat{\boldsymbol{p}}|\boldsymbol{k}\rangle \\
&= (i\hbar\frac{\partial}{\partial \boldsymbol{k}}\times \boldsymbol{k})\langle \boldsymbol{k}|\boldsymbol{k}\rangle \\
&= i\hbar\frac{\partial}{\partial \boldsymbol{k}}\times \boldsymbol{k} \\
&= i\frac{\partial}{\partial \boldsymbol{k}}\times \hbar\boldsymbol{k} \\
&= \boldsymbol{r}\times\hbar\boldsymbol{k} \\
&= \boldsymbol{r}\times\boldsymbol{p}
\end{aligned}$$

and

$$\hat{\boldsymbol{r}} \to \boldsymbol{r} = i\hbar\frac{\partial}{\partial \boldsymbol{p}} = i\frac{\partial}{\partial \boldsymbol{k}}$$

This is an expression for the orbital angular momentum for photon which we expect.

((**Note**))

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{k}}\times[\boldsymbol{k}\psi(\boldsymbol{k})] &= \nabla_{\boldsymbol{k}}\times[\boldsymbol{k}\psi(\boldsymbol{k})] \\
&= \nabla_{\boldsymbol{k}}\psi(\boldsymbol{k})\times\boldsymbol{k} + \psi(\boldsymbol{k})(\nabla_{\boldsymbol{k}}\times\boldsymbol{k}) \\
&= -\boldsymbol{k}\times\nabla_{\boldsymbol{k}}\psi(\boldsymbol{k})
\end{aligned}$$

or

$$i\frac{\partial}{\partial \boldsymbol{k}}\times\boldsymbol{k} = -\boldsymbol{k}\times i\frac{\partial}{\partial \boldsymbol{k}} \to \boldsymbol{r}\times\boldsymbol{k} = -\boldsymbol{k}\times\boldsymbol{r}\,,$$

We note that the orbital angular momentum along the propagation direction $\boldsymbol{k}$ should be zero for some mode in the sum. As $(\boldsymbol{r}\times\boldsymbol{k})\times\boldsymbol{k} = 0$, for a particular mode, there is no orbital angular momentum component in the direction of propagation.

8. Operators for circularly polarized light

The right-circularly polarized state:

$$
\begin{aligned}
|R\rangle &= \frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle) \\
&= \frac{1}{\sqrt{2}}(|1_{\boldsymbol{k},1}\rangle + i|1_{\boldsymbol{k},2}\rangle) \\
&= \frac{1}{\sqrt{2}}(\hat{a}^{+}_{\boldsymbol{k},1} + i\hat{a}^{+}_{\boldsymbol{k},2})|0\rangle) \\
&= -\hat{a}^{+}_{\boldsymbol{k},R}|0\rangle \\
&= -|1_{\boldsymbol{k},R}\rangle
\end{aligned}
$$

or

$$|R\rangle = -\hat{a}^{+}_{\boldsymbol{k},R}|0\rangle = -|1_{\boldsymbol{k},R}\rangle, \qquad \hat{a}^{+}_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R}|0\rangle = |1_{\boldsymbol{k},R}\rangle$$

with

$$\hat{a}^{+}_{\boldsymbol{k},R} = -\frac{1}{\sqrt{2}}(\hat{a}^{+}_{\boldsymbol{k},1} + i\hat{a}^{+}_{\boldsymbol{k},2}), \qquad \hat{a}_{\boldsymbol{k},R} = -\frac{1}{\sqrt{2}}(\hat{a}_{\boldsymbol{k},1} - i\hat{a}_{\boldsymbol{k},2})$$

