Change in phase of wave function for electrons in magnetic field:
Application of Feynman path integral

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1. Introduction

Both Dirac and Feynman first recognized the importance of the change in phase of the wave function in quantum mechanics. In particular, Feynman found that the phase in the electron wave function can be expressed in good approximation by the action for the classical Lagrangian even for the quantum mechanics.

When I taught the Feynman path integral at the class of Phys.422 (Quantum mechanics II) in Spring, 2021, I finally realized that the Onsager theory of de Haas-van Alphen effet (dHvA) published in 1952 (Philosophical Magazine) can be clearly understood in terms of the Feynman path integral. In spite of the quantum mechanics, the change in the phase of wave function for moving electrons along the path can be expressed in good approximation by

$$\exp(\frac{i}{\hbar}S_{cl}) = \exp(\frac{i}{\hbar}\int L_{cl}dt),$$

along the classical path, where S_{cl} is the action and L_{cl} is the Lagrangian where the classical motion of electrons is governed by the Lagrange equation. When the electron returns to the starting point after completing the closed orbit in the real space, the change of the phase in wave function should be equal to $2\pi n$, where n is an integer.

The quantization of the magnetic flux is experimentally observed in various kinds of electron systems (single electron or Cooper pair of two electrons) in the presence of an external magnetic field. The de Haas-van Alphen (dHvA) effect, Aharonov-Bohm (AB) effect, and superconductivity are among typical examples of such quantum phenomena. Note that the property of magnetic quantum flux is closely related to the unique and unusual role of vector potential $\boldsymbol{A}$;

(i) The phase in the wave function of electrons (single electron or Cooper pair) may undergo a change as

$$\psi' = \exp(\frac{iq\chi}{\hbar c})\psi,$$

after the gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \qquad \Phi' = \Phi - \frac{1}{c}\frac{\partial\chi}{\partial t}.$$

where $\boldsymbol{A}$ is the vector potential and Φ is a scalar potential.
(ii) We also notice that the canonical momentum $\boldsymbol{P}$ is related to the kinetic momentum $\mathbf{p} = m\mathbf{v}$ (satisfying the Newton second law for mechanics) with the relation

$$\mathbf{P} = \frac{\hbar}{i}\nabla = m\mathbf{v} + \frac{q}{c}\mathbf{A}.$$ (This relation will be discussed later in detail)

Here, we will present a possible explanation for these quantum behaviors, using the concept derived mainly from the Feynman path integral. We will also show that the approach of the Feynman path integral is actually equivalent to that based on the Schrödinger equation, in particular, the Ginzburg-Landau equation for the superconductivity. The magnetic quantum flux in the superconducting ring will be discussed on the basis of these models. The content of our discussion is as follows.

(a) de Haas-van Alphen effect (Onsager),
(b) Difference between canonical momentum and kinetic momentum,
(c) Aharonov-Bohm effect based on the Feynman path integral,
(d) Equivalence between methods of Feynman path integral and Ginzburg-Landau equation, for superconductor,
(e) Quantization of magnetic flux,
(f) London equation and magnetic field penetration depth (Meissner effect),
(g) Magnetic flux trapped in superconducting ring,
(h) Discussion of the measurement results of quantized magnetic flux both in the ZFC (zero-field cooled) state and FC (field-cooled state); Goodman et al (1971).

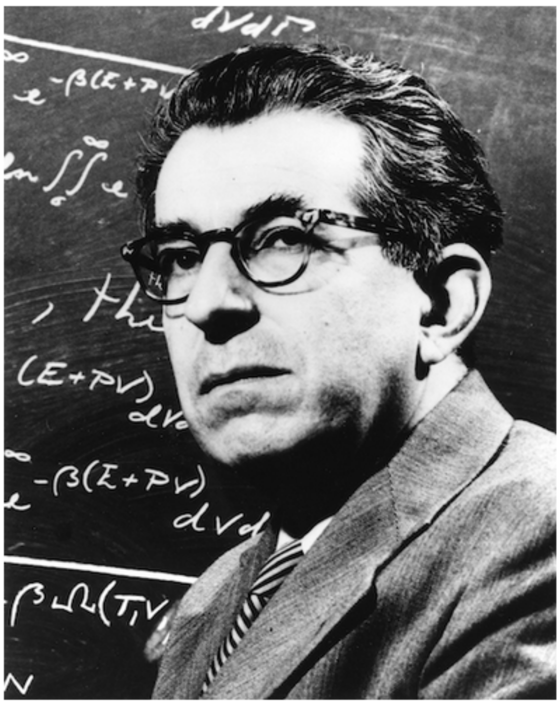
((**Note**)) **ZFC magnetization measurement**

We have experiences of doing experiments on spin glasses (aging dynamics) using the SQUID magnetometer (Quantum Design, MPMS XL-5) with an ultra low field capability. From a experimental view-point, it is very important to realize the noticeable difference between the ZFC and FC magnetization. In our case, first, a remnant magnetic field is reduced to zero field (exactly less than 3 mOe) at 298 K. Then the sample is cooled from 298 K to 1.9 K in zero field for the ZFC process. The Earth magnetic field at Binghamton, NY is 0.3 Oe (=3.0x10^{-5} T).

((**Fritz W. London**))

Fritz Wolfgang London (March 7, 1900 – March 30, 1954) was a German physicist and professor at Duke University. His fundamental contributions to the theories of chemical bonding and of intermolecular forces (London dispersion forces) are today considered classic and are discussed in standard textbooks of physical chemistry. With his brother Heinz London, he made a significant contribution to understanding electromagnetic properties of superconductors with the London equations and was nominated for the Nobel Prize in Chemistry on five separate occasions.

https://en.wikipedia.org/wiki/Fritz_London



Picture of Prof. Fritz London
https://physicstoday.scitation.org/pto/info/resources/whitepapers

((**Lars Onsager**))

Lars Onsager (November 27, 1903 – October 5, 1976) was a Norwegian-born American physical chemist and theoretical physicist. He held the Gibbs Professorship of Theoretical Chemistry at Yale University. He was awarded the Nobel Prize in Chemistry in 1968.
https://en.wikipedia.org/wiki/Lars_Onsager

Picture of Prof. Lars Onsager
https://en.wikipedia.org/wiki/Lars_Onsager#/media/File:Lars_Onsager2.jpg

2. Action in the Feynman path integral

The notion of *path integral* (sometimes also called *functional integral* or *integral over trajectories* or *integral over histories* or *continuous integral*) was introduced, for the first time, in the 1920s by Norbert Wiener (1921, 1923, 1924, 1930) as a method to solve problems in the theory of diffusion and Brownian motion. This integral, which is now also called the *Wiener integral*, has played a central role in the further development of the subject of path integration.

It was reinvented in a different form by Richard Feynman (1942, 1948) in 1942, for the reformulation of quantum mechanics (the so-called '*third formulation* of quantum mechanics' besides the Schrödinger and Heisenberg ones). The Feynman approach was inspired by Dirac's paper (1933) on the role of the Lagrangian and the least-action principle in quantum mechanics. This eventually led Feynman to represent the propagator of the Schrödinger equation by the complex-valued path integral which now bears his name. At the end of the 1940s Feynman (1950, 1951) worked out, on the basis of the path integrals, a new formulation of quantum electrodynamics and developed the well-known diagram technique for perturbation theory

We start with the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L_{cl}}{\partial \mathbf{v}}\right) = \frac{\partial L_{cl}}{\partial \mathbf{r}},$$

or

$$\frac{d}{dt}\mathbf{P} = \frac{\partial L_{cl}}{\partial \mathbf{r}},$$

The conjugate momentum ***P*** can be derived as

$$\mathbf{P} = \int \frac{\partial L_{cl}}{\partial \mathbf{r}} dt = \frac{\partial}{\partial \mathbf{r}} \int L_{cl} dt ,$$

or

$$\int \mathbf{P} \cdot d\mathbf{r} = \int L_{cl} dt .$$

Here we use a type of techniques which is used in the Feynman-Hellmann theorem;

$$\langle \psi(\lambda) | \frac{\partial \hat{H}}{\partial \lambda} | \psi(\lambda) \rangle = \frac{\partial}{\partial \lambda} \langle \psi(\lambda) | \hat{H} | \psi(\lambda) \rangle .$$

Thus, the action S_{cl} can be expressed by

$$S_{cl} = \int L_{cl} dt = \int \mathbf{P} \cdot d\mathbf{r} .$$

in the Feynman path integral. The change in phase of the wave function is

$$\Delta\theta = \frac{1}{\hbar} S_{cl} = \frac{1}{\hbar} \int \mathbf{P} \cdot d\mathbf{r} .$$

This simple form of the phase is essential point derived from the Feynman path integral.

3. Action S_{cl} of free particle

The Lagrangian for the free particles is

$$L = \frac{1}{2} m \mathbf{v}^2 ,$$

where *m* is the mass of particle. The Lagrange equation is

$$\frac{d}{dt}\mathbf{P} = \frac{d}{dt}(\frac{\partial L}{\partial \mathbf{v}}) = m\mathbf{v} = \frac{\partial L}{\partial \mathbf{r}} = 0,$$

where $\mathbf{P}$ is the canonical momentum (conjugate momentum),

$$\mathbf{P} = \frac{\partial L_{cl}}{\partial \mathbf{v}},$$

So that we have

$$L_{cl} = \mathbf{P} \cdot \mathbf{v} = m\mathbf{v}^2 = 2T,$$

where T is the kinetic energy of free particle. In the Feynman path integral, the phase is given by

$$\begin{aligned}
\exp(\frac{i}{\hbar}S_{cl}) &= \exp(\frac{i}{\hbar}\int L_{cl}dt) \\
&= \exp(\frac{i}{\hbar}\int \mathbf{P}\cdot \mathbf{v}dt) \\
&= \exp(\frac{i}{\hbar}\int \mathbf{P}\cdot d\mathbf{r}) \\
&= \exp(\frac{i}{\hbar}\int m\mathbf{v}\cdot d\mathbf{r}) \\
&= \exp(\frac{i}{\hbar}\int \mathbf{p}\cdot d\mathbf{r})
\end{aligned}$$

where $\mathbf{p} = m\mathbf{v}$. Note that for the free particle, we have $\mathbf{P} = \mathbf{p} = m\mathbf{v}$. Thus, we have an appropriate form of the phase as

$$\exp(\frac{i}{\hbar}S_{cl}) = \exp(\frac{i}{\hbar}\int \mathbf{p}\cdot d\mathbf{r}) \simeq \exp(\frac{i}{\hbar}\mathbf{p}\cdot \mathbf{r}).$$

This expression of the phase factor is the same as that of the plane wave and spherical wave. We also note that this is the same as Maupertuis' principle (1741) [Pierre Louis Maupertuis]

$$\delta S = \delta\int \mathbf{P}\cdot d\mathbf{r} = \delta\int 2Tdt,$$

(**Landau-Lifshitz**).

((Pierre Louis Maupertuis: (1698 – 1759) French.

Maupertuis is often credited with having invented the principle of least action; a version is known as Maupertuis's principle – an integral equation that determines the path followed by a physical system. His work in natural history is interesting in relation to modern science, since he touched on aspects of heredity and the struggle for life.

4. Change in phase of wave function for a particle in the presence of magnetic fiel*A*

From the theory of the Feynman path integral, it is found that the change in phase of the wave function is closely related to the action of classical Lagrangian, even for the phenomena of quantum mechanics. The change in phase in the presence of a vector potential ***A***, is given by

$$\Delta\theta = \frac{1}{\hbar}\int \mathbf{P}\cdot d\mathbf{r},$$

where ***P*** is the canonical momentum and ***p*** is the kinetic momentum, L_{cl} is the Lagrangian and is defined by

$$L_{cl} = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v},$$

where *e* is the charge of particle and *m* is the mass. The canonical momentum ***P*** is defined by

$$\mathbf{P} = \frac{\partial L_{cl}}{\partial \mathbf{v}} = m\mathbf{v} + \frac{q}{c}\mathbf{A}.$$

From the classical Lagrange theory. The equation of motion is governed by the Lagrange equation,

$$\frac{d}{dt}\mathbf{P} = \frac{d}{dt}\left(\frac{\partial L_{cl}}{\partial \mathbf{v}}\right) = \frac{\partial L_{cl}}{\partial \mathbf{r}}.$$

The time derivative of the kinetic momentum ***p*** is equal to an external force ***F*** such as Lorentz force,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \qquad \text{(Newton's second law)}$$

which will be shown later. In quantum mechanics, it is known that the canonical momentum ***P*** (but not ***p***) is expressed by the differential operator

$$\mathbf{P} = \frac{\hbar}{i}\nabla .$$

If we assume that the wave function is given by the form

$$\psi(\mathbf{r}) = |\psi_0| e^{i\theta(\mathbf{r})} ,$$

we get the expression,

$$\mathbf{P}\,\psi(\mathbf{r}) = \frac{\hbar}{i}\nabla |\psi_0| e^{i\theta(\mathbf{r})} = [\hbar\nabla\theta(\mathbf{r})]\psi(r) ,$$

leading to the relation

$$\frac{1}{\hbar}\mathbf{P} = \nabla\theta(\mathbf{r}) ,$$

in our cases where the amplitude of the wave function is constant. So that, the canonical momentum is directly related to the phase of the wave function. Using this relation, we get the change of phase as

$$\frac{1}{\hbar}S_{cl} = \frac{1}{\hbar}\int \mathbf{P}\cdot d\mathbf{r} = \int \nabla\theta(\mathbf{r})\cdot d\mathbf{r} = \Delta\theta .$$

In quantum mechanics, canonical momentum can be replaced by the quantum mechanical operator;

$$[\hat{P}_i, r_j] = \frac{\hbar}{i}\delta_{i,j}\hat{1} ,$$

or

$$\langle \mathbf{r}|\hat{\mathbf{P}}|\psi\rangle = \frac{\hbar}{i}\nabla\langle \mathbf{r}|\psi\rangle = \frac{\hbar}{i}\nabla\psi(\mathbf{r}) ,$$

in the $|\mathbf{r}\rangle$ representation. The conjugate momentum is represented by

$$\mathbf{P} = \frac{\hbar}{i}\nabla .$$

5. de Haas-van Alphen (dHvA) effect

It is well known that a conduction electron in metal undergoes a rotation around a closed orbit in the presence of an external magnetic field B. At low temperatures and strong magnetic field, the magnetization oscillates with $1/B$ with the frequency F which is closely related to the extremal cross section of the Fermi surface normal to the direction of magnetic field. This phenomenon is called the de Haas-van Alphen effect.

In 1952, Lar Onsager spent at the Cambridge University as a sabbatical stay from Yale University. He stayed at the Mont Laboratory of Cambridge University. During the times, David Shoenberg and his group (A. Brian Pippard) studied the Fermi surface of metals such as copper, bismuth, and so on, by using a de Haas-van Alpen effect. Just before Onsager returned to Yale University at the end of his sabbatical stay, he submitted his theory how to understand the principle of de Haas van Alphen effect. According to Pippard, experimentalists in the Mond laboratory did not understand the theory proposed by Onsager.

When I was a graduate student of the University of Tokyo, I had an opportunity to do experiments of de Haas-van Alphen (dHvA) effect of copper. In order to understand the principle, I read the book of Charles Kittel, (APPENDIX of Introduction to Solid State Physics, 4th edition). I understood the essential point of the dHvA effect. However, I did not understand the starting assumption of the paper of Onsager (or Kittel); the difference between the mechanical linear momentum ($\mathbf{p} = m\mathbf{v}$) and the canonical momentum ($\mathbf{P}$). However, once we learn the principle of the Feynman path integral we now clearly understand the difference between p and $\boldsymbol{P}$. Note that sometimes, $\mathbf{\Pi}$ is used instead of $\boldsymbol{P}$.

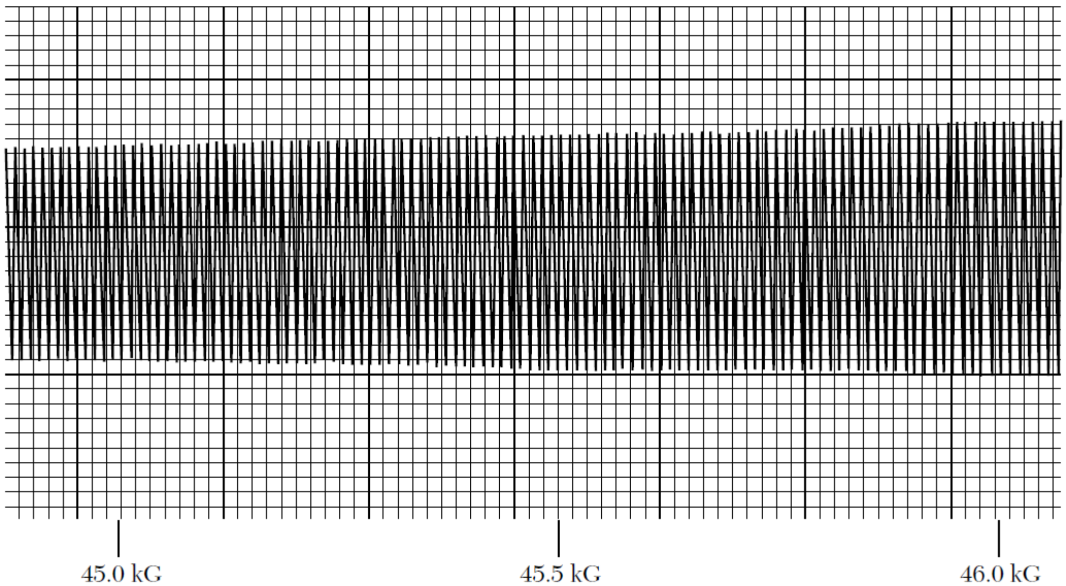


Fig.1 de Haas van Alphen effect in gold with B//[110]. The oscillation is from the dog's bone orbit. The signal is related to the second derivative of the magnetic moment with respect to field. (**C. Kittel**).

