The Binding energy of atoms and isotopes: magic number

Masatsugu Sei Suzuki and Itsuko S. Suzuki
Department of Physics
SUNY at Binghamton
Binghamton, New York, U.S.A.
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Magic number

In nuclear physics, a magic number is a number of nucleus (either protons or neutrons, separately) such that they are arranged into complete shells within atomic nucleus. As a result, atomic nuclei with a "magic" number of protons or neutrons are much more stable than other nuclei. The seven most widely recognized magnetic numbers are 2, 8, 20, 28, 50, 82, and 102. For protons, this corresponds to the elements such as helium, oxygen, nickel, tin, lead, although 126 is so far only known to be a magic number for neutrons. Atomic nuclei consisting of such a magic number of nucleons have a higher average binding energy per nucleon than one would expect based predictions such as the semi-empirical mass formula and are hence more stable against nuclear decay. Nuclei which have neutron number and proton number each equal to one of the magic numbers are called "doubly magic."

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1. Overview

The binding energy of atoms and isotopes as a function of atomic number is arguably the most important curve in science, responsible for our very existence and potentially our extinction as well. Moreover, it is the potential solution to many of the world's energy and environmental problems. In order to understand such problems, the nuclear shell model is a theoretical model that describes the structure of atomic nuclei in terms of energy levels occupied by protons and neutrons. One of the key features of the model is the concept of "magic numbers," (**Maria G. Mayer**), which are certain numbers of protons or neutrons that correspond to particularly stable configurations of nuclear matter. The magic numbers are 2, 8, 20, 28, 50, 82, and 126. These numbers represent the number of protons or neutrons that fill complete shells in the nucleus. For example, oxygen 16 has eight protons and eight neutrons (*doubly magic*), which corresponds to a complete shell of eight protons and eight neutrons. The magic numbers arise because of the way protons and neutrons interact with each other through the strong nuclear force, which binds the nucleus together. The strong force has a limited range, and it is most effective at short distances. As a result, the protons and neutrons in the nucleus tend to occupy discrete energy levels that are determined by the geometry of the nuclear potential well. When a shell is completely filled with protons or neutrons, the nucleus becomes particularly stable because the particles are less likely to interact with each other in ways that would disrupt the binding energy of the nucleus. This stability gives rise to the magic numbers and explains why nuclei with magnetic numbers tend to be more stable and have higher binding energies than nuclei with other numbers of protons and neutrons.

According to the Bohr's model, the velocity of electron in the ground state of the hydrogen-like atom with the atomic number Z is predicted as

$$v = \frac{e^2}{\hbar c} Zc = \alpha Zc = \frac{Z}{137.0360} c \, .$$

where α is the fine structure constant, $\alpha = \frac{1}{137.0360}$. Since $v < c$ (relativity), the value of Z should be lower than $Z \leq 137$. On the other hand, the number of neutrons for many heavy elements can be larger than this value: $N \geq Z$. For the stable elements with small atomic mass A, the number of protons (atomic number) Z is equal to the number of neutrons N; $A = Z + N$. Iron (Fe)-56 is a strongly and effectively bound nucleus, having a binding energy of 8.8 MeV per nucleon. The binding energy per nucleus vs A shows a maximum at the value of A for Fe. One can expect a fusion for lower than the value of A for Fe, and a fission for higher than the value of A for Fe. The binding energy of elements is different depending on the kinds of elements.

There are so many excellent books and articles which discuss the binding energy of atoms. These show that the binding energy of atoms as a function atomic mass A takes a maximum around the atomic mass of iron. Here we evaluate the binding energy of all elements and their isotopes based on the NIST atomic weight of all elements and isotopes (the latest version of data) with the use of Mathematica, although the binding energy per nucleus as a function of atomic mass A is seen in many standard textbooks in atomic and nuclear physics. We will also compare the experimental values of the binding energy with the theoretical prediction from the liquid drop model. It is clearly seen that the difference between these shows local maxima as a function of the number of neutrons, at the magic numbers of neutron numbers, N = 2, 8, 20, 28, 50, 82, and 126.

The binding energy per nucleus of atoms and their isotopes as a function of A are explained in terms of several models, including the following models.

1. Fermi gas model
2. Liquid droplet model (George Gamov, Niels Bohr, John Wheeler)
3. Spin-orbit interaction (Clebsch-Gordan coefficient)

((Maria G. Mayer))

Picture of Prof. Maria Goeppert Mayer

Maria Goeppert Mayer (June 28, 1906 – February 20, 1972) was a German-born American theoretical physicist, and Nobel laureate in Physics for proposing the nuclear shell model of the atomic nucleus. She was the second woman to win a Nobel Prize in physics, the first being Marie Curie. In 1986, the Maria Goeppert-Mayer Award for early-career women physicists was established in her honor.