The left-circularly polarized state:

$$
\begin{aligned}
|L\rangle &= \frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle) \\
&= \frac{1}{\sqrt{2}}(|1_{\boldsymbol{k},1}\rangle - i|1_{\boldsymbol{k},2}\rangle) \\
&= \frac{1}{\sqrt{2}}(\hat{a}^{+}{}_{\boldsymbol{k},1} - i\hat{a}^{+}{}_{\boldsymbol{k},2})|0\rangle) \\
&= \hat{a}^{+}_{\boldsymbol{k},L}|0\rangle \\
&= |1_{\boldsymbol{k},L}\rangle
\end{aligned}
$$

or

$$|L\rangle = |1_{\boldsymbol{k},L}\rangle = \hat{a}^+_{\boldsymbol{k},L}|0\rangle, \qquad \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L}|0\rangle = |1_{\boldsymbol{k},R}\rangle$$

with

$$\hat{a}^+_{\boldsymbol{k},L} = \frac{1}{\sqrt{2}}(\hat{a}^+_{\boldsymbol{k},1} - i\hat{a}^+_{\boldsymbol{k},2}), \qquad \hat{a}_{\boldsymbol{k},L} = \frac{1}{\sqrt{2}}(\hat{a}_{\boldsymbol{k},1} + i\hat{a}_{\boldsymbol{k},2})$$

We also have

$$\hat{a}_{\boldsymbol{k},1} = \frac{1}{\sqrt{2}}(-\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L}), \qquad \hat{a}_{\boldsymbol{k},2} = -\frac{i}{\sqrt{2}}(\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L}).$$

The commutation relation:

$$[\hat{a}_{\boldsymbol{k},R}, \hat{a}^+_{\boldsymbol{k}',R}] = \frac{1}{2}[\hat{a}_{\boldsymbol{k},1} + i\hat{a}_{\boldsymbol{k},2}, \hat{a}^+_{\boldsymbol{k}',1} - i\hat{a}^+_{\boldsymbol{k}',2}] = \delta_{\boldsymbol{k},\boldsymbol{k}'},$$

$$[\hat{a}_{\boldsymbol{k},L}, \hat{a}^+_{\boldsymbol{k}',L}] = \frac{1}{2}[\hat{a}_{\boldsymbol{k},1} - i\hat{a}_{\boldsymbol{k},2}, \hat{a}^+_{\boldsymbol{k}',1} + i\hat{a}^+_{\boldsymbol{k}',2}] = \delta_{\boldsymbol{k},\boldsymbol{k}'},$$

since

$$[\hat{a}_{\boldsymbol{k},s}, \hat{a}^+{}_{\boldsymbol{k}',s'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{s,s'}.$$

9. Helicity; spin angular momentum along the *k* direction

We start with the expression of the spin angular momentum,

$$\hat{\boldsymbol{S}}_{EM} = i\hbar\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k}(\hat{a}_{\boldsymbol{k},2}{}^+\hat{a}_{\boldsymbol{k},1} - \hat{a}_{\boldsymbol{k},1}{}^+\hat{a}_{\boldsymbol{k},2}).$$

We note that

$$\hat{a}_{\boldsymbol{k},2}{}^{+}\hat{a}_{\boldsymbol{k},1} - \hat{a}_{\boldsymbol{k},1}{}^{+}\hat{a}_{\boldsymbol{k},2} = \frac{i}{\sqrt{2}}(\hat{a}_{\boldsymbol{k},R}^{+} + \hat{a}_{\boldsymbol{k},L}^{+})\frac{1}{\sqrt{2}}(-\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L})$$

$$-\frac{1}{\sqrt{2}}(-\hat{a}_{\boldsymbol{k},R}^{+} + \hat{a}_{\boldsymbol{k},L}^{+})(-\frac{i}{\sqrt{2}})(\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L})$$

$$= \frac{i}{2}[(\hat{a}_{\boldsymbol{k},R}^{+} + \hat{a}_{\boldsymbol{k},L}^{+})(-\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L}) + (-\hat{a}_{\boldsymbol{k},R}^{+} + \hat{a}_{\boldsymbol{k},L}^{+})(\hat{a}_{\boldsymbol{k},R} + \hat{a}_{\boldsymbol{k},L})]$$

$$= i(\hat{a}_{\boldsymbol{k},L}^{+}\hat{a}_{\boldsymbol{k},L} - \hat{a}_{\boldsymbol{k},R}^{+}\hat{a}_{\boldsymbol{k},R})$$

Then we get

$$\hat{\boldsymbol{S}}_{EM} = \hbar\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}}{k}(\hat{a}_{\boldsymbol{k},R}^{+}\hat{a}_{\boldsymbol{k},R} - \hat{a}_{\boldsymbol{k},L}^{+}\hat{a}_{\boldsymbol{k},L}).$$