6. Action S_{cl} for a charged particle in the presence of vector potential A

We start with a Lagrangian given by

$$L = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v},$$

where $\boldsymbol{A}$ is the vector potential and $\boldsymbol{B}$ is the magnetic field ($\mathbf{B} = \nabla \times \mathbf{A}$). The canonical (conjugate) momentum is defined by

$$\mathbf{P} = \frac{\partial L_{cl}}{\partial \mathbf{v}} = m\mathbf{v} + \frac{q}{c}\mathbf{A} = \mathbf{p} + \frac{q}{c}\mathbf{A} \qquad \text{(conjugate momentum)}$$

Where $\mathbf{p} = m\mathbf{v}$ (the kinetic momentum) and is different from the conjugate momentum $\boldsymbol{P}$. Note that the Hamiltonian H is given by

$$
\begin{aligned}
H &= \mathbf{P}\cdot\mathbf{v} - L \\
&= (m\mathbf{v} + \frac{q}{c}\mathbf{A})\cdot\mathbf{v} - (\frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v}) \\
&= \frac{1}{2}m\mathbf{v}^2 \\
&= \frac{1}{2m}(\mathbf{P} - \frac{q}{c}\mathbf{A})^2
\end{aligned}
$$

In quantum mechanics, ***P*** corresponds to $\hat{\mathbf{P}}$ (operator) in quantum mechanics.

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{P}} - \frac{q}{c}\hat{\mathbf{A}})^2,$$

where we have the commutation relation,

$$[\hat{P}_i, \hat{x}_j] = \frac{\hbar}{i}\delta_{i,j}\hat{1},$$

and

$$\langle\mathbf{r}|\mathbf{P}|\psi\rangle = \frac{\hbar}{i}\nabla\langle\mathbf{r}|\psi\rangle = \frac{\hbar}{i}\nabla\psi(\mathbf{r}).$$

7. Role of mechanical momentum $\mathbf{p} = m\mathbf{v}$: Newton's second law

The Hamiltonian of the charged particle in the presence of vector potential $\hat{\mathbf{A}}$ and scalar potential Φ, is given by

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{P}} - \frac{q}{c}\hat{\mathbf{A}})^2 + q\Phi = \frac{1}{2}m\hat{\mathbf{v}}^2 + q\Phi,$$

with

$$\hat{\mathbf{p}} = m\hat{\mathbf{v}} = \hat{\mathbf{P}} - \frac{q}{c}\hat{\mathbf{A}}.$$

where $\hat{\mathbf{p}}$ is the operator of kinetic momentum. Note that the Lagrangian L is given by

$$L = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\Phi$$

Now we use the Heisenberg's equation of motion,

$$\begin{aligned}
\frac{d}{dt}m\hat{v}_x &= \frac{m}{i\hbar}\left[\hat{v}_x, \hat{H}\right] \\
&= \frac{m^2}{2i\hbar}\left[\hat{v}_x, \hat{v}_x{}^2 + \hat{v}_y{}^2 + \hat{v}_z{}^2\right] + \frac{mq}{i\hbar}\left[\hat{v}_x, \Phi\right] \\
&= \frac{m^2}{2i\hbar}\left[\hat{v}_x, \hat{v}_y{}^2 + \hat{v}_z{}^2\right] + \frac{mq}{i\hbar}\left[\hat{v}_x, \Phi\right] \\
&= \frac{m^2}{2i\hbar}\left(\frac{2iq\hbar B_z}{m^2c}\hat{v}_y - \frac{2iq\hbar B_y}{m^2c}\hat{v}_z\right) + \frac{q}{i\hbar}\left[\hat{P}_x - \frac{q}{c}\hat{A}_x, \Phi\right] \\
&= \frac{q}{c}(B_z\hat{v}_y - B_y\hat{v}_z) + \frac{q}{i\hbar}\left[\hat{P}_x, \Phi\right] \\
&= \frac{q}{c}(\hat{\mathbf{v}}\times\mathbf{B})_x - q\frac{\partial\Phi}{\partial\hat{x}}
\end{aligned}$$

where

$$\begin{aligned}
\left[\hat{v}_x, \hat{v}_y{}^2\right] &= \left[\hat{v}_x, \hat{v}_y\right]\hat{v}_y + \hat{v}_y\left[\hat{v}_x, \hat{v}_y\right] \\
&= \frac{2i\hbar q}{m^2c}B_z\hat{v}_y
\end{aligned}$$

$$\begin{aligned}
\left[\hat{v}_x, \hat{v}_z{}^2\right] &= \left[\hat{v}_x, \hat{v}_z\right]\hat{v}_z + \hat{v}_z\left[\hat{v}_x, \hat{v}_z\right] \\
&= -\frac{2i\hbar q}{m^2c}B_y\hat{v}_z
\end{aligned}$$

$$\left[\hat{P}_x, \Phi\right] = \frac{\hbar}{i}\frac{\partial}{\partial\hat{x}}\Phi$$

Note that the electric field and the magnetic field are defined by

$$\mathbf{E} = -\nabla\Phi, \qquad \mathbf{B} = \nabla\times\mathbf{A}.$$

Thus, we have the Newton's second law

$$\frac{d}{dt}m\hat{v}_x = \frac{q}{c}(\hat{\mathbf{v}}\times\mathbf{B})_x - q\frac{\partial}{\partial x}\Phi$$
$$= q[\mathbf{E}+\frac{1}{c}(\mathbf{v}\times\mathbf{B})]_x$$
$$= \mathbf{F}_x$$

where ***F*** is the Lorentz force. So that, we get the

$$m\frac{d\hat{\mathbf{p}}}{dt} = m\frac{d\hat{\mathbf{v}}}{dt} = \mathbf{F} = q(E+\frac{1}{c}\hat{\mathbf{v}}\times\mathbf{B}).$$

So that, the kinetic momentum ***p*** is directly related to the external force through the Newton's second law.

8. Commutation relation of velocity operators (Dyson)

We consider the commutation relations:

$$[m\hat{v}_x, m\hat{v}_y] = \left[\hat{P}_x - \frac{q}{c}\hat{A}_x, \hat{P}_y - \frac{q}{c}\hat{A}_y\right]$$
$$= \left[\hat{P}_x, \hat{P}_y\right] + \frac{q^2}{c^2}\left[\hat{A}_x, \hat{A}_y\right]$$
$$-\frac{q}{c}\left[\hat{P}_x, \hat{A}_y\right] + \frac{q}{c}\left[\hat{P}_y, \hat{A}_x\right]$$
$$= \frac{iq\hbar}{c}(\frac{\partial\hat{A}_y}{\partial\hat{x}} - \frac{\partial\hat{A}_x}{\partial\hat{y}})$$
$$= \frac{iq\hbar B_z}{c}\hat{1}$$

or

$$[\hat{v}_x, \hat{v}_y] = \frac{iq\hbar B_z}{m^2 c}\hat{1}$$

Similarly, we have

$$[\hat{v}_y, \hat{v}_z] = \frac{iq\hbar B_x}{m^2 c}\hat{1}, \qquad [\hat{v}_z, \hat{v}_x] = \frac{iq\hbar B_y}{m^2 c}\hat{1}$$

We also have the relations

$$[\hat{v}_i, \hat{x}_j] = \frac{1}{m}\left[\hat{P}_i - q\hat{A}_i, \hat{x}_j\right]$$
$$= \frac{1}{m}\left[\hat{P}_i, \hat{x}_j\right]$$
$$= \frac{\hbar}{im}\delta_{ij}\hat{1}$$

or

$$[\hat{v}_i, \hat{x}_j] = \frac{\hbar}{im}\delta_{ij}\hat{1}\,.$$

Similarly, we get

$$[x_i, \hat{v}_j] = \frac{1}{m}\left[\hat{x}_i, \hat{P}_j - \frac{q}{c}\hat{A}_j,\right]$$
$$= \frac{1}{m}\left[\hat{x}_i, \hat{P}_j\right]$$
$$= \frac{i\hbar}{m}\delta_{ij}\hat{1}$$

Derivative of the commutation relation:

$$[x_i, \frac{d}{dt}m\hat{v}_j] + [\hat{v}_i, m\hat{v}_j] = 0\,,$$

or

$$[x_i, \frac{\hat{F}_j}{m}] + [\hat{v}_i, \hat{v}_j] = 0\,.$$

9. Explanation of de Haas-van Alphen effect (Onsager, Kittel, Chambers)

Suppose that the electron moves along a closed orbit in the presence of a magnetic field along the *z* axis. According to the Feynman path integral, the phase factor is given by

$$\begin{aligned}\exp(\frac{i}{\hbar}S_{cl}) &= \exp(\frac{i}{\hbar}\int L_{cl}dt) \\ &= \exp(\frac{i}{\hbar}\int \mathbf{P}\cdot d\mathbf{r}) \\ &= \exp[\frac{i}{\hbar}\int (\mathbf{p}+\frac{q}{c}\mathbf{A})\cdot d\mathbf{r} \\ &= \exp(\frac{i}{\hbar}\int \mathbf{p}\cdot d\mathbf{r})\cdot\exp(\frac{iq}{c\hbar}\int \mathbf{A}\cdot d\mathbf{r})\end{aligned}$$

for the charged particle in the presence of magnetic field. We now consider the two parts, separately the first part and the second part. The equation of motion of a particle of charge q in a magnetic field $\boldsymbol{B}$ is

$$\frac{d\mathbf{p}}{dt} = \frac{q}{c}\frac{d\mathbf{r}}{dt}\times\mathbf{B},$$

(Lorentz force), where $\mathbf{p}$ $(= m\mathbf{v})$ is the kinetic momentum of the particle and $\boldsymbol{A}$ is the vector potential. This may be integrated with respect to time to give

$$\mathbf{p} = \frac{q}{c}\mathbf{r}\times\mathbf{B},$$

apart from an additive constant which does not contribute to the final result. Thus, one of the path integral is

$$\begin{aligned}\oint \mathbf{p}\cdot d\mathbf{r} &= \frac{q}{c}\oint(\mathbf{r}\times\mathbf{B})\cdot d\mathbf{r} \\ &= -\frac{q}{c}\mathbf{B}\cdot\oint(\mathbf{r}\times d\mathbf{r}) \\ &= -\frac{q}{c}\mathbf{B}\cdot 2a(\mathbf{r}) \\ &= -\frac{2q}{c}\Phi\end{aligned}$$

Φ is the magnetic flux contained within the orbit in real space

$$\Phi = Ba(\mathbf{r}),$$

where $a(\mathbf{r})$ is the area enclosed by closed orbit in the real space. The other path integral is

$$\frac{q}{c}\oint \mathbf{A}\cdot d\mathbf{r} = \frac{q}{c}\oint (\nabla\times\mathbf{A})\cdot d\mathbf{a}$$
$$= \frac{q}{c}\int \mathbf{B}\cdot d\mathbf{a}$$
$$= \frac{q}{c} Ba(\mathbf{r})$$
$$= \frac{q}{c}\Phi$$

using the Stokes' theorem, where $d\mathbf{a}$ is the area element (vector) in real space. The phase is given by

$$\Delta\phi = \frac{1}{\hbar}S_{cl}$$
$$= \frac{1}{\hbar}\int \mathbf{p}\cdot d\mathbf{r} + \frac{q}{c\hbar}\int \mathbf{A}\cdot d\mathbf{r}$$
$$= -\frac{2q}{c\hbar}\Phi + \frac{q}{c\hbar}\Phi$$
$$= -\frac{q}{c\hbar}\Phi$$

Suppose that the change of phase is $\Delta\phi = 2(n+\gamma)\pi$ after passing along the closed orbit, where γ is constant. Then the magnetic flux is quantized as

$$\Phi = (-\frac{2\pi c\hbar}{q})(n+\gamma)$$
$$= (-\frac{ch}{q})(n+\gamma) \quad ,$$
$$= 2\Phi_0(n+\gamma)$$

Or

$$\Phi = (n+\gamma)\frac{2\pi c\hbar}{|q|},$$

where

$$\Phi_0 = 2.067833848 \times 10^{-7}\,\text{G cm}^2 . \qquad \text{(magnetic flux quantum).}$$

The area of the orbit in the ***k***-space is related to the area of the orbit in the ***r***-space

$$a(\mathbf{r}) = (\frac{\hbar c}{qB})^2 a(\mathbf{k}) .$$

The magnetic flux is expressed in terms of

$$\Phi = Ba(\mathbf{r}) = B(\frac{\hbar c}{qB})^2 a(\mathbf{k}) = (\frac{\hbar c}{q})^2 \frac{1}{B} a(\mathbf{k}) ,$$

or

$$\Phi = (\frac{\hbar c}{e})^2 \frac{1}{B} a(\mathbf{k}) = \frac{2\pi\hbar c}{|e|}(n+\gamma) ,$$

or

$$a(\mathbf{k}) = \frac{2\pi|q|}{\hbar c} B(n+\gamma) .$$

In the Fermi surface experiments we may be interested in the increment $\Delta(\frac{1}{B})$

$$\Delta(\frac{1}{B}) = \frac{1}{F} = \frac{1}{a(k)}\frac{2\pi|q|}{\hbar c} ,$$

or

$$F = \frac{\hbar c}{2\pi|q|} a(k) .$$

In the experiment of dHvA effect, it is known that the magnetization of metals oscillates as a function of $1/B$ with the frequency as

$$M = \sin(\frac{2\pi F}{B}) ,$$

where F is directly related to the area of cross section of Fermi surface.

$$F = \frac{\hbar c}{2\pi |q|} a(\mathbf{k}) .$$

Experimentally, one can find the extremal cross section $a(\mathbf{k})$, which is perpendicular to the direction of magnetic field.

10. Mond Laboratory before and after the publication of Onsager's paper

I found a very interesting article in the book entitled Out of Crystal Maze by Hoddeson et al. In 1952, Lars Onsager stayed at the Mond Laboratory (Cambridge University), where the Fermi surfaces in metals such as copper and bismuth were experimentally determined by the de Haas-van Alphen (dHvA) effect, by David Shoenberg and A.B. Pippard, and others. Experimentalists were struggling to understand the principle of the dHvA effect. Onsager proposed his model on the quantization of magnetic quantum flux inside the closed orbits of conduction electrons.

((**Hoddeson et al.**))

This was basically the situation until at least the 1950/1951 academic year, when the Norwegian-American quantum chemist Lars Onsager visited Shoenberg's Mond Laboratory on a year's sabbatical from Yale. As Shoenberg recalls:

> For quite a while [even] before then he had made cryptic remarks. about a simple interpretation of the de Haas-van Alphen periodicities in terms of Fermi surface areas. But I could never understand just what he meant. It was only after repeated requests that he should write down his ideas that, practically on the eve of his return to Yale, he produced a three-page paper for *Phil. Mag.,* which has since become a classic. He showed that the period [of de Haas-van Alphen oscillations] was inversely proportional to the extremal area of cross-section of the Fermi surface. It turned out that I. M. Lifshitz had had the same idea independently but had not published it; eventually it was I. M. Lifshitz and his students who developed the general theory in detail.

Lifshitz's original contribution had come as early as 1950 "at a session of the Soviet Academy of Sciences [of the Ukrainian SSR in Kiev] when he read a paper that showed how the movement of an electron following a complex law of dispersion in a magnetic field can be quantized." As Shoenberg's former research student Brian Pippard recalls, Shoenberg and Onsager had shared an office during the latter's visiting year at Cambridge.

But even after Onsager had written his paper, at least for a year or two, Cambridge physicists tended to give it little importance. Pippard recalls:

> There wasn't a lovely lot of algebraic quantities and integrals and things which you could evaluate [in this paper] because Onsager was talking in *geometrical* terms—and I think David [Shoenberg] was disappointed to see "so little" coming out and failed to realize that Onsager had provided the complete clue. So nothing happened. The paper was published and nobody in Cambridge took any notice. They went on measuring the de Haas-van Alphen effect and fitting it with ellipsoidal shapes [in which the relation between energy and wave vector is assumed to be quadratic].

That was the situation at the end of 1952.

11. Demonstration of the notation $\mathbf{P} = \frac{\hbar}{i}\nabla$ (Richard Feynman)

I found a very interesting article on the canonical momentum (conjugate momentum). In his book of the Feynman Lectures on Physics, Richard Feynman demonstrated that

$$\mathbf{P} = \frac{\hbar}{i}\nabla .$$

using the concept of the gauge transformation. This is amazing to us. Here we will reproduce his discussion from his book. Note that the use of gauge transformation is essential to the demonstration of the above form of ***P***.

((**REFERENCES**))

R.P. Feynman, R.B. Leighton, and M. Sands; The Feynman Lectures on Physics, The New Millennium edition, Vol. III: Quantum Mechanics (Basic Books, 1964).

