# 3.3 Orbital angular momentum

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## 1. Orbital angular momentum

We discuss the connection between the differential operator $\mathbf{L}^2$ ($\hat{\mathbf{L}}$ is the orbital angular momentum in quantum mechanics) and the angular part of the Laplacian in the spherical coordinates.

Here we use the identity (epsilon-delta relation)

$$\sum_k \varepsilon_{ijk}\varepsilon_{lmk} = \delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l} .$$

Together with the commutation relations of the position and momentum operators and expression for the orbital angular momentum operators to verify that $\hat{\mathbf{L}}^2$ operator is expressed by

$$\hat{\mathbf{L}}^2 = (\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}}) = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} .$$

((**Proof**))

The proof is straightforward. To this end, we use the epsilon-delta relation (the proof is given later). Thus, we get

$$\begin{aligned}
\hat{\mathbf{L}}^2 &= (\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}}) \\
&= \sum_{i,j,k,l,m} \varepsilon_{ijk}\varepsilon_{lmk}\hat{x}_i\hat{p}_j\hat{x}_l\hat{p}_m \\
&= \sum_{i,j,l,m} (\delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l})\hat{x}_i\hat{p}_j\hat{x}_l\hat{p}_m \\
&= \sum_{i,j,l,m} [\delta_{i,l}\delta_{j,m}\hat{x}_i\hat{p}_j\hat{x}_l\hat{p}_m - \delta_{i,m}\delta_{j,l}\hat{x}_i\hat{p}_j\hat{x}_l\hat{p}_m] \\
&= \sum_{i,j,l,m} [\delta_{i,l}\delta_{j,m}\hat{x}_i(\hat{x}_l\hat{p}_j - i\hbar\delta_{j,l})\hat{p}_m - \delta_{i,m}\delta_{j,l}\hat{x}_i\hat{p}_j(\hat{p}_m\hat{x}_l + i\hbar\delta_{l,m})]
\end{aligned}$$

or

$$
\begin{aligned}
\hat{\mathbf{L}}^2 &= (\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}}) \\
&= \sum_{i,j,l,m}[\delta_{i,l}\delta_{j,m}\hat{x}_i(\hat{x}_l\hat{p}_j - i\hbar\delta_{j,l})\hat{p}_m - \delta_{i,m}\delta_{j,l}(\hat{x}_i\hat{p}_m\hat{p}_j\hat{x}_l + i\hbar\delta_{l,m}\hat{x}_i\hat{p}_j)] \\
&= \sum_{i,j,l,m}[\delta_{i,l}\delta_{j,m}(\hat{x}_i\hat{x}_l\hat{p}_j\hat{p}_m - i\hbar\delta_{j,l}\hat{x}_i\hat{p}_m) - \delta_{i,m}\delta_{j,l}(\hat{x}_i\hat{p}_m\hat{p}_j\hat{x}_l + i\hbar\delta_{l,m}\hat{x}_i\hat{p}_j)] \\
&= \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - \sum_{i,j,l,m}\delta_{i,m}\delta_{j,l}[\hat{x}_i\hat{p}_m(\hat{x}_l\hat{p}_j - i\hbar\delta_{l,j}) + i\hbar\delta_{l,m}\hat{x}_i\hat{p}_j) \\
&= \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - \sum_{i,j,l,m}\delta_{i,m}\delta_{j,l}(\hat{x}_i\hat{p}_m\hat{x}_l\hat{p}_j - i\hbar\delta_{l,j}\hat{x}_i\hat{p}_m + i\hbar\delta_{l,m}\hat{x}_i\hat{p}_j) \\
&= \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 + 3i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} \\
&= \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2
\end{aligned}
$$

or

$$
\hat{\mathbf{L}}^2 = (\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}}) = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 .
$$

Note that

$$
\sum_{i,j,l,m}\delta_{i,l}\delta_{j,m}\hat{x}_i\hat{x}_l\hat{p}_j\hat{p}_m = \sum_{i,j}\hat{x}_i{}^2\hat{x}_i\hat{p}_j{}^2 = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 ,
$$

$$
\sum_{i,j,l,m}\delta_{i,l}\delta_{j,m}\delta_{j,l}\hat{x}_i\hat{p}_m = \sum_{i,j,l}\delta_{i,l}\delta_{j,l}\hat{x}_i\hat{p}_j = \sum_{i,l}\delta_{i,l}\hat{x}_i\hat{p}_l = \sum_i\hat{x}_i\hat{p}_i = \hat{\mathbf{r}}\cdot\hat{\mathbf{p}} ,
$$

$$
\sum_{i,j,l,m}\delta_{i,m}\delta_{j,l}\delta_{l,j}\hat{x}_i\hat{p}_m = (\sum_{i,m}\delta_{i,m}\hat{x}_i\hat{p}_m)\sum_{j,l}\delta_{j,l}\delta_{l,j} = 3\sum_i\hat{x}_i\hat{p}_i = 3(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}) ,
$$

$$
\sum_{j,l}\delta_{j,l}\delta_{l,j} = \sum_{j,l}\langle j|l\rangle\langle l|j\rangle = \sum_j\langle j|j\rangle = 3
$$