We define the spin angular momentum along the $\boldsymbol{k}$ direction as the helicity,

$$\hat{\boldsymbol{S}}_{EM}\cdot\frac{\boldsymbol{k}}{k} = \hat{J}_z = i\hbar(\hat{a}_{\boldsymbol{k},2}^{+}\hat{a}_{\boldsymbol{k},1} - \hat{a}_{\boldsymbol{k},1}^{+}\hat{a}_{\boldsymbol{k},2}),$$

or

$$\hat{\boldsymbol{S}}_{EM}\cdot\frac{\boldsymbol{k}}{k} = \hat{J}_z = \hbar(\hat{a}_{\boldsymbol{k},R}^{+}\hat{a}_{\boldsymbol{k},R} - \hat{a}_{\boldsymbol{k},L}^{+}\hat{a}_{\boldsymbol{k},L}).$$

We note that

$$[\hat{J}_z, \hat{a}_{\boldsymbol{k},1}^{+}] = -i\hbar\hat{a}_{\boldsymbol{k},2}^{+}. \qquad [\hat{J}_z, \hat{a}_{\boldsymbol{k},2}^{+}] = i\hbar\hat{a}_{\boldsymbol{k},1}^{+}$$

$$[\hat{J}_z, \hat{a}_{\boldsymbol{k},R}^{+}] = \hbar\hat{a}_{\boldsymbol{k},R}^{+}, \qquad [\hat{J}_z, \hat{a}_{\boldsymbol{k},L}^{+}] = -\hbar\hat{a}_{\boldsymbol{k},L}^{+}$$

and

$$[\hat{J}_z, \hat{a}_{\boldsymbol{k},R}^{+}]\hat{a}_{\boldsymbol{k},R} + [\hat{J}_z, \hat{a}_{\boldsymbol{k},L}^{+}]\hat{a}_{\boldsymbol{k},L}) = \hbar(\hat{a}_{\boldsymbol{k},R}^{+}\hat{a}_{\boldsymbol{k},R} - \hat{a}_{\boldsymbol{k},L}^{+}\hat{a}_{\boldsymbol{k},L}).$$

(a) Matrix under the basis of $|x\rangle$ and $|y\rangle$

$$\hat{J}_z|x\rangle = \hat{J}_z|1_{\boldsymbol{k},1}\rangle = i\hbar(\hat{a}_{\boldsymbol{k},2}^{+}\hat{a}_{\boldsymbol{k},1} - \hat{a}_{\boldsymbol{k},1}^{+}\hat{a}_{\boldsymbol{k},2})|1_{\boldsymbol{k},1}\rangle = i\hbar(\hat{a}_{\boldsymbol{k},2}^{+}\hat{a}_{\boldsymbol{k},1})|1_{\boldsymbol{k},1}\rangle = i\hbar|1_{\boldsymbol{k},2}\rangle = i\hbar|y\rangle$$

$$\hat{J}_z|y\rangle = \hat{J}_z|1_{\boldsymbol{k},2}\rangle = i\hbar(\hat{a}^+_{\boldsymbol{k},2}\hat{a}_{\boldsymbol{k},1} - \hat{a}^+_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2})|1_{\boldsymbol{k},2}\rangle = -i\hbar(\hat{a}^+_{\boldsymbol{k},1}\hat{a}_{\boldsymbol{k},2})|1_{\boldsymbol{k},2}\rangle = -i\hbar|1_{\boldsymbol{k},1}\rangle = -i\hbar|x\rangle$$

Then we have

$$\hat{J}_z = \hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hbar\hat{\sigma}_y ,$$

under the basis of $|x\rangle$ and $|y\rangle$. The eigenkets of $\hat{J}_z$ are obtained as

$$\frac{1}{\sqrt{2}}(|x\rangle + i|y\rangle) = |R\rangle, \qquad \text{with the eigenvalue } (+\hbar).$$

and

$$\frac{1}{\sqrt{2}}(|x\rangle - i|y\rangle) = |L\rangle, \qquad \text{with the eigenvalue } (-\hbar).$$