11.1 Gauge transformation (definition)

Here we use the notation q for the charge of electron, for convenience. According to the theory of the gauge transformation, the vector potential ***A*** and a scalar potential Φ

$$\mathbf{A}' = \mathbf{A} + \nabla\chi , \qquad \Phi' = \Phi - \frac{1}{c}\frac{\partial \chi}{\partial t} ,$$

where χ is an arbitrary function of *x*, *y*, *z*, and *t*. We note that Schrödinger equation is given by

$$[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^2+q\Phi]\psi=i\hbar\frac{\partial\psi}{\partial t},$$

for $\mathbf{A}$ and Φ

$$[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A}')^2+q\Phi']\psi'=i\hbar\frac{\partial\psi'}{\partial t},$$

for $\mathbf{A}'$ and Φ', where the wave function changes from ψ to ψ' as

$$\psi'=\exp(\frac{iq}{\hbar c}\chi)\psi.$$

11.2 Gauge used in the Aharonov-Bohm effect

In the above equations, we assume that $\mathbf{A}'=0$ such that $\chi=-\chi_0$; χ_0 is independent of t.

$$\mathbf{A}'=\mathbf{A}-\nabla\chi_0=0, \qquad \Phi'=\Phi,$$

leading to

$$\chi_0=\int\mathbf{A}\cdot d\mathbf{r},$$

$$\psi'=\exp(-\frac{iq}{\hbar c}\chi_0)\psi=\exp(-\frac{iq}{\hbar c}\int\mathbf{A}\cdot d\mathbf{r})\psi_0,$$

where the new wave function is expressed by $\psi'=\psi_0$. Thus, we have the Schrödinger equation

$$(\frac{1}{2m}\mathbf{P}^2+q\Phi)\psi_0=i\hbar\frac{\partial\psi_0}{\partial t}.$$

When $\Phi=0$, ψ_0 is the wave function of the free particle. So that, we can conclude that

$$\mathbf{P}=\frac{\hbar}{i}\nabla.$$

11.3 Probability current *J* and probability density ρ

For the original gauge, we still have the same Schrödinger equation,

$$[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^2+q\Phi]\psi=i\hbar\frac{\partial\psi}{\partial t}.$$

In quantum mechanics, the probability density is given by

$$\rho=\psi^*\psi,$$

and the continuity equation is

$$\frac{d}{dt}\rho=-\nabla\cdot\mathbf{J},$$

where ***J*** is the probability current. We note that

$$\frac{d}{dt}\rho=\frac{\partial\psi^*}{\partial t}\psi+\psi^*\frac{\partial\psi}{\partial t}.$$

Using the Schrödinger equation, we get

$$i\hbar\frac{\partial\psi}{\partial t}=[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^2+q\Phi]\psi,$$

and its complex conjugate,

$$-i\hbar\frac{\partial\psi^*}{\partial t}=[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^{*2}+q\Phi]\psi^*.$$

Then we have

$$\begin{aligned}
\frac{d}{dt}\rho &= \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t} \\
&= \frac{1}{i\hbar}\psi^*[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^2 + q\Phi]\psi - \frac{1}{i\hbar}\psi[\frac{1}{2m}(\mathbf{P}-\frac{q}{c}\mathbf{A})^{*2} + q\Phi]\psi^* \\
&= \frac{1}{2mi\hbar}[\psi^*(\mathbf{P}-\frac{q}{c}\mathbf{A})^2\psi - \psi(\mathbf{P}-\frac{q}{c}\mathbf{A})^{*2}\psi^*] \\
&= \frac{1}{2mi\hbar}[\psi^*(\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\cdot(\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\psi \\
&\quad -\psi(-\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\cdot(-\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\psi^*]
\end{aligned}$$

or

$$\frac{d}{dt}\rho = \frac{1}{2mi\hbar}[\psi^*(\mathbf{S}\cdot\mathbf{S})\psi - \psi(\mathbf{S}^*\cdot\mathbf{S}^*)\psi^*],$$

where

$$\mathbf{S} = \frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A} \qquad \mathbf{S}^* = -\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A}.$$

Note that

$$\begin{aligned}
I_1 &= \psi^*(\mathbf{S}\cdot\mathbf{S})\psi - \psi(\mathbf{S}^*\cdot\mathbf{S}^*)\psi^* \\
&= \psi^*(\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\cdot\mathbf{S}\psi - \psi(-\frac{\hbar}{i}\nabla-\frac{q}{c}\mathbf{A})\cdot\mathbf{S}^*\psi^* \\
&= \frac{\hbar}{i}(\psi^*\nabla\cdot\mathbf{S}\psi + \psi\nabla\cdot\mathbf{S}^*\psi^*) - \frac{q}{c}(\psi^*\mathbf{A}\cdot\mathbf{S}\psi - \psi\mathbf{A}\cdot\mathbf{S}^*\psi^*) \\
&= \frac{\hbar}{i}[\psi^*\nabla\cdot\mathbf{S}\psi + \psi\nabla\cdot\mathbf{S}^*\psi^* - \frac{q}{c}(\psi^*\mathbf{A}\cdot\nabla\psi - \psi\mathbf{A}\cdot\nabla\psi^*)]
\end{aligned}$$

where

$$\begin{aligned}
\psi^*\mathbf{A}\cdot\mathbf{S}\psi - \psi\mathbf{A}\cdot\mathbf{S}^*\psi^* &= \psi^*\mathbf{A}\cdot(\frac{\hbar}{i}\nabla\psi - \frac{q}{c}\mathbf{A}\psi) + \psi\mathbf{A}\cdot(-\frac{\hbar}{i}\nabla\psi^* + \frac{q}{c}\mathbf{A}\psi^*) \\
&= \frac{\hbar}{i}(\psi^*\mathbf{A}\cdot\nabla\psi - \psi\mathbf{A}\cdot\nabla\psi^*)
\end{aligned}$$

We now calculate the term defined by $\nabla\cdot(\psi^*\mathbf{S}\psi + \psi\mathbf{S}^*\psi^*)$

$$
\begin{aligned}
I_2 &= \nabla \cdot (\psi^* \mathbf{S} \psi + \psi \mathbf{S}^* \psi^*) \\
&= \psi^* \nabla \cdot (\mathbf{S} \psi) + \mathbf{S} \psi \cdot \nabla \psi^* + \psi \nabla \cdot (\mathbf{S}^* \psi) + (\mathbf{S}^* \psi^*) \cdot \nabla \psi \\
&= \psi^* \nabla \cdot (\mathbf{S} \psi) + \nabla \psi^* \cdot (\mathbf{S} \psi) + \psi \nabla \cdot (\mathbf{S}^* \psi^*) + \nabla \psi \cdot (\mathbf{S}^* \psi^*) \\
&= \psi^* \nabla \cdot (\mathbf{S} \psi) + \psi \nabla \cdot (\mathbf{S}^* \psi^*) + \mathbf{S} \psi \cdot \nabla \psi^* + \mathbf{S}^* \psi^* \cdot \nabla \psi \\
&= \psi^* \nabla \cdot (\mathbf{S} \psi) + \psi \nabla \cdot (\mathbf{S}^* \psi^*) - \frac{q}{c} (\mathbf{A} \psi \cdot \nabla \psi^* + \mathbf{A} \psi^* \cdot \nabla \psi)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{S} \psi \cdot \nabla \psi^* + \mathbf{S}^* \psi^* \cdot \nabla \psi &= (\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A}) \psi \cdot \nabla \psi^* + (-\frac{\hbar}{i} \nabla - \frac{q}{c} \mathbf{A}) \psi^* \cdot \nabla \psi \\
&= \frac{\hbar}{i} \nabla \psi \cdot \nabla \psi^* - \frac{\hbar}{i} \nabla \psi^* \cdot \nabla \psi \\
&\quad - \frac{q}{c} (\mathbf{A} \psi \cdot \nabla \psi^* + \mathbf{A} \psi^* \cdot \nabla \psi) \\
&= -\frac{q}{c} (\mathbf{A} \psi \cdot \nabla \psi^* + \mathbf{A} \psi^* \cdot \nabla \psi)
\end{aligned}
$$

Thus, we get the relation $I_1 = \frac{\hbar}{i} I_2$, leading to

$$
\begin{aligned}
\frac{d}{dt} \rho &= \frac{1}{2mi\hbar} [\psi^* (\mathbf{S} \cdot \mathbf{S}) \psi - \psi (\mathbf{S}^* \cdot \mathbf{S}^*) \psi^*] \\
&= \frac{1}{2mi\hbar} I_1 \\
&= \frac{1}{2mi\hbar} \frac{\hbar}{i} I_2 \\
&= -\frac{1}{2m} \nabla \cdot (\psi^* \mathbf{S} \psi + \psi \mathbf{S}^* \psi^*) \\
&= -\nabla \cdot \mathbf{J}
\end{aligned}
$$

The Probability current is obtained as

$$
\begin{aligned}
\mathbf{J} &= \frac{1}{2m} (\psi^* \mathbf{S} \psi + \psi \mathbf{S}^* \psi^*) \\
&= \mathrm{Re}[\psi^* \frac{\mathbf{S}}{m} \psi] \\
&= \mathrm{Re}[\psi^* \mathbf{v} \psi]
\end{aligned}
$$

The current density is related to the velocity $\boldsymbol{v}$ as

$$\mathbf{J} = \mathrm{Re}[\psi^* \mathbf{v} \psi],$$

$$m\mathbf{v} = \mathbf{S} = \mathbf{P} - \frac{q}{c}\mathbf{A} = \frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A},$$

$$\mathbf{P} = m\mathbf{v} + \frac{q}{c}\mathbf{A}.$$

12. Two kinds of momenta (Richard P. Feynman)

In his book, Feynman pointed out the difference between the canonical momentum $\boldsymbol{P}$ and $\mathbf{p} = m\mathbf{v}$. The difference of these momenta is clearly explained as follows by Feynman.

REFERENCES

R.P. Feynman, R. Leighton, and M. Sands, The Feynman Lectures on Physics, The New Millennium Edition Vol. III: Quantum Mechanics (Basic Books, 2010)

((**Feynman**))

The equation for the current is rather interesting, and sometimes causes a certain amount of worry. You would think the current would be something like the density of particles times the velocity. The density should be something like $\psi^*\psi$, which is o.k. And each term in

$$\mathbf{J} = \frac{1}{2}\{\psi^* (\frac{\hat{\mathbf{P}} - \frac{q}{c}A}{m})\psi + \psi(\frac{\hat{\mathbf{P}} - \frac{q}{c}A}{m})^* \psi^*\}$$

looks like the typical form for the average-value of the operator

$$\frac{1}{m}(\hat{\mathbf{P}} - \frac{q}{c}\mathbf{A}), \qquad (1)$$

so maybe we should think of it as the velocity of flow. It looks as though we have two suggestions for relations of velocity to momentum, because we would also think that momentum divided by mass, $\hat{\mathbf{P}}/m$, should be a velocity. The two possibilities differ by the vector potential. It happens that these two possibilities were also discovered in classical physics, when it was found that momentum could be defined in two ways. One of them is called "kinematic momentum," but for absolute clarity I will in this lecture call it the "*mv*-momentum." This is the momentum obtained by multiplying mass by velocity. The other is a more mathematical, more abstract momentum, some times called the "dynamical momentum," which I'll call "*p*-momentum." The two possibilities are

mv-momentum = $m\boldsymbol{v}$,

$$p\text{-momentum: } \mathbf{P} = m\mathbf{v} + \frac{q}{c}\mathbf{A} \qquad (2)$$

It turns out that in quantum mechanics with magnetic fields it is the *p*-momentum which is connected to the gradient operator $\hat{\mathbf{P}}$, so it follows that Eq.(1) is the operator of a velocity. I'd like to make a brief digression to show you what this is all about—why there must be something like Eq.(2) in the quantum mechanics. The wave function changes with time according to the Schrödinger equation in Eq.(3)

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi = [\frac{1}{2m}(\frac{\hbar}{i}\nabla - \frac{q}{c}\mathbf{A})^2 + q\phi]\psi \ . \qquad (3)$$

If I would suddenly change the vector potential, the wave function would not change at the first instant; only its rate of change changes. Now think of what would happen in the following circumstance. Suppose I have a long solenoid, in **Fig.2**. And there is a charged particle sitting nearby. Suppose this flux nearly instantaneously builds up from zero to something. I start with zero vector potential and then I turn on a vector potential **A**.

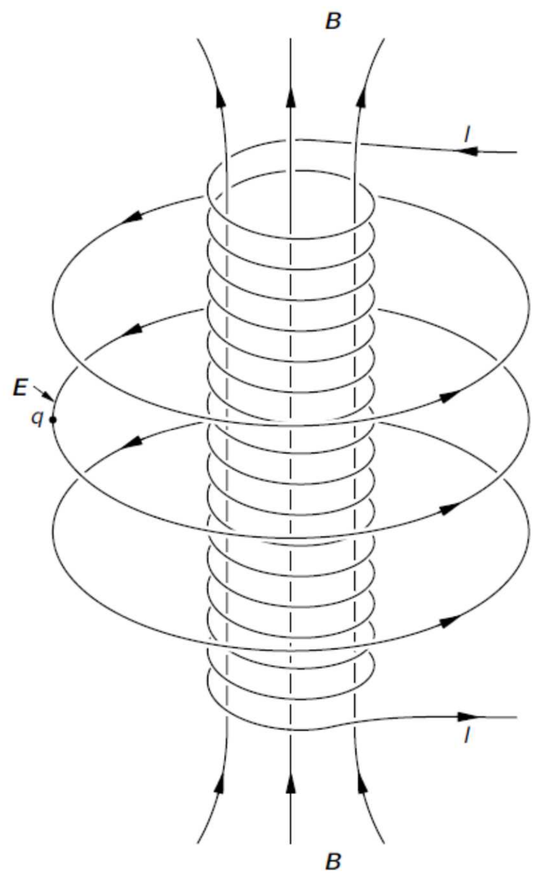


Fig.2 The electric field outside a solenoid with an increasing current.

That means that I produce suddenly a circumferential vector potential ***A***. You'll remember that the line integral of ***A*** around a loop is the same as the flux of ***B*** through the loop. Now what happens if I suddenly turn on a vector potential? According to the quantum mechanical equation the sudden change of ***A*** does not make a sudden change of ψ; the wave function is still the same. So the gradient is also unchanged.

But remember what happens electrically when I suddenly turn on a flux. During the short time that the flux is rising, there is an electric field generated whose line integral is the rate of change of the flux with time:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A} .$$

That electric field is enormous if the flux is changing rapidly, and it gives a force on the particle. The force is the charge times the electric field, and so during the build up of the flux the particle obtains a total impulse (that is, a change in *m**v***) equal to $-q\mathbf{A}/c$. In other words, if you suddenly turn on a vector potential at a charge, this charge immediately picks up an *mv*-momentum equal to $-q\mathbf{A}/c$. But there is something that is not changed immediately and that's the difference between *m**v*** and $-q\mathbf{A}/c$. And so the sum $\mathbf{P} = m\mathbf{v} + \frac{q}{c}\mathbf{A}$ is something which is not changed when you make a sudden change in the vector potential. This quantity ***P*** is what we have called the *p*-momentum and is of importance in classical mechanics in the theory of dynamics, but it also has a direct significance in quantum mechanics. It depends on the character of the wave function, and it is the one to be identified with the operator

$$\mathbf{P} = \frac{\hbar}{i}\nabla = m\mathbf{v} + \frac{q}{c}\mathbf{A}.$$

13. Aharonov-Bohm effect

Here we discuss the Aharonov-Bohn effect by using the Feynman path integral.

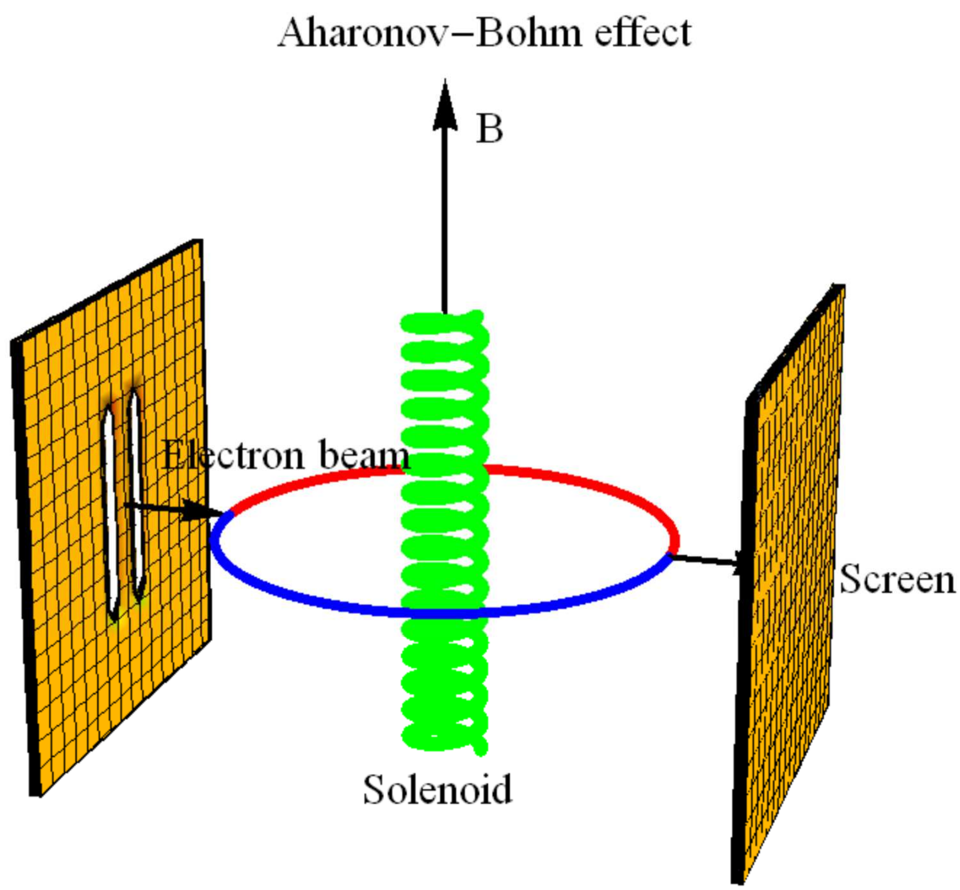


Fig.3 Schematic diagram of the Aharonov-Bohm experiment. Electron beams are split into two paths surrounding the long solenoid. The beams are brought together at a screen, and the resulting quantum interference pattern depends upon the magnetic flux strength, despite the fact that the electrons only encounter a zero magnetic field. Path denoted by red (counterclockwise). Path denoted by blue (clockwise).