A graduate of the University of Göttingen, Goeppert Mayer wrote her doctoral thesis on the theory of possible two-photon absorption by atoms. At the time, the chances of experimentally verifying her thesis seemed remote, but the development of the laser in the 1960s later permitted this. Today, the unit for the two-photon absorption cross section is named the Goeppert Mayer (GM) unit.
https://en.wikipedia.org/wiki/Maria_Goeppert_Mayer

((George Gamov))

Picture of Prof. George Gamov

George Gamow (March 4, 1904 – August 19, 1968), was a Russian-born American polymath, theoretical physicist and cosmologist. He was an early advocate and developer of Big Bang theory. He discovered a theoretical explanation of alpha decay by quantum tunneling, invented the liquid drop model and the first mathematical model of the atomic nucleus, and worked on radioactive decay, star formation, stellar nucleosynthesis and Big Bang nucleosynthesis (which he collectively called nucleocosmogenesis), and molecular genetics.
https://en.wikipedia.org/wiki/George_Gamow

2. Typical examples of atomic elements with magic number

(a) He (Helium)

$A = 4, Z = 2, N = 2$

The number of neutrons (2) and protons (2) is the magic number. It is *doubly magic*.

Isotope	Half Life
He-3	Stable
He-4	Stable

(b) O (Oxygen)

$A = 16, Z = 8, N = 8$

The number of neutrons (8) and protons (8) is the magic number. It is *doubly magic*.

Isotope	Half Life
O-15	122.2 seconds
O-16	Stable
O-17	Stable
O-18	Stable

(c) Ca (Calcium)

$A = 40, Z = 20, N = 20$

The number of neutrons (20) and protons (20) is the magic number. It is *doubly magic*.

Isotope	Half Life
Ca-40	Stable
Ca-41	103000.0 years
Ca-42	Stable
Ca-43	Stable
Ca-44	Stable
Ca-45	162.7 days
Ca-46	Stable
Ca-47	4.5 days
Ca-48	Stable
Ca-49	8.7 minutes

(d) Ni (Nickel)

$A = 59, Z = 28, N = 31$

The number of proton (28) is the magic number.

Isotope	Half Life
Ni-56	6.1 days
Ni-57	35.6 hours
Ni-58	Stable
Ni-59	76000.0 years
Ni-60	Stable
Ni-61	Stable
Ni-62	Stable
Ni-63	100.0 years
Ni-64	Stable
Ni-65	2.51 hours

(e) Sn (tin)

$A = 119$, $Z = 50$, $N = 69$

The number of proton (50) is the magic number.

Isotope	Half Life
Sn-112	Stable
Sn-113	115.1 days
Sn-114	Stable
Sn-115	Stable
Sn-116	Stable
Sn-117	Stable
Sn-117m	13.6 days
Sn-118	Stable
Sn-119	Stable
Sn-119m	293.0 days
Sn-120	Stable
Sn-121	1.12 days
Sn-121m	55.0 years
Sn-122	Stable
Sn-123	129.2 days
Sn-123m	40.1 minutes
Sn-124	Stable
Sn-125	9.63 days
Sn-125m	9.5 minutes
Sn-126	100000.0 years

(f) Pb (lead)

$A = 208$, $Z = 82$, $N = 126$

The number of neutrons (126) and protons (82) is the magic number

Isotope	Half Life
Pb-202	53000.0 years
Pb-203	2.16 days
Pb-204	Stable
Pb-204m	1.12 hours
Pb-205	1.5E7 years
Pb-206	Stable
Pb-207	Stable
Pb-208	Stable
Pb-209	3.25 hours
Pb-210	22.3 years
Pb-211	36.1 minutes
Pb-212	10.64 hours
Pb-214	27.0 minutes

3. The atomic number *Z*

Why does *Z* stand for the atomic number? Why does *A* stand for mass number instead of atomic number? We learn from the Wikipedia that the letter *Z* denotes "Zahl (number in English), which, before the modern synthesis of ideas from chemistry and physics, merely denoted an element's numerical place in the periodic table, whose order was then approximately, but not completely, consistent with the order of the elements by atomic weights. Only after 1915, with the suggestion

and evidence that this *Z* number was also the nuclear charge and a physical characteristic of atoms, did the word *Atomzahl* (and its English equivalent *atomic number*) come into common use in this context.
https://en.wikipedia.org/wiki/Atomic_number

The **atomic number** or **proton number** (symbol *Z*) of a chemical element is the number of protons found in the nucleus of every atom of that element. The atomic number uniquely identifies a chemical element. It is identical to the charge number of the nucleus. In an uncharged atom, the atomic number is also equal to the number of electrons. The sum of the **atomic number *Z*** and the **number of neutrons *N*** gives the **mass number *A*** of an atom.

$$A = Z + N\ .$$

Since protons and neutrons have approximately the same mass (and the mass of the electrons is negligible for many purposes) and the mass defect of nucleon binding is always small compared to the nucleon mass, the atomic mass of any atom, when expressed in unified atomic mass units (making a quantity called the "relative isotopic mass"), is within 1% of the whole number *A*.