(b) Matrix $\hat{J}_z$ under the basis of $\{|R\rangle$ and $|L\rangle\}$,

$$\hat{J}_z|R\rangle = -\hat{J}_z|1_{\boldsymbol{k},R}\rangle = -\hbar(\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} - \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L})|1_{\boldsymbol{k},R}\rangle = -\hbar|1_{\boldsymbol{k},R}\rangle = \hbar|R\rangle ,$$

$$\hat{J}_z|L\rangle = \hat{J}_z|1_{\boldsymbol{k},L}\rangle = \hbar(\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} - \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L})|1_{\boldsymbol{k},L}\rangle = -\hbar|1_{\boldsymbol{k},L}\rangle = -\hbar|L\rangle ,$$

Then we have

$$\hat{J}_z = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hbar\hat{\sigma}_z ,$$

under the basis of $|R\rangle$ and $|L\rangle$, which is diagonal. The eigenkets of $\hat{J}_z$ are obtained as

$$\hat{J}_z|R\rangle = \hbar|R\rangle ,$$

$$\hat{J}_z|L\rangle = -\hbar|L\rangle .$$

((**Note**)) **Summary**

The total energy is

$$\hat{H} = \sum_k \hbar\omega_{\boldsymbol{k}} (\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} + \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L}) .$$

The total linear momentum is

$$\hat{\boldsymbol{P}}_{EM} = \sum_k \hbar\boldsymbol{k} (\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} + \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L}) .$$

The total number of photons is

$$\hat{N} = \sum_k (\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} + \hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R}) .$$

The helicity is given by

$$\hat{J}_z = \hbar(\hat{a}^+_{\boldsymbol{k},R}\hat{a}_{\boldsymbol{k},R} - \hat{a}^+_{\boldsymbol{k},L}\hat{a}_{\boldsymbol{k},L}) .$$

11. $\{\hat{a}^+_{\boldsymbol{k},1}, \hat{a}^+_{\boldsymbol{k},2}\}$ as a vector operator

$\hat{J}_z$ is the helicity of the photon. We note that

$$[\hat{J}_z, \hat{a}^+_{\boldsymbol{k},1}] = i\hbar\hat{a}^+_{\boldsymbol{k},2} . \qquad [\hat{J}_z, \hat{a}^+_{\boldsymbol{k},2}] = -i\hbar\hat{a}^+_{\boldsymbol{k},1} .$$

This means that $\{\hat{a}^+_{\boldsymbol{k},1}, \hat{a}^+_{\boldsymbol{k},2}\}$ is a vector operator. Then we have

$$\hat{R}\hat{a}^+_{\boldsymbol{k},1}\hat{R}^+ = \hat{a}^+_{\boldsymbol{k},1}\Re_{11} + \hat{a}^+_{\boldsymbol{k},2}\Re_{21} = \hat{a}^+_{\boldsymbol{k},1}\cos\alpha + \hat{a}^+_{\boldsymbol{k},2}\sin\alpha ,$$

$$\hat{R}\hat{a}^+_{\boldsymbol{k},2}\hat{R}^+ = \hat{a}^+_{\boldsymbol{k},1}\Re_{12} + \hat{a}^+_{\boldsymbol{k},2}\Re_{22} = -\hat{a}^+_{\boldsymbol{k},1}\sin\alpha + \hat{a}^+_{\boldsymbol{k},2}\cos\alpha ,$$

where $\hat{R}$ is the rotation operator around the *z* axis, with the rotation angle α.

$$\hat{R}\hat{a}^+_{\boldsymbol{k},R}\hat{R}^+ = -\frac{1}{\sqrt{2}}\hat{R}(\hat{a}^+_{\boldsymbol{k},1} + i\hat{a}^+_{\boldsymbol{k},2})\hat{R}^+ = e^{-i\alpha}\hat{a}^+_{\boldsymbol{k},R} .$$

and

$$\hat{R}\hat{a}^{+}_{\boldsymbol{k},L}\hat{R}^{+} = \frac{1}{\sqrt{2}}\hat{R}(\hat{a}^{+}_{\boldsymbol{k},1} - i\hat{a}^{+}_{\boldsymbol{k},2})\hat{R}^{+} = e^{i\alpha}\hat{a}^{+}_{\boldsymbol{k},L}\,.$$