We now consider the phase difference between two paths in the Aharonov-Bohm effect. A beam of electrons is split into two paths, and pass either side of a long solenoid, before being recombined. The beams are kept well away from the solenoid, so they encounter only regions where $\boldsymbol{B} = 0$. However, the potential vector A is not zero. We consider the two paths; path denoted by red (counterclockwise) and path denoted by blue (clockwise). The change of the phase along the red path

$$\theta_{red} = \frac{1}{\hbar}\int_{red} \mathbf{P}\cdot d\mathbf{r}$$
$$= \frac{1}{\hbar}\int_{red} (m\mathbf{v} + \frac{q}{c}\mathbf{A})\cdot d\mathbf{r}$$
$$= \frac{1}{\hbar}\int_{red} (\hbar\mathbf{k} + \frac{q}{c}\mathbf{A})\cdot d\mathbf{r}$$

Here we note that the kinetic momentum is given by

$$\mathbf{p} = \hbar\mathbf{k}$$

since there is no external force on electron along the red path;

$$\mathbf{F} = 0 = m\frac{d\mathbf{v}}{dt}.$$

The change of the phase along the blue path,

$$\theta_{blue} = \frac{1}{\hbar}\int_{blue} \mathbf{P}\cdot d\mathbf{r}$$
$$= \frac{1}{\hbar}\int_{blue} (m\mathbf{v} + \frac{q}{c}\mathbf{A})\cdot d\mathbf{r}$$
$$= \frac{1}{\hbar}\int_{blue} (\hbar\mathbf{k} + \frac{q}{c}\mathbf{A})\cdot d\mathbf{r}$$

The phase difference between two paths is obtained as

$$\Delta\theta = \theta_{red} - \theta_{blue}$$
$$= (\int_{red} \mathbf{k}\cdot d\mathbf{r} - \int_{blue} \mathbf{k}\cdot d\mathbf{r}) + \frac{q}{c\hbar}\oint \mathbf{A}\cdot d\mathbf{r}$$
$$= \frac{2\pi}{\lambda}\Delta d + \frac{q}{c\hbar}\int (\nabla\times\mathbf{A})\cdot d\mathbf{a}$$

Using the Stokes' theorem. Noting that $\mathbf{B} = \nabla\times\mathbf{A}$, we get the change of phase is

$$\Delta\theta = \frac{2\pi}{\lambda}\Delta d + \frac{q}{c\hbar}\int \mathbf{B}\cdot d\mathbf{a}$$
$$= \frac{2\pi}{\lambda}\Delta d + \frac{q}{c\hbar}\Phi$$

or

$$\Delta\theta = 2\pi \frac{\Delta d}{\lambda} - \pi \frac{\Phi}{\Phi_0}$$

where $|q| = e$ for single electron,

$$\Phi_0 = \frac{2\pi c\hbar}{2|q|} \qquad \text{with} \quad |q| = e.$$

The first term is the phase change without magnetic field and the second term is the phase change in the presence of magnetic field.

14. Magnetic flux quantization in superconducting ring

As a typical example, we apply the concept of quantum phase derived above, to the superconductivity, in particular superconducting ring in the presence of magnetic field.

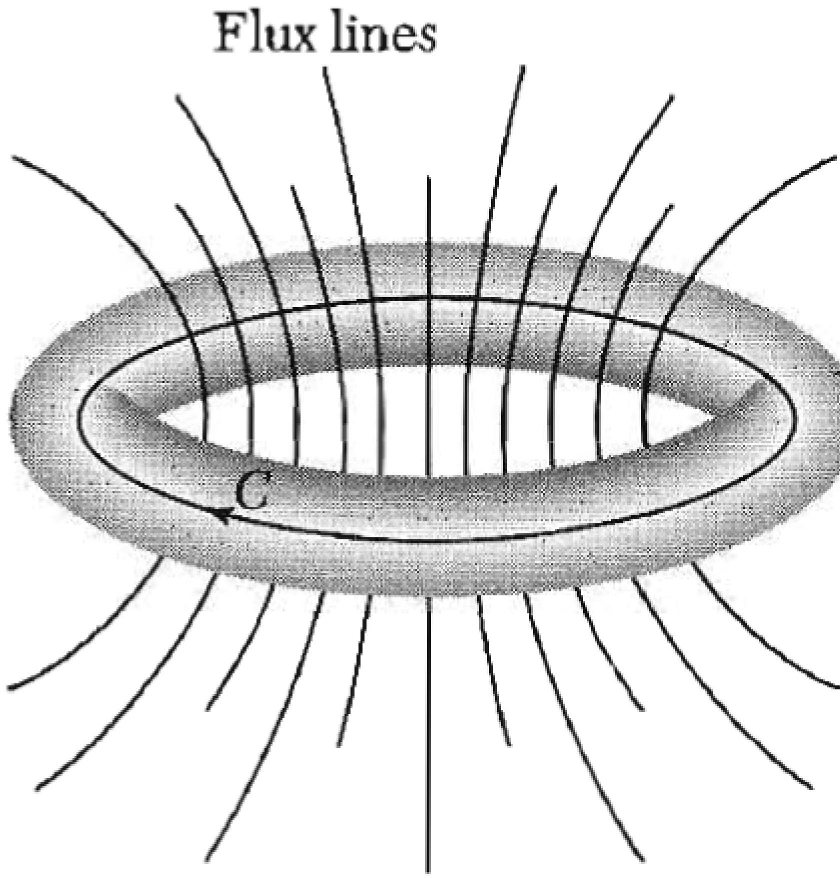


Fig.4 Superconducting ring. The magnetic field is applied along the *z* axis, normal to the ring. There is no current inside of the superconducting ring (C. Kittel). There are inner and outer edge currents only over the penetration depth from surfaces.

We now consider the magnetic flux quantization in superconductivity. According to the Feynman path integral, the change of the phase is expressed by

$$\Delta\theta = \frac{1}{\hbar}\oint \mathbf{P}\cdot d\mathbf{r} = \int \nabla\theta\cdot d\mathbf{r} = \frac{1}{\hbar}\oint (m^{*}\mathbf{v} + \frac{q^{*}}{c}\mathbf{A})\cdot d\mathbf{r}\,.$$

We note that the current density is related to the velocity $\mathbf{v}$ as $\mathbf{j} = n^{*}q^{*}\mathbf{v}$. Inside the superconductor ring, we have $\mathbf{J} = 0$. So that the change of the phase is rewritten as

$$\begin{aligned}\Delta\theta &= \frac{q^*}{c\hbar}\oint \mathbf{A}\cdot d\mathbf{l} \\ &= \frac{q^*}{c\hbar}\int \nabla\times\mathbf{A}\cdot d\mathbf{a} \\ &= \frac{q^*}{c\hbar}\int B\cdot d\mathbf{a} \\ &= \frac{q^*}{c\hbar}\Phi\end{aligned}$$

When $\Delta\theta = 2\pi$, the magnetic quantum flux is quantized as

$$\Phi = \frac{2\pi c\hbar}{q^*} = \frac{2\pi ch}{2|e|} = \Phi_0 .$$

Note that n^* the number density of Copper-pairs, and it is estimated as $n^+ = n/2$. $m^* = 2m$, and $q^* = -2e$ (e>).

15. London equation and Meissner effect in superconductor

From the equation,

$$\Delta\theta = \frac{1}{\hbar}\oint \mathbf{P}\cdot d\mathbf{r} = \int \nabla\theta\cdot d\mathbf{r} = \frac{1}{\hbar}\oint (m^*\mathbf{v} + \frac{q^*}{c}\mathbf{A})\cdot d\mathbf{r} ,$$

the London equation of superconductors can be derived as

$$m^*\mathbf{v} + \frac{q^*}{c}\mathbf{A} = \hbar\nabla\theta ,$$

or

$$\mathbf{v} = \frac{1}{m^*}(\hbar\nabla\theta - \frac{q^*}{c}\mathbf{A}) ,$$

or

$$\mathbf{J} = n^* q^* \mathbf{v} = \frac{n^* q^*}{m^*}(\hbar\nabla\theta - \frac{q^*}{c}\mathbf{A}).$$

We may take the rot of both sides to obtain the London equation

$$\nabla\times\mathbf{J} = -\frac{n^* q^{*2}}{m^* c}\nabla\times\mathbf{A} = -\frac{n^* q^{*2}}{m^* c}\mathbf{B}.$$

U sing the Maxwell's equation; $\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$ and $\nabla\cdot\mathbf{B} = 0$, we can derive the London equation

$$\nabla\times\mathbf{J} = \frac{c}{4\pi}\nabla\times(\nabla\times\mathbf{B}) = -\frac{n^* q^{*2}}{m^* c}\mathbf{B}$$

or

$$,\nabla\times(\nabla\times\mathbf{B}) = -\frac{4\pi n^* q^{*2}}{m^* c^2}\mathbf{B},$$

or

$$\nabla^2\mathbf{B} = \frac{4\pi n^* q^{*2}}{m^* c^2}\mathbf{B} = \frac{1}{\lambda_L{}^2}B,$$

where λ_L is the London penetration depth,

$$\lambda_L = \sqrt{\frac{m^* c^2}{4\pi n^* q^{*2}}} = \sqrt{\frac{mc^2}{4\pi ne^2}}.$$

The magnetic field reduces to zero over the penetration depth from the surface of the system (**Meissner effect**).

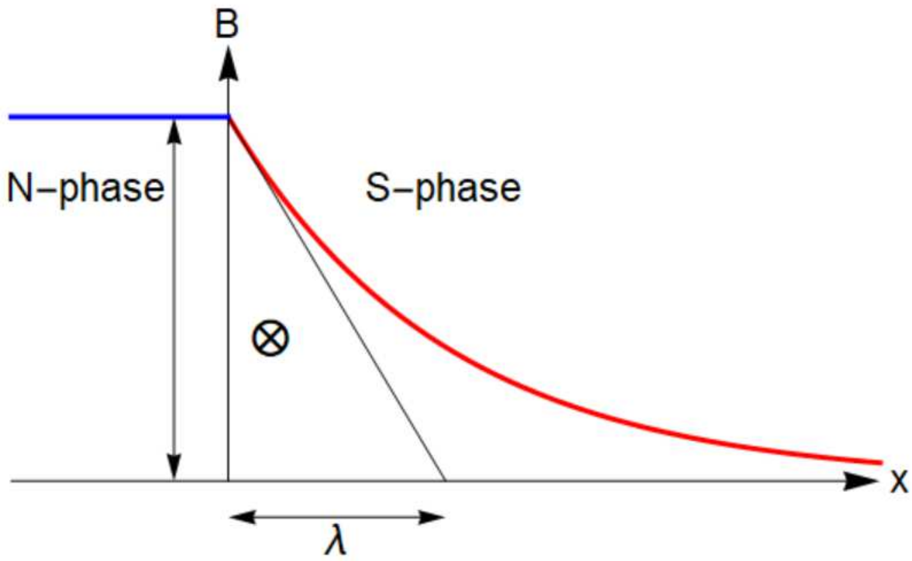


Fig.5 The magnetic field penetrates into the superconducting phase, over the London penetration depth. The current becomes zero inside the superconducting phase.

16. Derivation from Schrödinger equation: current density *J*

Heisenberg's equation of motion:

$$\hat{\mathbf{v}} = \frac{d}{dt}\hat{\mathbf{r}} = \frac{1}{i\hbar}[\hat{\mathbf{r}}, \hat{H}] = \frac{1}{2m^* i\hbar}[\hat{\mathbf{r}}, (\hat{\mathbf{P}} - \frac{q^*}{c}\mathbf{A}\hat{1})^2]$$

or

$$\hat{\mathbf{v}} = \frac{1}{2m^* i\hbar} i\hbar \frac{\partial}{\partial \hat{\mathbf{P}}}(\hat{\mathbf{P}} - \frac{q^*}{c}\mathbf{A}\hat{1})^2] = \frac{1}{m^*}(\hat{\mathbf{P}} - \frac{q^*}{c}\mathbf{A}\hat{1})$$

The probability current density is given by

$$\begin{aligned}\mathbf{J} &= \mathrm{Re}(\psi^* \mathbf{v} \psi) \\ &= \frac{1}{m^*}\mathrm{Re}[\psi^*(\mathbf{P} - \frac{q^*}{c}\mathbf{A})\psi] \\ &= \frac{1}{2m^*}[\psi^*(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi + \psi^*(-\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi]\end{aligned}$$

The current density is derived as

$$\boldsymbol{J} = \frac{q^* \hbar}{2m^* i}[\psi^* \nabla \psi - \psi \nabla \psi^*] - \frac{q^{*2} |\psi|^2}{m^* c} \boldsymbol{A} .$$

Now we assume that

$$\psi = |\psi| e^{i\theta} .$$

where $|\psi|$ is independent of $\boldsymbol{r}$ and t. Since

$$\psi^* \nabla \psi - \psi \nabla \psi^* = 2i |\psi|^2 \nabla \theta ,$$

we have

$$\boldsymbol{J} = \frac{q^* \hbar}{m^*} |\psi|^2 (\nabla \theta - \frac{q^*}{c\hbar} \boldsymbol{A}) = q^* |\psi|^2 \boldsymbol{v}_s$$

or

$$\hbar \nabla \theta = \frac{q^*}{c} \boldsymbol{A} + m^* \boldsymbol{v}_s .$$

This equation is generally valid. Note that $\boldsymbol{J}$ is gauge-invariant. Under the gauge transformation, the wave function is transformed as

$$\psi'(\boldsymbol{r}) = \exp(\frac{iq^* \chi}{\hbar c}) \psi(\boldsymbol{r}) .$$

This implies that

$$\theta \to \theta' = \theta + \frac{q^* \chi}{\hbar c} ,$$

Since $\boldsymbol{A}' = \boldsymbol{A} + \nabla \chi$, we have

$$\boldsymbol{J}' = \hbar(\nabla \theta' - \frac{q^*}{c\hbar} \boldsymbol{A}') = \hbar[\nabla(\theta + \frac{q^* \chi}{\hbar c}) - \frac{q^*}{c\hbar}(\boldsymbol{A} + \nabla \chi)] = \hbar(\nabla \theta - \frac{q^*}{c\hbar} \boldsymbol{A}) .$$

So the current density is invariant under the gauge transformation. Here we note that

$$\psi(\boldsymbol{r}) = |\psi(\boldsymbol{r})| \exp[i\theta(\boldsymbol{r})] .$$

and

$$\boldsymbol{P}\psi(\boldsymbol{r}) = \frac{\hbar}{i}\nabla\{\exp[i\theta(\boldsymbol{r})]|\psi(\boldsymbol{r})|\} = \frac{\hbar}{i}[i\psi(\boldsymbol{r})\nabla\theta(\boldsymbol{r}) + \exp[i\theta(\boldsymbol{r})]\nabla|\psi(\boldsymbol{r})|].$$

If $|\psi(\boldsymbol{r})|$ is independent of $\boldsymbol{r}$, we have

$$\boldsymbol{P}\psi(\boldsymbol{r}) = [\hbar\nabla\theta(\boldsymbol{r})]\psi(\boldsymbol{r})$$

or

$$\boldsymbol{P} = \hbar\nabla\theta(\boldsymbol{r}).$$

Then we have the following relation

$$\boldsymbol{P} = \hbar\nabla\theta = \frac{q^*}{c}\boldsymbol{A} + m^*\boldsymbol{v}_s$$

when $|\psi(\mathbf{r})|$ is independent of $\boldsymbol{r}$.

In summary, the approach from the Ginzburg-Landau equation leads top the same results derived from the Feynman path integral.