Atoms with the same atomic number but different neutron numbers, and hence different mass numbers, are known as isotopes. A little more than three-quarters of naturally occurring elements exist as a mixture of isotopes (see monoisotopic elements), and the average isotopic mass of an isotopic mixture for an element (called the relative atomic mass) in a defined environment on Earth, determines the element's standard atomic weight. Historically, it was these atomic weights of elements (in comparison to hydrogen) that were the quantities measurable by chemists in the 19th century.

Each element can be represented by the notation ${}^{A}_{Z}X$, where *A*, the mass number, is the sum of the number of protons and the number of neutrons, and *Z*, the atomic number, is the number of protons. The protons and neutrons that make up the nucleus of an atom are called **nucleons**, and an atom with a particular number of protons and neutrons is called a **nuclide**. Nuclides with the same number of protons but different numbers of neutrons are called **isotopes**. Isotopes can also be represented by an alternative notation that uses the name of the element followed by the mass number, such as carbon-12.

4. Formula of binding energy

We make a plot of the binding energy per nucleus for each element as a function of the atomic number *A*. The value of the binding energy per nucleus can be evaluated using the formula.

$$\frac{E_B(atom)}{A} = \frac{1}{A}[Z\ m(^1H) + N\ m_n - M(atom)]c^2\ ,$$

with

$$A = Z + N \, .$$

In **Table-1 (APPENDIX-A)**, we show the values of A, Z, N, M, the experimental binding energy $E_B(atom)/A$ and the theoretical binding energy for each element, are listed.

A: atomic mass
Z: number of protons (atomic number)
N: number of neutrons

We use the following data, which are obtained from NIST web site.

$c = 2.99792458 \text{ x } 10^8$ m (speed of light)

Ry=13.60569312299426 eV (Rydberg unit of energy)

$$\frac{Ry}{c^2} = 1.46063 \times 10^{-8} \; u$$

$u = 1.66053906660 \times 10^{-27}$ kg (atomic mass unit)

$m_e = 0.0005485799007 \; u$ (mass of electron)

$m_p = 1.007276466 \; u$ (mass of proton)

$m_n = 1.00866491595 \; u$ (mass of neutron)

$$m_p + m_n = 2.015941382 \; u$$

$$m_p + m_e = 1.007825046 \; u$$

$m(^1\text{H}) = 1.00782503223 \; u$ (mass of hydrogen atom)

$m(^2\text{D}) = 2.01410177812 \; u$ (mass of deuterium)

$m(^3\text{T}) = 3.0160492779 \; u$ (mass of tritium)

5. Binding energy for hydrogen atom ^{1}H

It is well known that the binding energy of hydrogen atom is

$$E_B = \text{Ry}=13.60569312299426 \text{ eV} \qquad \text{for } ^1\text{H},$$

where Ry is the Rydberg unit of energy. The hydrogen atom consists of one proton and one electron. So, the mass of hydrogen atom is given by

$$\begin{aligned} m_H &= m_p + m_e - \frac{R_y}{c^2 u} \\ &= 1.007825046\ u - 1.46063\times 10^{-8} u \\ &= 1.007825031\ u \end{aligned}$$

taking into account of the binding energy (Ry). This value of m_H is in good agreement with that reported by NIST;

$$m(^1\text{H}) = 1.00782503223\ u.$$

6. Binding energy for Deuterium ^{2}D

Deuterium consists of one proton, one neutron and one electron. Using Eq.(1), the binding energy of deuterium is evaluated as

$$\begin{aligned} E_B &= [1\ m_P + 1\ m_n - m(^2\text{D})]c^2 \\ &= 2.22458 \text{ MeV} \end{aligned}$$

or

$$\begin{aligned} E_B &= [1\ m(^1\text{H}) + 1\ m_n - m(^2\text{D})]c^2 \\ &= 2.22457 \text{ MeV} \end{aligned}$$

where $m(^2\text{D})$ is the molecular mass of deuterium

$$m(^2\text{D}) = 2.01410177812\ u$$

These values of E_B are the same. Hereafter, we use the latter value of E_B to evaluate the binding energy. The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{2.22457 \text{ MeV}}{2} = 1.11228 \text{ MeV}$$ for ^2D.