10. The uncertainty in the electric field and magnetic field

We calculate the expectation of $\boldsymbol{E}$ and $\boldsymbol{E}^2$, $\boldsymbol{B}$, and $\boldsymbol{B}^2$ in the state $\left|n_{\boldsymbol{k},s}\right\rangle$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}[\frac{\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)}{k}][\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}]$$

$$\begin{aligned}
\hat{\boldsymbol{E}}^2(\boldsymbol{r},t) &= -\frac{2\pi\hbar}{V}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s')[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} - \hat{a}^{+}_{\boldsymbol{k},s}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}[\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - \hat{a}^{+}_{\boldsymbol{k}'s'}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= -\frac{2\pi\hbar}{V}\sum_{\boldsymbol{k},s}\sum_{\boldsymbol{k}',s'}\sqrt{\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}'}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)\cdot\boldsymbol{\varepsilon}(\boldsymbol{k}',s') \\
&\quad [\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - \hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}'s'}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} - \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}',s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)} \\
&\quad + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k}'s'}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}e^{-i(\boldsymbol{k}'\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}'}t)}] \\
&= \frac{2\pi\hbar}{V}\omega_{\boldsymbol{k}}(\hat{a}_{\boldsymbol{k},s}\hat{a}^{+}_{\boldsymbol{k},s} + \hat{a}^{+}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}) + ...
\end{aligned}$$

$$\begin{aligned}
\left\langle n_{\boldsymbol{k},s}\right|\hat{\boldsymbol{E}}^2(\boldsymbol{r},t)\left|n_{\boldsymbol{k},s}\right\rangle &= \frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}\left\langle n_{\boldsymbol{k},s}\right|(\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}{}^{+} + \hat{a}_{\boldsymbol{k},s}{}^{+}\hat{a}_{\boldsymbol{k},s})\left|n_{\boldsymbol{k},s}\right\rangle \\
&= \frac{4\pi\hbar\omega_{\boldsymbol{k}}}{V}(n_{k,s} + \frac{1}{2})
\end{aligned}$$

$$\left\langle n_{\boldsymbol{k},s}\right|\hat{\boldsymbol{E}}(\boldsymbol{r},t)\left|n_{\boldsymbol{k},s}\right\rangle = 0$$

$$\begin{aligned}
\hat{\boldsymbol{B}}^2(\boldsymbol{r},t) &= \frac{2\pi\hbar\omega_{\boldsymbol{k}}}{Vk^2}[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)]\cdot[\boldsymbol{k}\times\boldsymbol{\varepsilon}(\boldsymbol{k},s)](\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}{}^{+} + \hat{a}_{\boldsymbol{k},s}{}^{+}\hat{a}_{\boldsymbol{k},s}) + ... \\
&= \frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}(\hat{a}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k},s}{}^{+} + \hat{a}_{\boldsymbol{k},s}{}^{+}\hat{a}_{\boldsymbol{k},s}) + ...
\end{aligned}$$

$$\langle n_{\mathbf{k},s}|\hat{\mathbf{B}}^2(\mathbf{r},t)|n_{\mathbf{k},s}\rangle = \frac{2\pi\hbar\omega_{\mathbf{k}}}{V}\langle n_{\mathbf{k},s}|(\hat{a}_{\mathbf{k},s}\hat{a}_{\mathbf{k},s}{}^{+} + \hat{a}_{\mathbf{k},s}{}^{+}\hat{a}_{\mathbf{k},s})|n_{\mathbf{k},s}\rangle$$
$$= \frac{4\pi\hbar\omega_{\mathbf{k}}}{V}(n_{\mathbf{k},s} + \frac{1}{2})$$

$$\langle n_{\mathbf{k},s}|\hat{\mathbf{B}}(\mathbf{r},t)|n_{\mathbf{k},s}\rangle = 0$$