17. Approach from the Ginzburg-Landau equation: Flux quantization

We start with the current density

$$\boldsymbol{J}_s = \frac{q^*\hbar}{m^*}|\psi|^2(\nabla\theta - \frac{q^*}{c\hbar}\boldsymbol{A}) = q^*|\psi|^2\boldsymbol{v}_s.$$

Suppose that $n_s^* = |\psi|^2$=constant, then we have

$$\nabla\theta = \frac{m^*}{q^*\hbar n_s^*}\boldsymbol{J}_s + \frac{q^*}{c\hbar}\boldsymbol{A},$$

or

$$\oint\nabla\theta\cdot d\boldsymbol{l} = \frac{m^*}{q^*\hbar n_s^*}\oint\boldsymbol{J}_s\cdot d\boldsymbol{l} + \frac{q^*}{c\hbar}\oint\boldsymbol{A}\cdot d\boldsymbol{l}.$$

The path of integration can be taken inside the penetration depth where $\boldsymbol{J}_s$=0.

$$\oint\nabla\theta\cdot d\boldsymbol{l} = \frac{q^*}{c\hbar}\oint\boldsymbol{A}\cdot d\boldsymbol{l} = \frac{q^*}{c\hbar}\int(\nabla\times\boldsymbol{A})\cdot d\boldsymbol{a} = \frac{q^*}{c\hbar}\int\boldsymbol{B}\cdot d\boldsymbol{a} = \frac{q^*}{c\hbar}\Phi,$$

where Φ is the magnetic flux. Then we find that

$$\Delta\theta = \theta_2 - \theta_1 = 2\pi n = \frac{q^*}{c\hbar}\Phi ,$$

where n is an integer. The phase θ of the wave function must be unique, or differ by a multiple of 2π at each point,

$$\Phi = \frac{2\pi c\hbar}{|q^*|} n .$$

The flux is quantized. When $|q^*| = 2|e$ (for |Cooper pair of electrons), we have a magnetic quantum fluxoid;

$$\Phi_0 = \frac{2\pi c\hbar}{2|e|} = \frac{ch}{2|e|} = 2.06783372 \times 10^{-7} \text{ Gauss cm}^2.$$

18. Magnetic field penetration depth λ

We start with the current density (Ginzburg-Landau theory)

$$\boldsymbol{J} = \frac{q^*\hbar}{2m^* i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^* c}\boldsymbol{A} .$$

We assume that $\psi = \psi_\infty$ (real).

$$\boldsymbol{J} = -\frac{q^{*2}\psi_\infty^{\ 2}}{m^* c}\boldsymbol{A} \qquad \text{(London's equation),}$$

$$\nabla\times\boldsymbol{J} = -\frac{q^{*2}\psi_\infty^{\ 2}}{m^* c}\nabla\times\boldsymbol{A} = -\frac{q^{*2}\psi_\infty^{\ 2}}{m^* c}\boldsymbol{B} ,$$

where

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} , \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A} , \qquad \nabla\cdot\boldsymbol{B} = 0 .$$

Then we have

$$\nabla\times(\frac{c}{4\pi}\nabla\times\boldsymbol{B}) = -\frac{q^{*2}\psi_\infty^{\ 2}}{m^* c}\boldsymbol{B} ,$$

or

$$\nabla\times(\nabla\times\boldsymbol{B})=\nabla(\nabla\cdot\boldsymbol{B})-\nabla^2\boldsymbol{B}=-\frac{4\pi q^{*2}\psi_\infty{}^2}{m^*c^2}\boldsymbol{B}.$$

Then we have London's equation

$$\nabla^2\boldsymbol{B}=\frac{1}{\lambda^2}\boldsymbol{B},$$

where

$$\lambda^{-2}=\frac{4\pi q^{*2}\psi_\infty{}^2}{m^*c^2}.$$

λ is the penetration depth

$$\lambda=\sqrt{\frac{m^*c^2}{4\pi q^{*2}\psi_\infty{}^2}}=\sqrt{\frac{m^*c^2\beta}{4\pi q^{*2}|\alpha|}}.$$

The solution of the above differential equation is given by

$$B_Z(x)=B_Z(x=0)\exp(-x/\lambda),$$

where the magnetic field is directed along the z axis.

$$\boldsymbol{J}=\frac{c}{4\pi}\nabla\times\boldsymbol{B}=\frac{c}{4\pi}\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & B_z(x)\end{vmatrix}=\frac{c}{4\pi}(0,-\frac{\partial}{\partial x}B_z(x),0).$$

The current $\boldsymbol{J}$ flows along the y direction.

$$J_y=\frac{cB_z(x=0)}{4\pi\lambda}\exp(-x/\lambda).$$

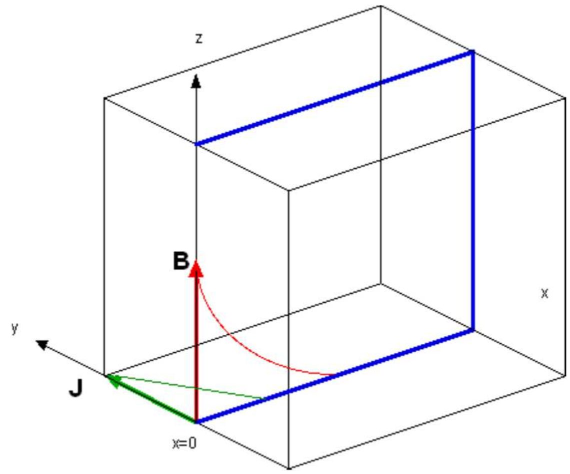


Fig.6 The distribution of the magnetic induction $B(x)$ (along the z axis) and the current density (along the y axis) near the boundary between the normal phase and the superconducting phase. The plane with $x = 0$ is the boundary.

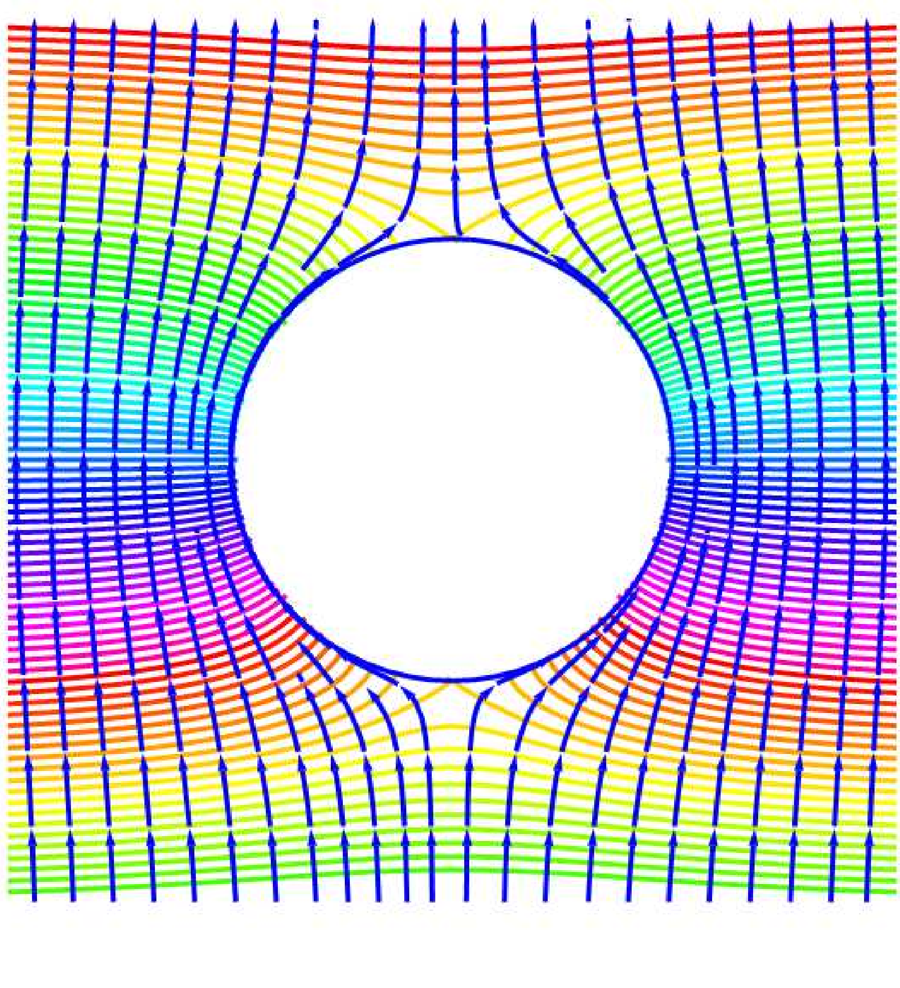


Fig.7 Meissner effect, which can be observed in the ZFC state. $B = 0$ inside the sphere (superconductor). We use Mathematica (ContourPlot and

StreamPlot). See the detail in Superconductivity (M.S. Suzuki and I.S. Suzuki).

19. Comment by Parks on the superconducting current in superconducting ring

Long ago, it was suggested by Fritz London that if a supercurrent flows in a superconducting ring, the magnetic flux maintained in the hole by the current should be quantized (in the FC state). In his article, Parks discussed the current inside the superconducting ring. We show his interesting discussion on the quantized magnetic flux as follows.

R.D. Parks, Quantized Magnetic Flux in Superconductors, Science, Dec. 11, 1964. New Series Vol. 146, No. 3650 (Dec.11, 1964) p.1429-1435 (American Association for the Advancement of Science).
https://www.jstor.org/stable/1714816

For the Superconducting cylinder with very thin wall, we have

$$nh = \oint \mathbf{P}\cdot d\mathbf{l} = \oint m^* \mathbf{v}_s \cdot d\mathbf{l} + \frac{q^*}{c}\oint \mathbf{A}\cdot d\mathbf{l}$$

$$\frac{c}{q^*}nh = \frac{c}{q^*}\oint \mathbf{P}\cdot d\mathbf{l} = \frac{c}{q^*}\oint m^* \mathbf{v}_s \cdot d\mathbf{l} + \oint \mathbf{A}\cdot d\mathbf{l}$$

$$\frac{c}{q^*}nh = c\Lambda\oint \mathbf{J}\cdot d\mathbf{l} + \int \mathbf{B}\cdot d\mathbf{a}$$

where

$$\mathbf{J} = n_s q^* \mathbf{v}_s, \qquad \Lambda = \frac{m^*}{n_s q^{*2}},$$

$$\Phi_0 = \frac{hc}{|q^*|} = \frac{2\pi\hbar c}{|q^*|} = \frac{\pi\hbar c}{e}$$

The London penetration depth λ_L:

$$\lambda_L^{-2} = \frac{4\pi n_s q^{*2}}{m^* c^2} = \frac{4\pi}{c^2\Lambda}, \qquad \Lambda = \frac{4\pi\lambda_L^2}{c^2}$$

$$\frac{c}{q^*}nh = -\Phi_0 n = c\Lambda J_s 2\pi R + \Phi$$

$$J_s = \frac{1}{2\pi Rc\Lambda}(-\Phi_0 n - \Phi) = \frac{\Phi_0}{2\pi Rc\Lambda}(-n - \frac{\Phi}{\Phi_0})$$

For convenience, we change n to $-n$. Thus, we get

$$J_s = \frac{\Phi_0}{2\pi Rc\Lambda}(n - \frac{\Phi}{\Phi_0}) = J_{s0}(n - \frac{\Phi}{\Phi_0}).$$

The kinetic energy of the system is

$$\begin{aligned} f_{KE} &= \frac{1}{2}n_s m^* \mathbf{v}_s^{\ 2} \\ &= \frac{1}{2}\Lambda \mathbf{J}_s^{\ 2} \\ &= \frac{\Phi_0^{\ 2}}{8\pi^2 R^2 c^2 \Lambda}(n - \frac{\Phi}{\Phi_0})^2 \\ &= f_0(n - \frac{\Phi}{\Phi_0})^2 \end{aligned}$$

where

$$\Phi = \pi R^2 B,$$

and B is the magnetic field in the hole of the ring or cylinder).

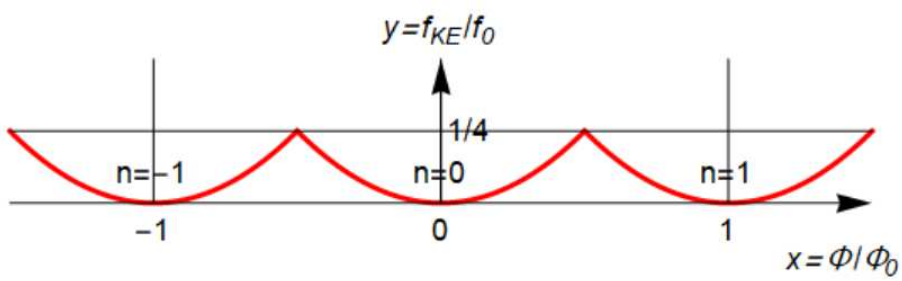


Fig.8 Plot of the normalized kinetic energy $y = f_{KE}/f_0$ as a function of normalized magnetic flux $x = \Phi/\Phi_0$. n is a quantum number (integer).

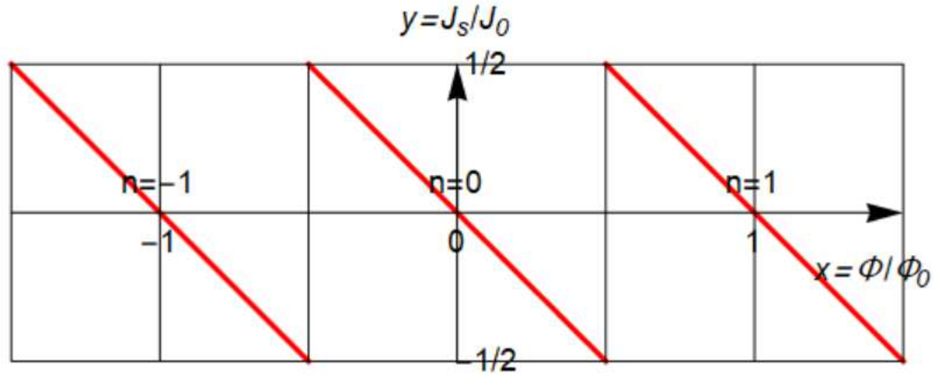


Fig.9 Plot of the normalized current density $y = J_s/J_0$ as a function of normalized magnetic flux $x = \Phi/\Phi_0$. n is a quantum number (integer).

20. Magnetic flux in the ZFC state and FC state

(a) FC state with $B \neq 0$

At $T > T_c$, the sample is in the absence of external magnetic field. Next, the external magnetic field (B_0) is applied to the sample (Sn, type-1 superconductor). Then, the sample is cooled down to the lower temperature below T_c. We call such a state as the **FC** (field cooled) state. At the fixed temperature T ($T < T_c$), the magnetic field is gradually removed. The magnetic flux is now trapped in the superconducting ring (or cylinder) without magnetic field.

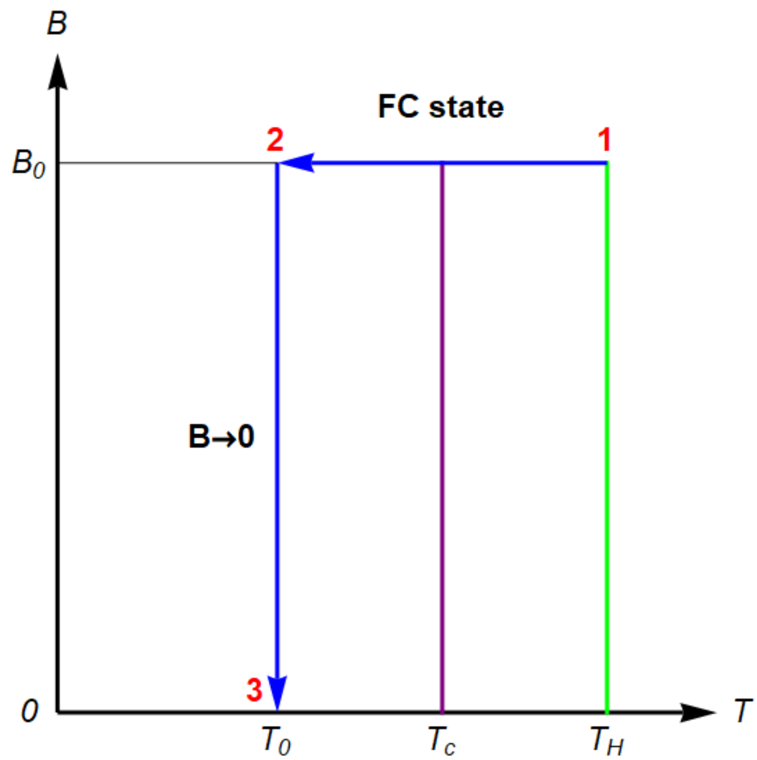


Fig.10 The measurement of magnetic flux of superconducting ring in the **FC state**. The process between the point 2 and the point 3 is reversible for the path 2→3 and path 3→2 at $T = T_0$.