## 2. $\varepsilon-\delta$ relation

In the above proof we use the relation given by

$$
\sum_k\varepsilon_{ijk}\varepsilon_{lmk} = \delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l}
$$

((**Proof**))

We use the formula in vector analysis (Cartesian coordinates, 3D)

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})\cdot(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})\cdot(\mathbf{B}\cdot\mathbf{C})$$

where

$$(\mathbf{A}\times\mathbf{B})_k=\sum_{i,j}\varepsilon_{ijk}A_iB_j\ ,$$

$$(\mathbf{C}\times\mathbf{D})_k=\sum_{l.m}\varepsilon_{lmk}C_lD_m\ .$$

This we have

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=\sum_{i.j.k.l.m}\varepsilon_{ijk}\varepsilon_{lmk}A_iB_jC_lD_m\ ,$$

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{C})\cdot(\mathbf{B}\cdot\mathbf{D})&=\sum_{i,l}\delta_{i,l}A_iC_l\sum_{j,m}\delta_{j,m}B_jD_m\\&=\sum_{i,j,l,m}\delta_{i,l}\delta_{j,m}A_iB_jC_lD_m\end{aligned}$$

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{D})\cdot(\mathbf{B}\cdot\mathbf{C})&=\sum_{i,m}\delta_{i,m}A_iD_m\sum_{j,m}\delta_{j,l}B_jC_l\\&=\sum_{i,j,l,m}\delta_{i,m}\delta_{j,l}A_iB_jC_lD_m\end{aligned}$$

So that, we get the epsilon-delta relation

$$\sum_k\varepsilon_{ijk}\varepsilon_{lmk}=\delta_{i,l}\delta_{j,m}-\delta_{i,m}\delta_{j,l}\ .$$

## 3. The expression: $\mathbf{p}^2=p_r{}^2+\dfrac{\mathbf{L}^2}{r^2}$

We note that

$$\hat{\mathbf{L}}^2=(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\hat{\mathbf{p}})=\hat{\mathbf{r}}^2\hat{\mathbf{p}}^2+i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}-(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2$$

In the $|\mathbf{r}\rangle$ representation, we get the following expressions,

$$\langle\mathbf{r}|\hat{\mathbf{L}}^2|\psi\rangle=\mathbf{L}^2\langle\mathbf{r}|\psi\rangle=\mathbf{L}^2\psi(\mathbf{r})$$

where $\mathbf{L}^2$ is the differential operator.

$$\begin{aligned}\langle \mathbf{r}|\hat{\mathbf{r}}^2\hat{\mathbf{p}}^2|\psi\rangle &= -\hbar^2\mathbf{r}^2\nabla^2\langle \mathbf{r}|\psi\rangle \\ &= -\hbar^2\mathbf{r}^2\nabla^2\psi(\mathbf{r})\end{aligned}$$

where $\mathbf{p} = \frac{\hbar}{i}\nabla$ is the differential operator of the momentum $\hat{\mathbf{p}}$ of quantum mechanics.

$$\begin{aligned}\langle \mathbf{r}|i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}}|\psi\rangle &= i\hbar\mathbf{r}\cdot\frac{\hbar}{i}\nabla\langle \mathbf{r}|\psi\rangle \\ &= \hbar^2(\mathbf{r}\cdot\nabla)\langle \mathbf{r}|\psi\rangle \\ &= \hbar^2 r\frac{\partial}{\partial r}\psi(\mathbf{r})\end{aligned}$$

$$\begin{aligned}\langle \mathbf{r}|(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})|\psi\rangle &= (\mathbf{r}\cdot\frac{\hbar}{i}\nabla)(\mathbf{r}\cdot\frac{\hbar}{i}\nabla)\langle \mathbf{r}|\psi\rangle \\ &= -\hbar^2(r\frac{\partial}{\partial r})(r\frac{\partial}{\partial r})\langle \mathbf{r}|\psi\rangle \\ &= -\hbar^2(r^2\frac{\partial^2}{\partial r^2}+r\frac{\partial}{\partial r})\psi(\mathbf{r})\end{aligned}$$

Thus, we have

$$\mathbf{L}^2\psi(\mathbf{r}) = -\hbar^2 r^2\nabla^2\psi(\mathbf{r}) + \hbar^2(r^2\frac{\partial^2}{\partial r^2}+2r\frac{\partial}{\partial r})\psi(\mathbf{r})$$

or

$$\begin{aligned}\frac{\mathbf{p}^2}{2\mu} &= -\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{r}) \\ &= -\frac{\hbar^2}{2\mu}(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r})\psi(\mathbf{r}) + \frac{1}{2\mu r^2}\mathbf{L}^2\psi(\mathbf{r}) \\ &= \frac{1}{2\mu}(p_r{}^2+\frac{1}{r^2}\mathbf{L}^2)\psi(\mathbf{r})\end{aligned}$$

where $\mu$ is the mass of particle and $p_r$ is the radial linear momentum.