The uncertainty for the electric field and magnetic field are obtained as

$$\sqrt{\langle n_{\mathbf{k},s}|\hat{\mathbf{E}}^2(\mathbf{r},t)|n_{\mathbf{k},s}\rangle - \langle n_{\mathbf{k},s}|\hat{\mathbf{E}}(\mathbf{r},t)|n_{\mathbf{k},s}\rangle^2} = \sqrt{\frac{4\pi\hbar\omega_{\mathbf{k}}}{V}(n_{\mathbf{k},s} + \frac{1}{2})},$$

$$\sqrt{\langle n_{\mathbf{k},s}|\hat{\mathbf{B}}^2(\mathbf{r},t)|n_{\mathbf{k},s}\rangle - \langle n_{\mathbf{k},s}|\hat{\mathbf{B}}(\mathbf{r},t)|n_{\mathbf{k},s}\rangle^2} = \sqrt{\frac{4\pi\hbar\omega_{\mathbf{k}}}{V}(n_{\mathbf{k},s} + \frac{1}{2})}.$$

We also note that

$$\sqrt{\langle 0_{\mathbf{k},s}|\hat{\mathbf{E}}^2(\mathbf{r},t)|0_{\mathbf{k},s}\rangle - \langle 0_{\mathbf{k},s}|\hat{\mathbf{E}}(\mathbf{r},t)|0_{\mathbf{k},s}\rangle^2} = \sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}$$

$$\sqrt{\langle 0_{\mathbf{k},s}|\hat{\mathbf{B}}^2(\mathbf{r},t)|0_{\mathbf{k},s}\rangle - \langle 0_{\mathbf{k},s}|\hat{\mathbf{B}}(\mathbf{r},t)|0_{\mathbf{k},s}\rangle^2} = \sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}$$

$$\langle 0_{\mathbf{k},s}|\hat{\mathbf{E}}^2(\mathbf{r},t)|0_{\mathbf{k},s}\rangle = \frac{2\pi\hbar\omega_{\mathbf{k}}}{V},$$

$$\langle 0_{\mathbf{k},s}|\hat{\mathbf{B}}^2(\mathbf{r},t)|0_{\mathbf{k},s}\rangle = \frac{2\pi\hbar\omega_{\mathbf{k}}}{V}$$

The full vacuum state $|0\rangle$ is the vacuum of each mode, namely

$$|0\rangle = |0_{\mathbf{k}_1,s_1}\rangle \otimes |0_{\mathbf{k}_1,s_1}\rangle \otimes |0_{\mathbf{k}_1,s_1}\rangle \otimes |0_{\mathbf{k}_1,s_1}\rangle$$

Thus

$$\langle 0|\hat{\mathbf{E}}^2(\mathbf{r},t)|0\rangle = \sum_{\mathbf{k},s}\frac{2\pi\hbar\omega_{\mathbf{k}}}{V} = \frac{4\pi}{V}\sum_{\mathbf{k},s}\frac{\hbar\omega_{\mathbf{k}}}{2} = \frac{4\pi}{V}\varepsilon_0$$

Similarly, we have

$$\langle 0|\hat{\boldsymbol{B}}^2(\boldsymbol{r},t)|0\rangle = \sum_{\boldsymbol{k},s}\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V} = \frac{4\pi}{V}\sum_{\boldsymbol{k},s}\frac{\hbar\omega_{\boldsymbol{k}}}{2} = \frac{4\pi}{V}\varepsilon_0$$

where ε_0 is the sum of the zero-point energies for each mode. Thus we get

$$\langle 0|\hat{H}|0\rangle = \frac{1}{8\pi}\int d\boldsymbol{r}\langle 0|\hat{\boldsymbol{E}}^2(\boldsymbol{r},t)+\hat{\boldsymbol{B}}^2(\boldsymbol{r},t)|0\rangle = \varepsilon_0$$

((**Note**))

The quantum fluctuations of the electromagnetic field have important physical consequences. In addition to the Casimir effect, they also lead to a splitting between the $2S_{1/2}$ and $2P_{1/2}$ levels of the hydrogen atom, which are degenerate in the approximation of the relativistic Dirac theory. This is called the *Lamb shift*. The quantum fluctuations are also responsible for the anomalous magnetic moment of the electron.