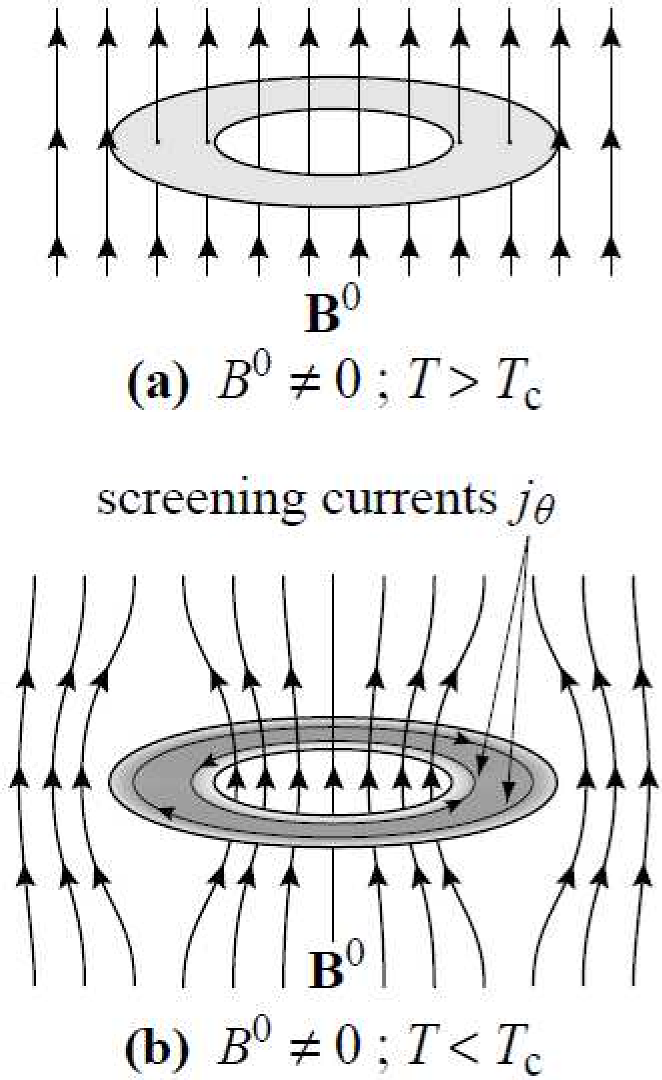


Fig.11 **FC state.** The magnetic field distribution near the superconducting ring, and the edge current. Figures taken from P. Mangin and R. Kahn, Superconductivity An Introduction (Springer International Publishing AG, 2017).

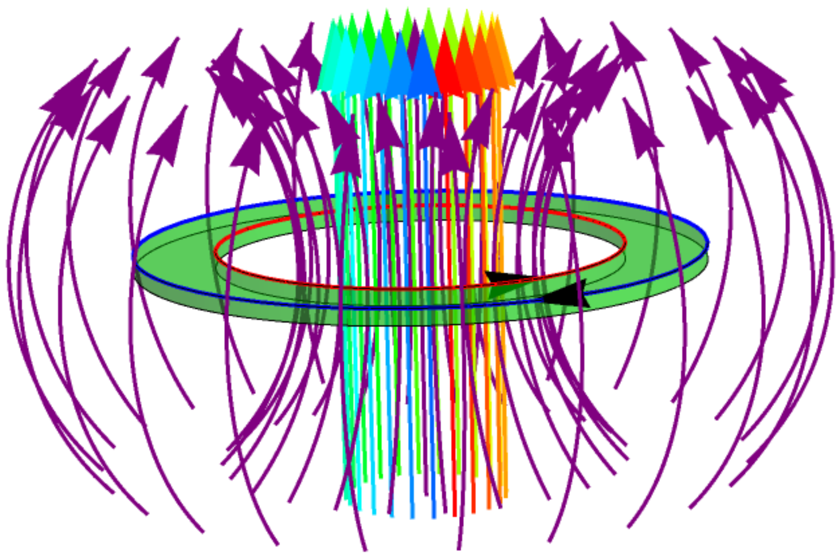


Fig.12. **FC state**. The magnetic field distribution near the superconducting ring and edge currents inside the ring. $T<T_c$ $0<B<B_0$. The direction of the outer edge current is opposed to that of the inner edge current.

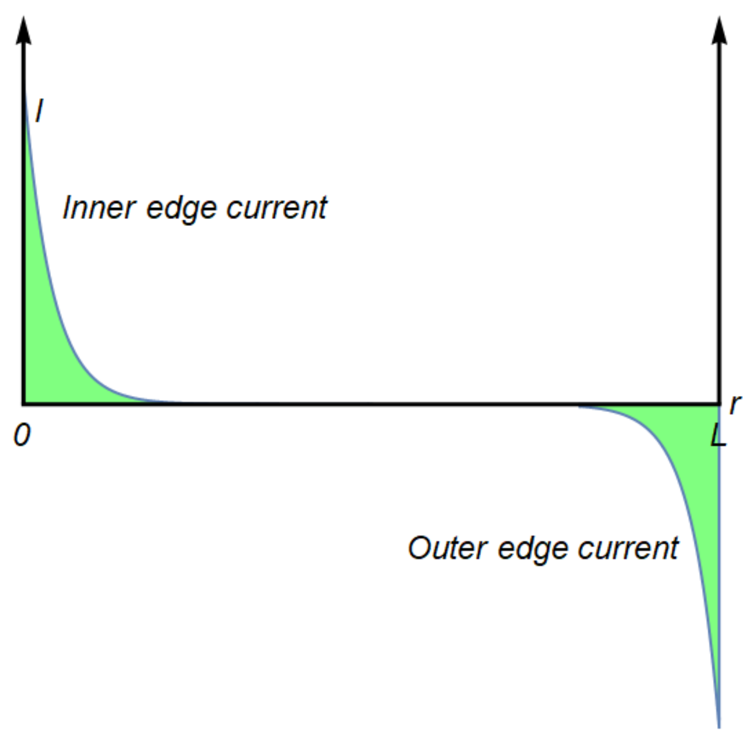


Fig.13 **FC state.** The outer edge current and inner edge current below T_c at $B = 0$, as a function of the distance. The system is cooled in the presence of magnetic field (**FC state**). Below T_c the magnetic field is decreased to zero.

(b) FC (field cooled) state: $T < T_c$ and $B = 0$

As we pass the transition temperature under applied field, superconducting currents appear near the inner and outer surfaces in order to expel the magnetic field from the superconductor. We note that they turn in opposite directions. The field lines are pushed towards the exterior and interior.

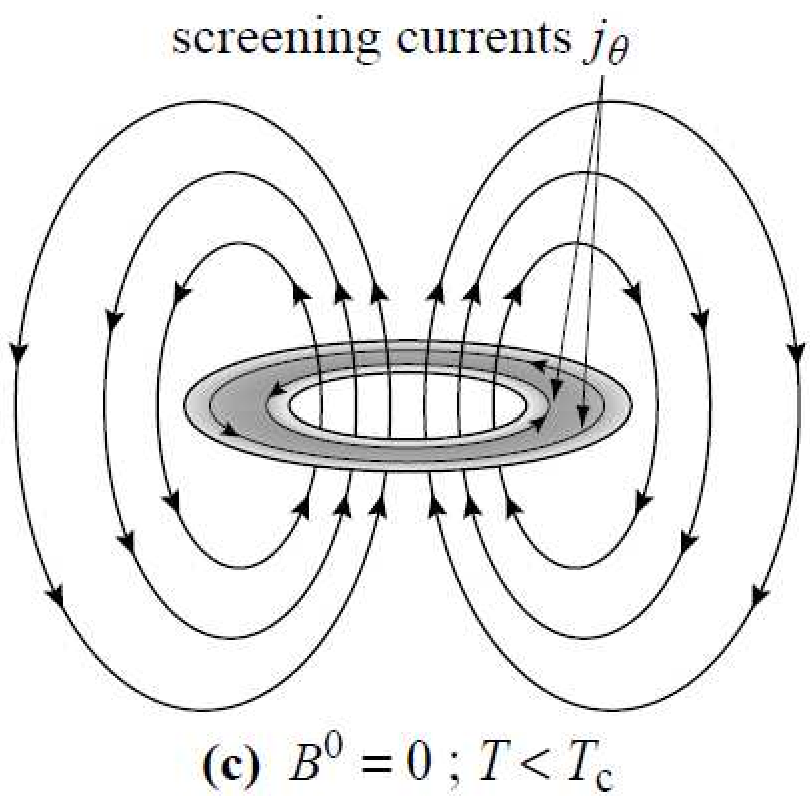
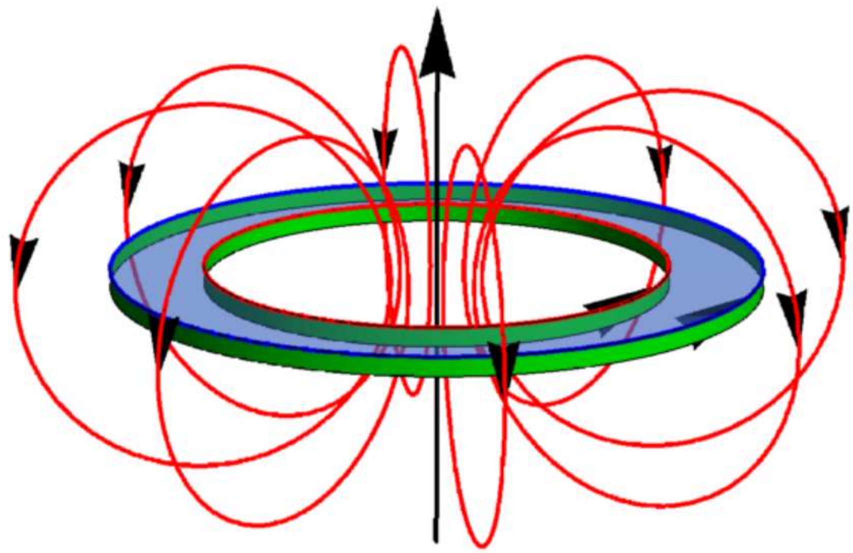
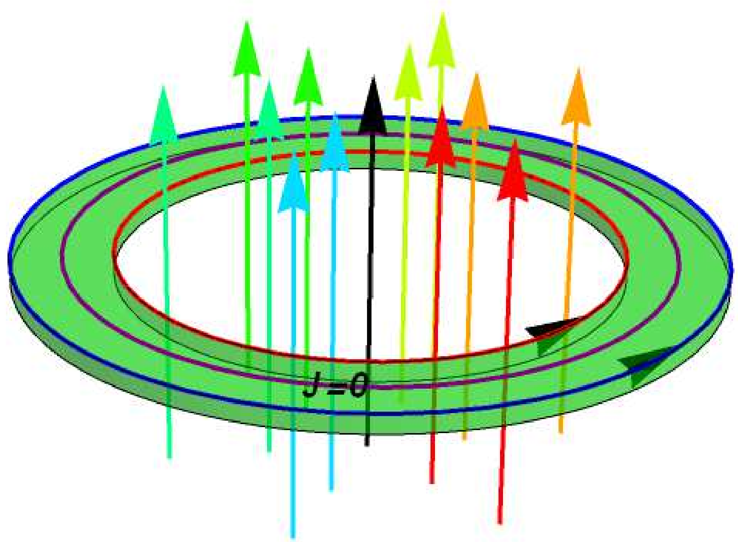


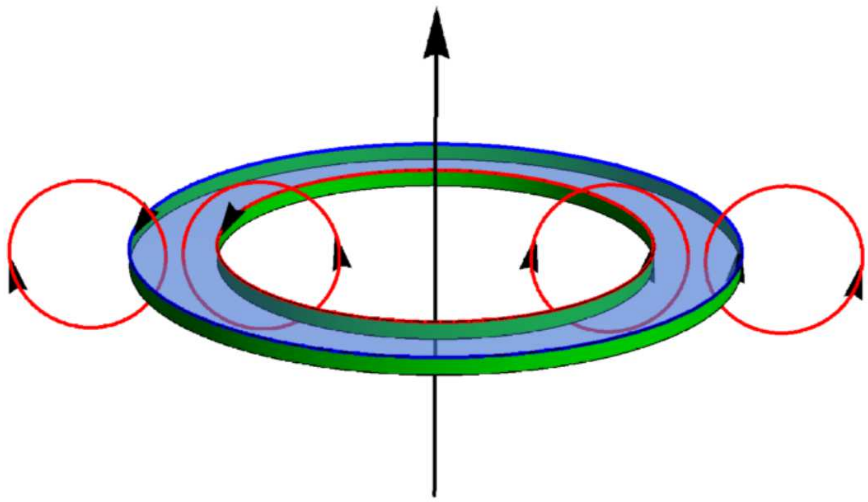
Fig.14 **FC state.** The magnetic field distribution. The quantized magnetic flux is trapped in the superconducting ring. Figures from P. Mangin and R.Kahn, Superconductivity An Introduction (Springer International Publishing AG 2017).



(a)



(b)



(c)

Fig.15 **FC state.** The system is cooled from the normal phase ($T > T_c$) to the superconducting phase ($T < T_c$) in the presence of external magnetic field (**FC state**). Below T_c, the external magnetic field is decreased to zero. (a), (b) and (c) The magnetic flux is trapped in the superconducting ring. $B = 0$. The magnetic field distribution and the edge currents. The direction of the outer edge current is the same as that of the inner edge current.

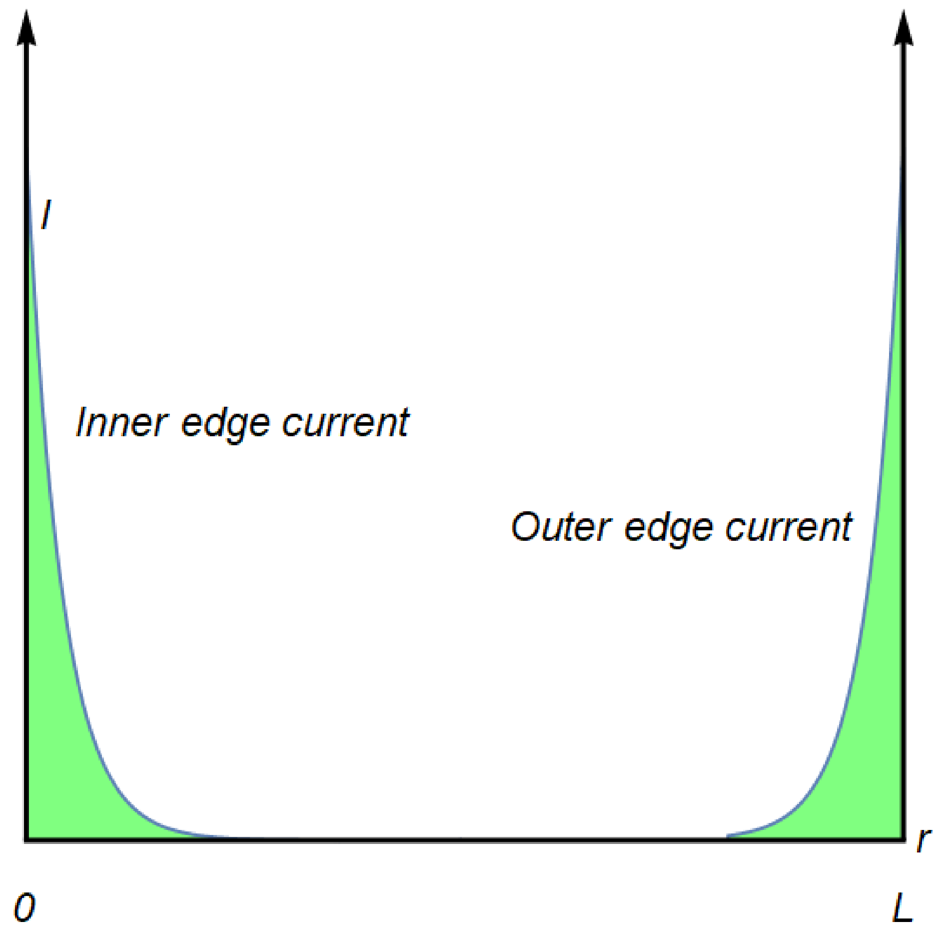


Fig.16 **FC state.** Outer edge current and inner edge current below T_c at $B = 0$. The system is cooled in the presence of magnetic field (**FC state**). Below T_c, then the magnetic field is decreased to zero. $B = 0$.

(c) Zero-field cooled (ZFC) state

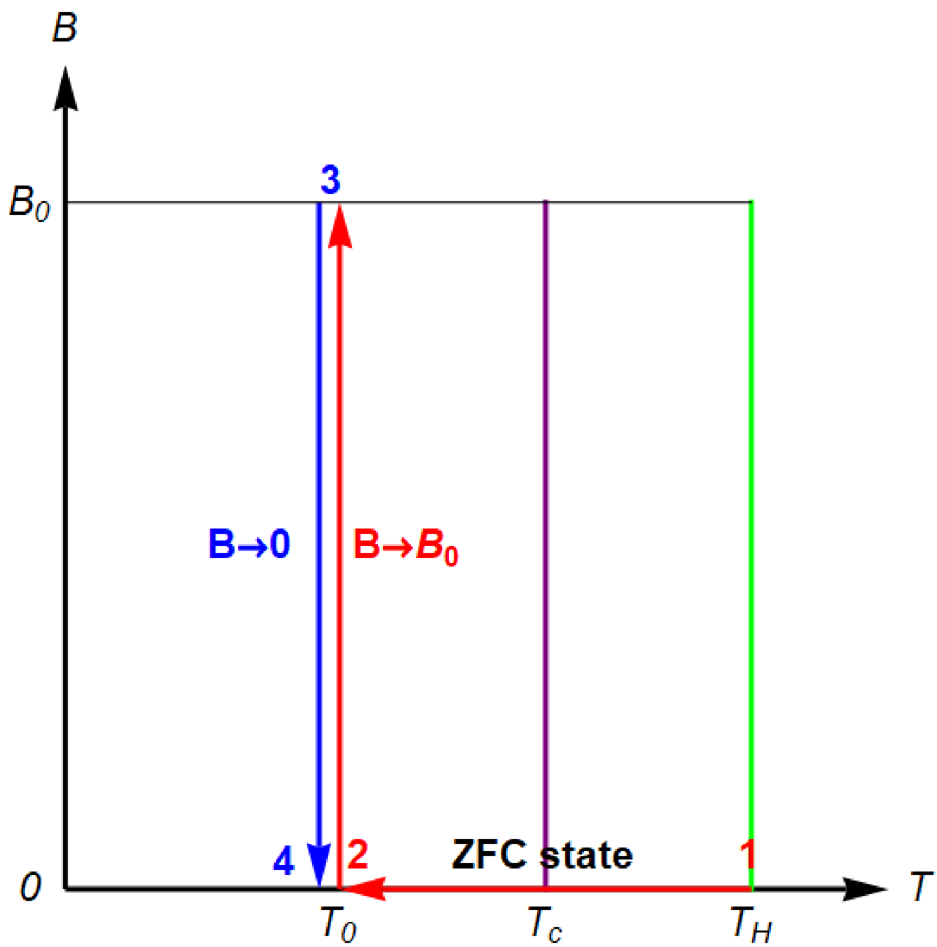


Fig.17 **ZFC state.** The system is cooled from the high temperature T_H (the point 1) to T_0 ($<T_c$) (the point 2, below Tc) in the absence of external magnetic field. At the temperature T_0, an external magnetic field is increases from $B = 0$ to B_0 (the point 3). The path between the point 3 and the point 4 is reversible (the path $3 \rightarrow 4$ and the path $4 \rightarrow 3$) at $T = T_0$.