$$
\begin{aligned}
p_r{}^2\psi(\mathbf{r}) &= \frac{\hbar}{ir}\frac{\partial}{\partial r}(r\frac{\hbar}{ir}\frac{\partial}{\partial r}r)\psi(\mathbf{r}) \\
&= -\frac{\hbar^2}{r}\frac{\partial^2}{\partial r^2}[r\psi(\mathbf{r})] \\
&= -\frac{\hbar^2}{r}\frac{\partial}{\partial r}[r\frac{\partial}{\partial r}\psi(\mathbf{r})+\psi(\mathbf{r})] \\
&= -\hbar^2(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r})\psi(\mathbf{r})]
\end{aligned}
$$

## 4. Proof using the Mathematica

Using the Mathematica (the differential operator in the spherical coordinate) we show that

$$
\mathbf{p}^2 = -\hbar^2\nabla^2 = p_r{}^2 + \frac{1}{r^2}\mathbf{L}^2
$$

or

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\mathbf{L}^2}{\hbar^2 r^2}
$$

with $p_r = \frac{\hbar}{ir}\frac{\partial}{\partial r}r$

((**Mathematica**))

Orbital angular momentum in the spherical coordinates

```
Clear["Global`"];
ux = {Sin[θ] Cos[ϕ], Cos[θ] Cos[ϕ], -Sin[ϕ]};
uy = {Sin[θ] Sin[ϕ], Cos[θ] Sin[ϕ], Cos[ϕ]};
uz = {Cos[θ], -Sin[θ], 0};
ur = {1, 0, 0};
Lap := Laplacian[#, { r, θ, ϕ}, "Spherical"] &;
Gra := Grad[#, { r, θ, ϕ}, "Spherical"] &;
Diva := Div[#, {r, θ, ϕ}, "Spherical"] &;
Curla := Curl[#, {r, θ, ϕ}, "Spherical"] &;
L := (-ⅈ ℏ ( Cross[(ur r), Gra[#]]) &) // Simplify;
Lx := (ux.L[#] &) // Simplify;
Ly := (uy.L[#] &) // Simplify;
Lz := (uz.L[#] &) // Simplify;
LP := (Lx[#] + ⅈ Ly[#]) & // Simplify;
LM = (Lx[#] - ⅈ Ly[#]) & // Simplify;
prq := ((-ⅈ ℏ)/2 ur .Gra[#] + (-ⅈ ℏ)/2 Diva[# ur]) &;
```

```
prq := ((-ⅈ ℏ/2) ur . Gra[#] + (-ⅈ ℏ/2) Diva[# ur]) &;
```

eq1=prq$^2\psi(r,\theta,\phi)$; prq is the radial linear ,omentum

```
eq1 = Nest[prq, ψ[r, θ, ϕ], 2] // FullSimplify
```

$$-\frac{\hbar^2\left(2\,\psi^{(1,0,0)}[r,\theta,\phi]+r\,\psi^{(2,0,0)}[r,\theta,\phi]\right)}{r}$$

eq2 = $\frac{1}{r^2}L^2\psi(r,\theta,\phi)$

```
eq2 =
 1/r^2 (Nest[Lx, ψ[r, θ, ϕ], 2] + Nest[Ly, ψ[r, θ, ϕ], 2] +
     Nest[Lz, ψ[r, θ, ϕ], 2]) // Simplify
```

$$-\frac{1}{r^2}\hbar^2\left(\mathrm{Csc}[\theta]^2\,\psi^{(0,0,2)}[r,\theta,\phi]+\mathrm{Cot}[\theta]\,\psi^{(0,1,0)}[r,\theta,\phi]+\psi^{(0,2,0)}[r,\theta,\phi]\right)$$

eq3 = $-\hbar^2\Delta\psi(r,\theta,\phi)$

```
eq3 = - ℏ^2 Lap[ψ[r, θ, ϕ]] // Simplify
```

$$-\frac{1}{r^2}\hbar^2\left(\mathrm{Csc}[\theta]^2\,\psi^{(0,0,2)}[r,\theta,\phi]+\mathrm{Cot}[\theta]\,\psi^{(0,1,0)}[r,\theta,\phi]+\psi^{(0,2,0)}[r,\theta,\phi]+2\,r\,\psi^{(1,0,0)}[r,\theta,\phi]+r^2\,\psi^{(2,0,0)}[r,\theta,\phi]\right)$$

eq1 = eq2 - eq3 = 0; Proof of $\boldsymbol{p}^2$=prq$^2$+$\boldsymbol{L}^2/r^2$

```
eq1 + eq2 - eq3 // FullSimplify
0
```

**REFERENCES**


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