11. Average of electric field over the coherent state

The coherent state $|\alpha\rangle$ is defined by

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n_{\boldsymbol{k},s}}^{\infty}\frac{\alpha^{n_{\boldsymbol{k},s}}}{\sqrt{n_{\boldsymbol{k},s}!}}|n_{\boldsymbol{k},s}\rangle,$$

where $\alpha = |\alpha|e^{i\delta}$. We calculate the average of $\hat{\boldsymbol{E}}$ over the state $|\alpha\rangle$.

$$\begin{aligned}\langle\alpha|\hat{\boldsymbol{E}}|\alpha\rangle &= i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\langle\alpha|\hat{a}_{\boldsymbol{k},s}|\alpha\rangle e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \langle\alpha|\hat{a}_{\boldsymbol{k},s}^{+}|\alpha\rangle e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\\ &= i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\alpha e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \alpha^* e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]\\ &= i\sqrt{\frac{2\pi\hbar\omega_k}{V}}|\alpha|\boldsymbol{\varepsilon}(\boldsymbol{k},s)[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t+\delta)} - e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t+\delta)}]\\ &= -2\sqrt{\frac{2\pi\hbar\omega_k}{V}}|\alpha|\boldsymbol{\varepsilon}(\boldsymbol{k},s)\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t+\delta)\end{aligned}$$

which is just the form of a classical travelling wave.

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = i\sum_{\boldsymbol{k},s}\sqrt{\frac{2\pi\hbar\omega_{\boldsymbol{k}}}{V}}\boldsymbol{\varepsilon}(\boldsymbol{k},s)[\hat{a}_{\boldsymbol{k},s}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} - \hat{a}_{\boldsymbol{k},s}^{+}e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}]$$

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APPENDIX I Spin operator for photon

The spin angular momentum for photon is given by

$$\begin{aligned}\left(\hat{\boldsymbol{S}}_{EM}\right)_i &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}(-i\hbar\varepsilon_{ijk}\hat{E}_j\hat{A}_k) \\ &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}\hat{E}_j(-i\hbar\varepsilon_{ijk})\hat{A}_k \\ &= \frac{i}{4\pi c\hbar}\int d\boldsymbol{r}\hat{E}_j(\hat{S}_i)_{jk}\hat{A}_k\end{aligned}$$

$(S_i)_{jk}$ comes from the cross product operator, however, it can be seen to be a quantum spin operator, that couples different components of ***E*** and ***A***. This operator is defined here by its matrices, one for each component *i*, where *j* and *k* are the column and row

$$(\hat{S}_i)_{jk} = -i\hbar\varepsilon_{ijk},$$

with

$$\hat{S}_x = -i\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \hat{S}_y = -i\hbar\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{S}_z = -i\hbar\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We note that

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \qquad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \qquad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

$$\hat{S}_x^{\ 2} + \hat{S}_y^{\ 2} + \hat{S}_z^{\ 2} = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2\hbar^2\hat{1}$$

None of the matrices $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are diagonal when expressed in Cartesian. This just means that the Cartesian axes, to which these correspond, are not the good quantization axes.

((Eigenvalue problems))

(a) $\hat{S}_x$

Eigenvalue: $(+\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$

Eigenvalue: (0) Eigenket: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Eigenvalue: $(-\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$

(b) $\hat{S}_y$

Eigenvalue: $(-\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$

Eigenvalue: (0) Eigenket: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Eigenvalue: $(+\hbar)$ Eigenket: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$

(c) $\hat{S}_z$

Eigenvalue: $(+\hbar)$ Eigenket: $|u_+\rangle = |+\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$

Eigenvalue: (0) Eigenket: $|u_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Eigenvalue: $(-\hbar)$ Eigenket: $|u_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$

The unitary operator $\hat{U}_z$ and its Hermite conjugate $\hat{U}_z^{\,+}$ are defined by

$$\hat{U}_z = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}, \qquad \hat{U}_z^{\,+} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

We have

$$\hat{U}_z^{\ +}\hat{S}_z\hat{U}_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \hat{J}_z$$

$$\hat{U}_z^{\ +}\hat{S}_x\hat{U}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \hat{J}_x$$

$$\hat{U}_z^{\ +}\hat{S}_y\hat{U}_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \hat{J}_y$$

where $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_x$ are the conventional angular momentum with the magnitude $\hbar$.