After cooling in zero field, a magnetic field is applied and then turned off while staying in the superconducting phase (Meissner effect)

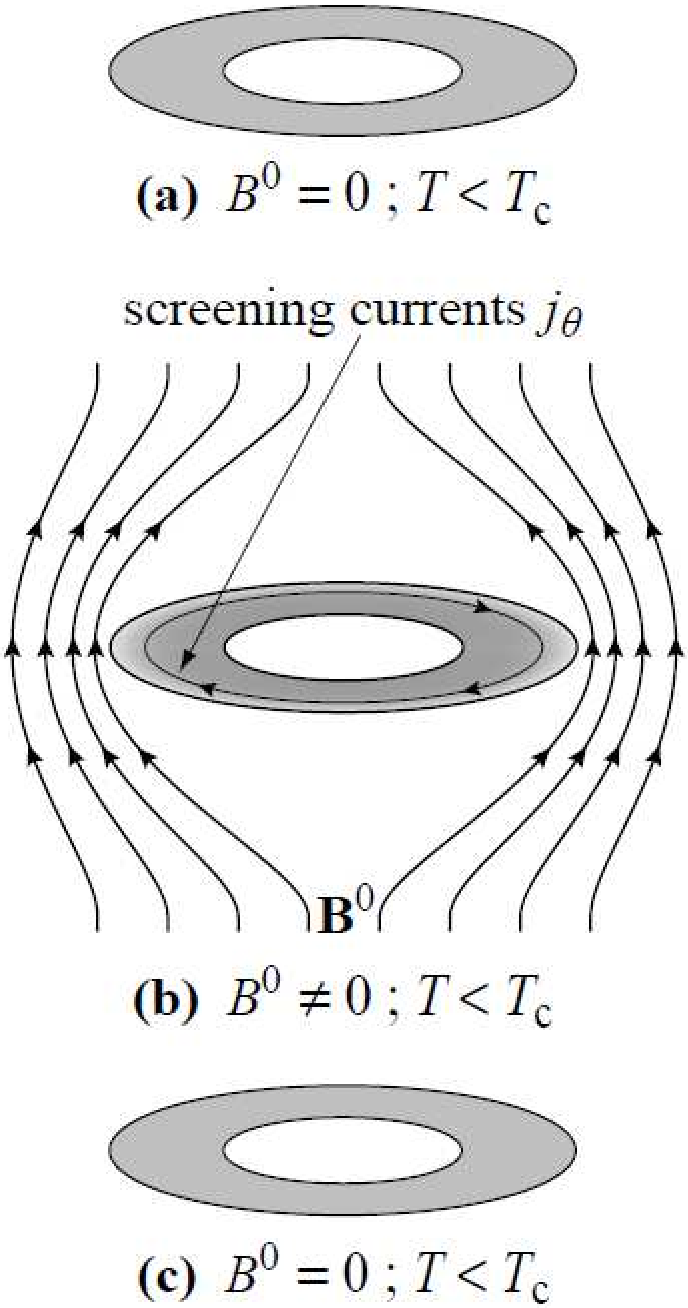


Fig.18 **ZFC state**. Figures taken from P. Mangin and R. Kahn, Superconductivity An Introduction (Springer International Publishing AG, 2017). The Meissner effect is observed for $T<T_c$ and $0<B<B_0$.

22. Expression for the Magnetic flux for the FC state and ZFC state

We start with the expression

$$B = B_a + 4\pi M \qquad \text{(cgs units)}$$

Where B_a is the external magnetic field and *M* is the magnetization. The magnetic flux is obtained as

$$Ba = B_a a + 4\pi M a,$$

where *a* is the area of superconducting ring. The flux through the ring, Φ, is the sum of the flux Φ_{ext} from the external source and the flux Φ_{sc} from the superconducting currents which flow in the surface of the ring:

$$\Phi = \Phi_{ext} + \Phi_{sc}.$$

(a) ZFC state (Meissner effect)

Because of the Meissner effect, in ZFC state we have

$$\Phi = 0 = \Phi_{ext} + (\Phi_{sc})_{ZFC},$$

or

$$(\Phi_{sc})_{ZFC} = -\Phi_{ext}.$$

(b) FC state

In FC state, we have

$$\Phi = n\Phi_0 = \Phi_{ext} + (\Phi_{sc})_{FC}, \qquad (n\text{: integer})$$

or

$$(\Phi_{sc})_{FC} = n\Phi_0 - \Phi_{ext}.$$

(c) The difference $\Delta\Phi = (\Phi_{sc})_{FC} - (\Phi_{sc})_{ZFC}$

Thus, we have the difference between

$\Delta\Phi = (\Phi_{sc})_{FC} - (\Phi_{sc})_{ZFC} = n\Phi_0$.

Schematically, we make a plot of $(\Phi_{sc})_{FC}$, $(\Phi_{sc})_{ZFC}$, $\Delta\Phi$, as a function of $(\Phi_{sc})_{FC}$ as follows.

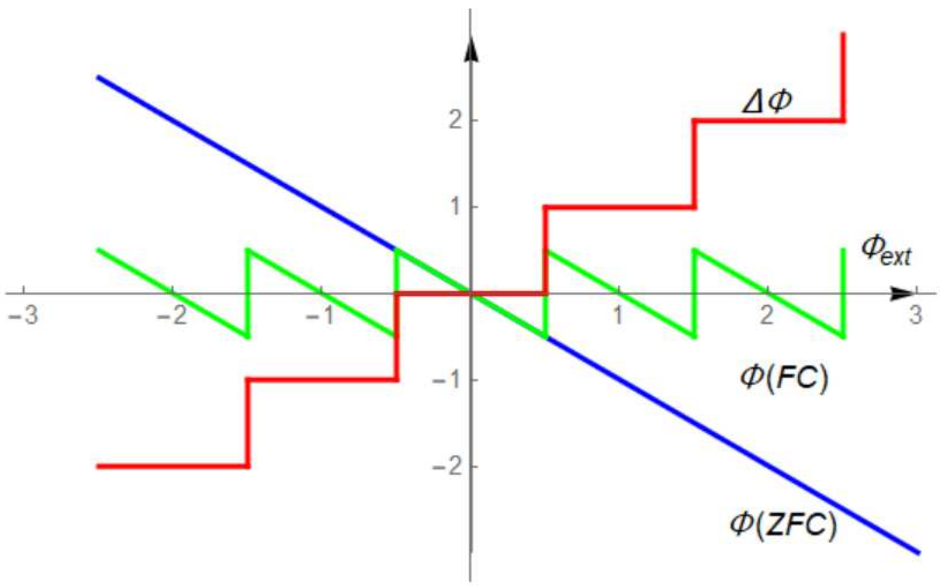


Fig.19 Plot of $\Delta\Phi = (\Phi_{sc})_{FC} - (\Phi_{sc})_{ZFC}$ (red line), $(\Phi_{sc})_{FC}$ (green line) and $(\Phi_{sc})_{ZFC}$ (blue line) as a function of Φ_{ext}. The negative slope in $(\Phi_{sc})_{FC}$ with $(\Phi_{sc})_{FC}$ indicates the diamagnetic behavior due to the Meissner effect. Note that $(\Phi_{sc})_{FC} \geq (\Phi_{sc})_{ZFC}$ for $\Phi_{ext} \geq 0$

23. Experiments by Goodman et al (1971) for quantum magnetic flux

The magnetic flux, represented by the symbol **Φ**, threading some contour or loop is defined as the magnetic field **B** multiplied by the loop area **S**, i.e. **Φ** = **B**·**S**. Both **B** and **S** can be arbitrary, meaning **Φ** can be as well. However, if one deals with the superconducting loop or a hole in a bulk superconductor, the magnetic flux threading such a hole/loop is actually quantized. The (superconducting) **magnetic flux quantum** $\Phi_0 = h/(2e) \approx 2.067833848... \times 10^{-15}$ Wb is a combination of fundamental physical constants: the Planck constant h and the electron charge e. Its value is, therefore, the same for any superconductor. The phenomenon of flux quantization was discovered experimentally by B. S. Deaver and W. M. Fairbank and, independently, by R. Doll and M. Näbauer, in 1961. The quantization of magnetic flux is closely related to the Little–Parks effect, but was predicted earlier by Fritz London in 1948 using a phenomenological model.

https://en.wikipedia.org/wiki/Magnetic_flux_quantum

$$\Phi_0 = \frac{hc}{2e} = \frac{2\pi\hbar c}{2e} = 2.067833848\times10^{-7} \text{ Gauss cm}^2 \text{ (cgs units)}$$

$$\Phi_0 = \frac{h}{2e} = \frac{2\pi\hbar}{2e} = 2.067833848\times10^{-15} \text{ Tesla m}^2 \text{ (SI units)}$$

((Experiment)) **Superconducting ring**
W.L. Goodman, W.D. Willis, D.A. Vincent, and B.S. Deaver, Jr., *Phys. Rev.* B 4, 1530 (1971).

Here we show the excellent experimental results of the superconducting cylinder by Goodman et al.

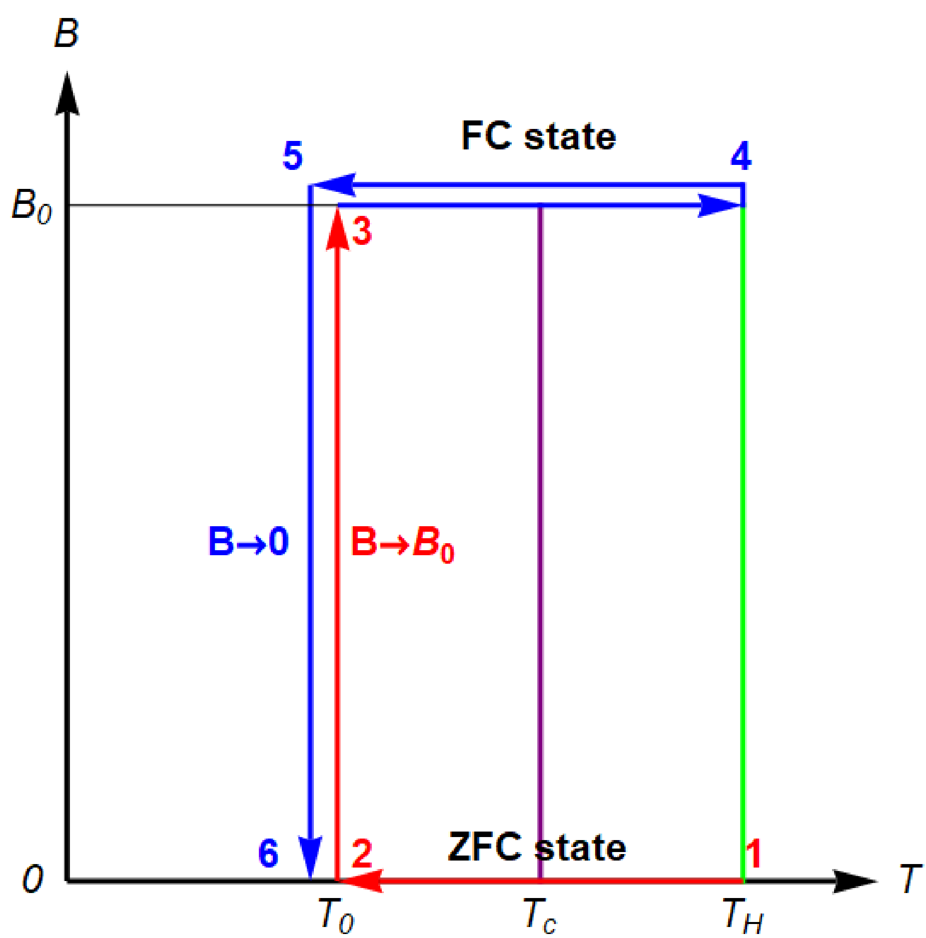


Fig.20 (i) Experiment-1 ($1 \rightarrow 2$) ZFC state (cooling down from high temperature T_H above T_c in the absence of a magnetic field). (ii) Experiment-2 (point 3): Meissner effect (ZFC state).

(iii) After the preparation of FC state ($3 \rightarrow 4 \rightarrow 5$). Measurement of magnetic flux in the FC state (5 to 6, and 6 to 5)

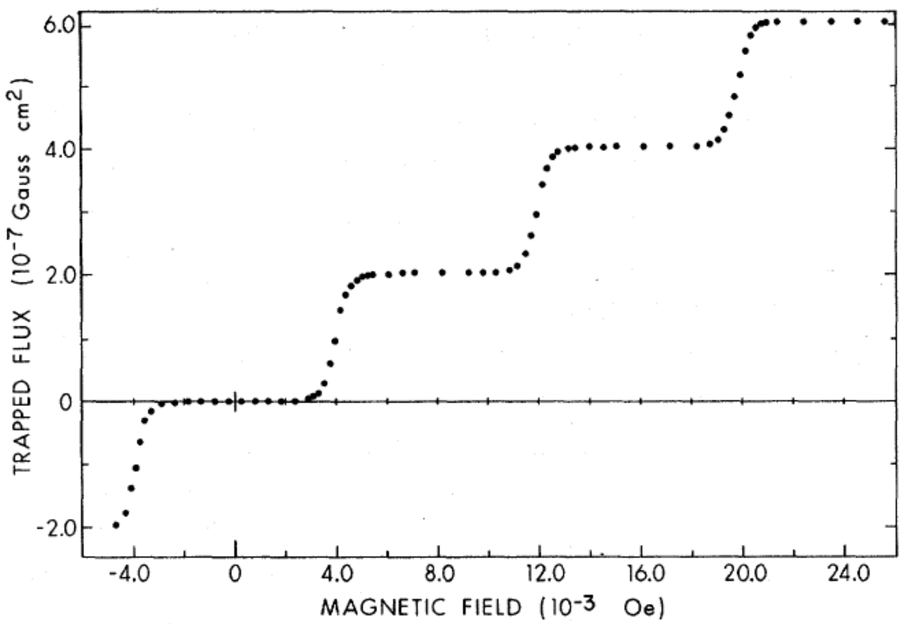


Fig.21 Trapped flux as a function of the magnetic field in which the cylinder was cooled below its critical temperature T_c. $\Delta\Phi = (\Phi_{sc})_{FC} - (\Phi_{sc})_{ZFC}$ as a function of the external magnetic field B_a.

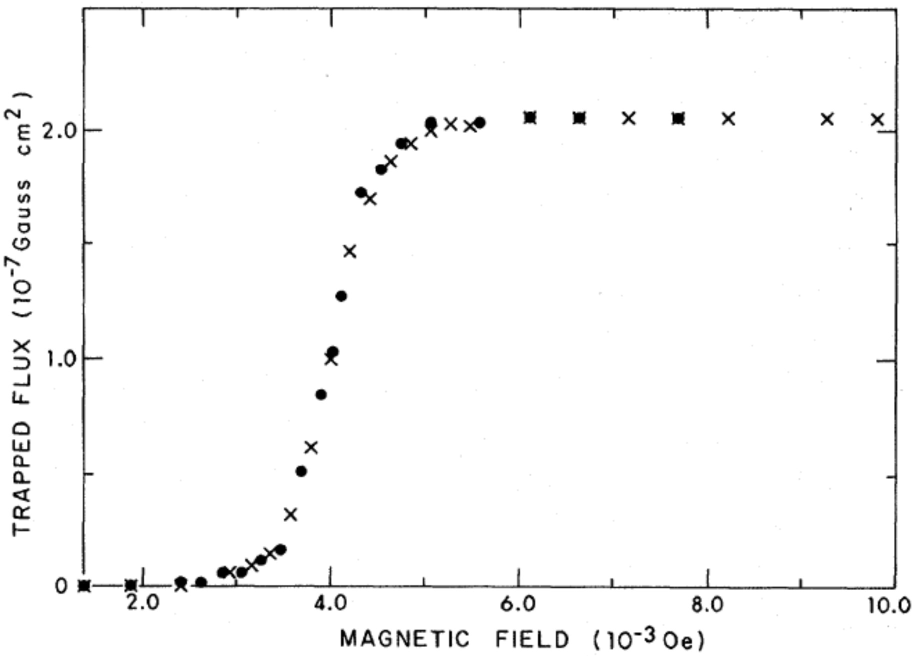


Fig.28 Trapped flux as a function of the field in which the cylinder was cooled through its critical temperature. The dots and crosses are data from two different runs at 3.68 and 3.60 K, respectively. The critical temperature of Sn is 3.72 K.

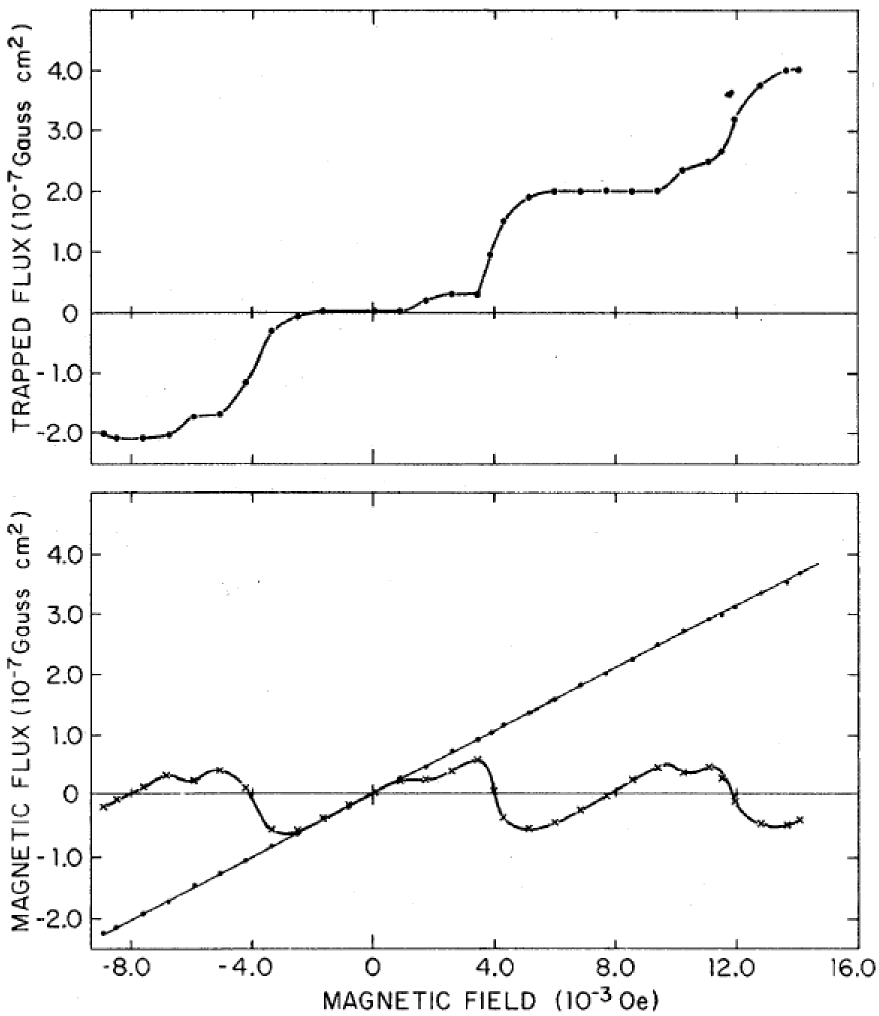


Fig.29

Upper: Trapped flux as a function of the magnetic field in which the cylinder was cooled below critical temperature (FC state). The difference between the straight line and the lower curve gives exactly the trapped flux curve. When the cylinder is cooled through T_c in the field and is unchanged when the field is turned off.

Lower: flux produced by the cylinder in the field in which it was cooled below the critical temperature (FC state). Note that the lower figure (which may correspond to Φ_{FC} and Φ_{ZFC}) is different from that in **Fig.19,** in sign.

The straight line shows the flux produced by the cylinder when it is inserted into the applied field already superconducting and with no flux trapped (ZFC state).

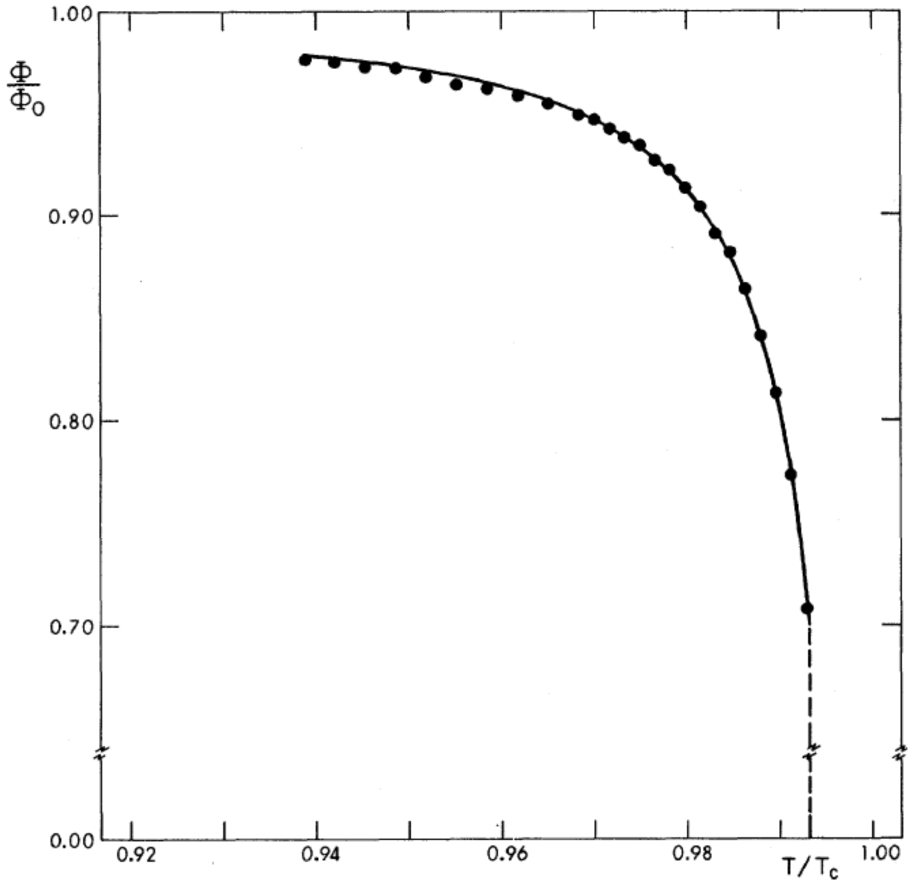


Fig.30 Trapped flux as a function of temperature. The dashed line indicates a jump to the zero flux state.

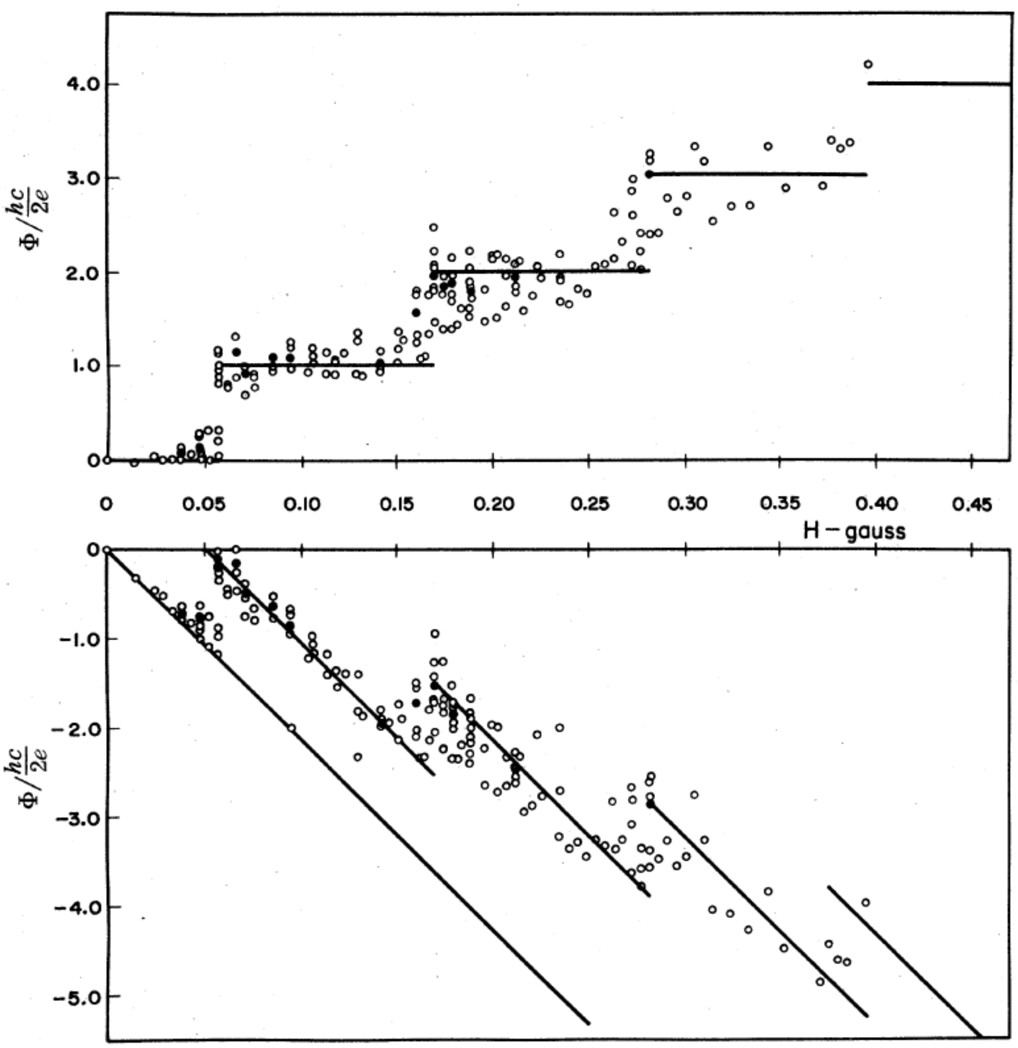


Fig.31 Net flux in cylinder before the applied field was turned off as a function of the applied field. The circles and triangles are for opposite directed fields. The solid lines are separated by hc/2e in the vertical direction. (B.S. Deaver Jr. and W.M. Fairbank, Phys. Rev. Lett. **7**, 43, (1961). Note that the lower figure (which may correspond to Φ_{FC} as shown in **Fig.19)** is somewhat different from the lower part of **Fig.29**.

24. Summary

Until 1940's, the amplitude of the wave function in Schrödinger equation has been sufficiently understood. The square of the amplitude indicates the probability of finding a quantum system in corresponding state (Born). How about the phase in the wave function? After 1940's, physicists gradually came to understand the significance of the quantum phase. They tried to explain the strange nature of superconductivity in metals and superfluidity of liquid ^{4}He. Here are several important articles contributing to the understanding of quantum phase in 1950's and early 1960's. It is amazing that the following theories and experiments appeared in such short times.

1. Dirac: the significance of Lagrangian in quantum mechanics.
2. Feynman; Feynman path integral; relation between action and phase.
3. London: discovery of London equation for the superconductivity.
4. Ginzburg-Landau theory for the superconductivity.
5. Onsager's theory on the de Haas van Alphen effect in metals.
6. Onsager and Feynman: quantized circulation in rotating He II of liquid ^{4}He.
7. BCS (Bardeen, Cooper, and Schrieffer) theory for the superconductivity; existence of Cooper pairs.
8. Abrikosov: mixed phase with vortex lines in type II superconductors.
9. Aharonov-Bohm effect; significance of the role of vector potential.
10. Evidence of the quantization of magnetic flux in superconducting ring.
11. Observation of quantization of circulation in liquid ^{4}He (Vinen)
12. Josephson: discovery of Josephson effect.

((Nomenclature))

Continuity equation
Probability current
Probability density
Canonical momentum (conjugate momentum)
Kinematic momentum (Feynman)
Dynamical momentum (Feynman)

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APPENDIX

Classical approach: A particle in the presence of electric field and magnetic field:

$$L = T - V = \frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\Phi .$$

Conjugate momentum

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + \frac{q}{c}\mathbf{A} .$$

Hamiltonian:

$$\begin{aligned} H &= \mathbf{P}\cdot\mathbf{v} - L \\ &= (m\mathbf{v} + \frac{q}{c}\mathbf{A})\cdot\mathbf{v} - (\frac{1}{2}m\mathbf{v}^2 + \frac{q}{c}\mathbf{A}\cdot\mathbf{v} - q\Phi) \\ &= \frac{1}{2}m\mathbf{v}^2 + q\Phi \\ &= \frac{1}{2m}(\mathbf{P} - \frac{q}{c}\mathbf{A})^2 + q\Phi \end{aligned}$$

Equation of motion (Hamilton's principle)

$$-\frac{d}{dt}\mathbf{P} = \frac{\partial}{\partial \mathbf{r}}H = q\nabla\phi + \frac{1}{2}m\nabla\mathbf{v}^2 ,$$

$$\frac{d}{dt}\mathbf{r} = \frac{\partial}{\partial \mathbf{P}}H = \mathbf{v} .$$

H is independent of $\boldsymbol{v}$;

$$0 = \frac{\partial}{\partial \mathbf{v}}H = \mathbf{P} - \frac{\partial}{\partial \mathbf{v}}L ,$$

leading to

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + \frac{q}{c}\mathbf{A} .$$

Here we use the formula of vector analysis. The proof of this formula will be given later.

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})$$

We now calculate

$$\frac{1}{2}m\nabla\mathbf{v}^2 = m(\mathbf{v}\cdot\nabla)\mathbf{v} + m\mathbf{v}\times(\nabla\times\mathbf{v})$$

for $\mathbf{A} = \mathbf{B} = \mathbf{v}$.

$$m\mathbf{v} = \mathbf{P} - \frac{q}{c}\mathbf{A}$$

We note that

$$\begin{aligned} m\nabla\times\mathbf{v} &= \nabla\times(\mathbf{P} - \frac{q}{c}\mathbf{A}) \\ &= -\frac{q}{c}\nabla\times\mathbf{A} \\ &= -\frac{q}{c}\mathbf{B} \end{aligned}$$

and

$$\begin{aligned} m(\mathbf{v}\cdot\nabla)\mathbf{v} &= (\mathbf{v}\cdot\nabla)(\mathbf{P} - \frac{q}{c}\mathbf{A}) \\ &= -\frac{q}{c}(\mathbf{v}\cdot\nabla\mathbf{A}) \end{aligned}$$

since $\mathbf{P}$ is independent of $\boldsymbol{r}$, and $\mathbf{B} = \nabla\times\mathbf{A}$. Thus, we get

$$-\frac{d}{dt}\mathbf{P} = q\nabla\Phi - \frac{q}{c}(\mathbf{v}\cdot\nabla)\mathbf{A} - \frac{q}{c}\mathbf{v}\times\mathbf{B} \qquad (1)$$

We also have the relation

$$\frac{d}{dt}\mathbf{P} = m\frac{d}{dt}\mathbf{v} + \frac{q}{c}\frac{d}{dt}\mathbf{A} \qquad (2)$$

From Eqs.(1) and (2), we have

$$m\frac{d}{dt}\mathbf{v} + \frac{q}{c}\frac{d}{dt}\mathbf{A} = -q\nabla\phi + \frac{q}{c}(\mathbf{v}\cdot\nabla)\mathbf{A} + \frac{q}{c}\mathbf{v}\times\mathbf{B}$$

Noting that

$$\frac{d}{dt}\mathbf{A} = \frac{\partial\mathbf{A}}{\partial\mathbf{r}}\frac{d\mathbf{r}}{dt} = \frac{\partial\mathbf{A}}{\partial\mathbf{r}}\frac{\partial H}{\partial\mathbf{P}} = (\mathbf{v}\cdot\nabla)\mathbf{A} \ ,$$

we get

$$m\frac{d}{dt}\mathbf{v} = -q\nabla\Phi + \frac{e}{c}\mathbf{v}\times\mathbf{B} \qquad \text{(Lorentz force)}$$

((**Matheamtica**))

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Cartesian coordinate system

Proof:

$\nabla(A\cdot B)=(A\cdot\nabla)B+(B\cdot\nabla)A+A\times(\nabla\times B)+B\times(\nabla\times A)$

```
Clear["Global`"];
Lap := Laplacian[#, { x, y, z}, "Cartesian"] &;
Gra := Grad[#, { x, y, z}, "Cartesian"] &;
Cur := Curl[#, {x, y,  z}, "Cartesian"] &;
Diva := Div[#, {x, y,  z}, "Cartesian"] &;
```

```
K11 :=
  (Bx[x, y, z] × D[#, x] + By[x, y, z] × D[#, y] +
     Bz[x, y, z] × D[#, z]) &;
K12 :=
  (Ax[x, y, z] × D[#, x] + Ay[x, y, z] × D[#, y] +
     Az[x, y, z] × D[#, z]) &;
```

```
A1 = {Ax[x, y, z], Ay[x, y, z], Az[x, y, z]};
B1 = {Bx[x, y, z],  By[x, y, z],  Bz[x, y, z]};
f1 = A1.B1;

s11 = Gra[f1] // FullSimplify;

s12 = Cross[B1, Cur[A1]] // FullSimplify;

s13 = Cross[A1, Cur[B1]] // FullSimplify;

s14 = {K11[Ax[x, y, z]], K11[Ay[x, y, z]],
    K11[Az[x, y, z]]} // FullSimplify;

s15 = {K12[Bx[x, y, z]], K12[By[x, y, z]],
    K12[Bz[x, y, z]]} // FullSimplify;

s11 - (s12 + s13 + s14 + s15) // Simplify

{0, 0, 0}
```