We now calculate the rotation operators

$$\hat{R}_z(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_z\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z(\theta)\left|u_+\right\rangle = e^{-i\theta}\left|u_+\right\rangle, \quad \hat{R}_z(\theta)\left|u_0\right\rangle = \left|u_0\right\rangle, \qquad \hat{R}_z(\theta)\left|u_-\right\rangle = e^{i\theta}\left|u_-\right\rangle$$

Note that

$$\hat{R}_z^{\ +}(\theta)\hat{S}_x\hat{R}_z(\theta) = \hat{S}_x\cos\theta - \hat{S}_y\sin\theta$$

$$\hat{R}_z^{\ +}(\theta)\hat{S}_y\hat{R}_z(\theta) = \hat{S}_x\sin\theta + \hat{S}_y\cos\theta$$

We also have

$$\hat{R}_x(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_x\theta] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{R}_y(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_y\theta] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

If we choose the eigenvectors of the spin operator $\hat{S}_z$,

$$\begin{aligned}\hat{E}_j(\hat{S}_z)_{jk}\hat{A}_k &= E_+\langle u_+|\hat{S}_z|u_+\rangle A_+ + E_-\langle u_-|\hat{S}_z|u_-\rangle A_- + E_0\langle u_0|\hat{S}_z|u_0\rangle A_0 \\ &= \hbar E_+A_+ - \hbar E_-A_-\end{aligned}$$

The two states $|+\rangle$ and $|-\rangle$ correspond to states where the *A* and *E* fields are rotating around the *z* axis. It is typical to consider waves propagating along the *z* axis.

APPENDIX II Rotation operator

Rotation operator:

$$\hat{R}_z(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_z\theta] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z^+(\theta) = \exp[\frac{i}{\hbar}\hat{S}_z\theta] = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{R}_z^+(\theta)\hat{S}_x\hat{R}_z(\theta) = \hat{S}_x\cos\theta - \hat{S}_y\sin\theta,$$

$$\hat{R}_z^+(\theta)\hat{S}_x\hat{R}_z(\theta) = \hat{S}_x\sin\theta + \hat{S}_y\cos\theta$$

$$\hat{R}_z(\theta)|u_+\rangle = e^{-i\theta}|u_+\rangle, \qquad \hat{R}_z(\theta)|u_-\rangle = e^{i\theta}|u_-\rangle$$

$$\hat{R}_x(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_x\theta] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$\hat{R}_y(\theta) = \exp[-\frac{i}{\hbar}\hat{S}_y\theta] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

((Mathematica))

```
Clear["Global`*"]; Sx = -ⅈ ℏ (0  0  0
                             0  0  1
                             0 -1  0);

Sy = -ⅈ ℏ (0  0  -1
           0  0   0
           1  0   0);

Sz = -ⅈ ℏ ( 0  1  0
           -1  0  0
            0  0  0);

Sx.Sx + Sy.Sy + Sz.Sz // MatrixForm
```

$$\begin{pmatrix} 2\hbar^2 & 0 & 0 \\ 0 & 2\hbar^2 & 0 \\ 0 & 0 & 2\hbar^2 \end{pmatrix}$$

```
Sx.Sy - Sy.Sx - ⅈ ℏ Sz
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```
Sy.Sz - Sz.Sy - ⅈ ℏ Sx
```

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```
Sz.Sx - Sx.Sz - ⅈ ℏ Sy
{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Eigensystem[Sx]
{{-ℏ, ℏ, 0}, {{0, ⅈ, 1}, {0, -ⅈ, 1}, {1, 0, 0}}}

Eigensystem[Sy]
{{-ℏ, ℏ, 0}, {{-ⅈ, 0, 1}, {ⅈ, 0, 1}, {0, 1, 0}}}

Eigensystem[Sz]
{{-ℏ, ℏ, 0}, {{ⅈ, 1, 0}, {-ⅈ, 1, 0}, {0, 0, 1}}}
```