7. Binding energy for tritium ^3T

Deuterium consists of one proton, one neutron and one electron. The binding energy of deuterium is

$$\begin{aligned} E_B &= [1m(^1\text{H}) + 2m_n - m(^3\text{T})]c^2 \\ &= 8.4818 \text{ MeV} \end{aligned}$$

where $m(^3\text{T})$ is the mass of tritium ^3T,

$$m(^3\text{T}) = 3.0160492779 \ u$$

The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{8.4818 \text{ MeV}}{3} = 2.8272 \text{ MeV}$$ for ^3T

8. Binding energy for ^3He

Helium 3 consists of two protons, one neutron, and two electrons. The binding energy of ^3He is

$$\begin{aligned} E_B &= [2m(^1\text{H}) + m_n - m(^3\text{He})]c^2 \\ &= 7.71804 \text{ MeV} \end{aligned}$$

where $m(^3\text{He}) = 3.0160293201 \ u$ is the mass of ^3He. The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{7.71804 \text{ MeV}}{3} = 2.57268 \text{ MeV}$$ for ^3He.

9. Binding energy for ^4He

Helium 4 consists of two proton, two neutrons and two electrons. The binding energy of ^4He is

$$\begin{aligned} E_B &= [2m(^1\text{H}) + 2m_n - m(^4\text{He})]c^2 \\ &= 28.2957 \text{ MeV} \end{aligned}$$

where $m(^4\text{He}) = 4.00260325413\ u$ is the mass of ^4He. The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{28.2957\ \text{MeV}}{4} = 7.07392\ \text{MeV} \qquad \text{for } ^4\text{He}.$$

10. Binding energy for ^{6}Li

^{6}Li consists of three proton, three neutrons and three electrons. The binding energy of ^{6}Li is

$$\begin{aligned} E_B &= [3m(^1\text{H}) + 3m_n - m(^6\text{Li})]c^2 \\ &= 31.994\ \text{MeV} \end{aligned}$$

where $m(^6\text{Li}) = 6.0151228874\ u$ is the mass of ^{6}Li. The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{31.994\ \text{MeV}}{6} = 5.33233\ \text{MeV} \quad \text{for } ^6\text{Li}.$$

11. Binding energy for ^{7}Li

^{7}Li consists of three proton, four neutrons and three electrons. The binding energy of ^{7}Li is

$$\begin{aligned} E_B &= [3\ m(^1\text{H}) + 4\ m_n - m(^7\text{Li})]c^2 \\ &= 39.2451\ \text{MeV} \end{aligned}$$

where $m(^7\text{Li}) = 7.0160034366\ u$ is the mass of ^{7}Li. The binding energy per nucleus is

$$\frac{E_B}{A} = \frac{39.2451\ \text{MeV}}{7} = 5.60644\ \text{MeV} \quad \text{for } ^7\text{Li}.$$

We note that the binding energy per nucleus of ^{4}He is much larger than that of ^{3}T, ^{3}He, and ^{7}Li.

12. The relation between *Z* vs *N* for experimental results and theory

We start with the energy for the liquid drop model (the detail of this model will be discussed later).

$$E_B = a_v A - a_s A^{2/3} - a_c \frac{1}{A^{1/3}} Z(Z-1) - a_{sy} \frac{1}{A}(N-Z)^2 + \delta,$$

with $A = Z + N$. We evaluate $\frac{dE_B}{dZ} = 0$ with A being hold constant.

$$\frac{dE_B}{dZ} = -a_c \frac{1}{A^{1/3}}(2Z-1) + a_{sy}\frac{1}{A}4(A-2Z) = 0,$$

leading the value of Z and N as a function of A,

$$Z = \frac{\frac{a_c}{A^{1/3}} + 4a_{sy}}{\frac{2a_c}{A^{1/3}} + \frac{8a_{sy}}{A}}$$
$$= \frac{A}{2}\frac{1 + \frac{a_c}{4a_{sy}}\frac{1}{A^{1/3}}}{1 + \frac{a_c}{4a_{sy}}A^{2/3}}$$

and

$$N = A - Z$$
$$= \frac{A}{2}\frac{1 + \frac{a_c}{2a_{sy}}A^{2/3}(1 - \frac{1}{2A})}{1 + \frac{a_c}{4a_{sy}}A^{2/3}},$$

with

$$\alpha = \frac{a_c}{a_{sy}} = \frac{0.714}{23.2} = 0.0307758, \qquad \text{from the liquid drop model.}$$

We make a plot of Z vs N by using the ParametricPlot of Mathematica, where A is changed as parameter.

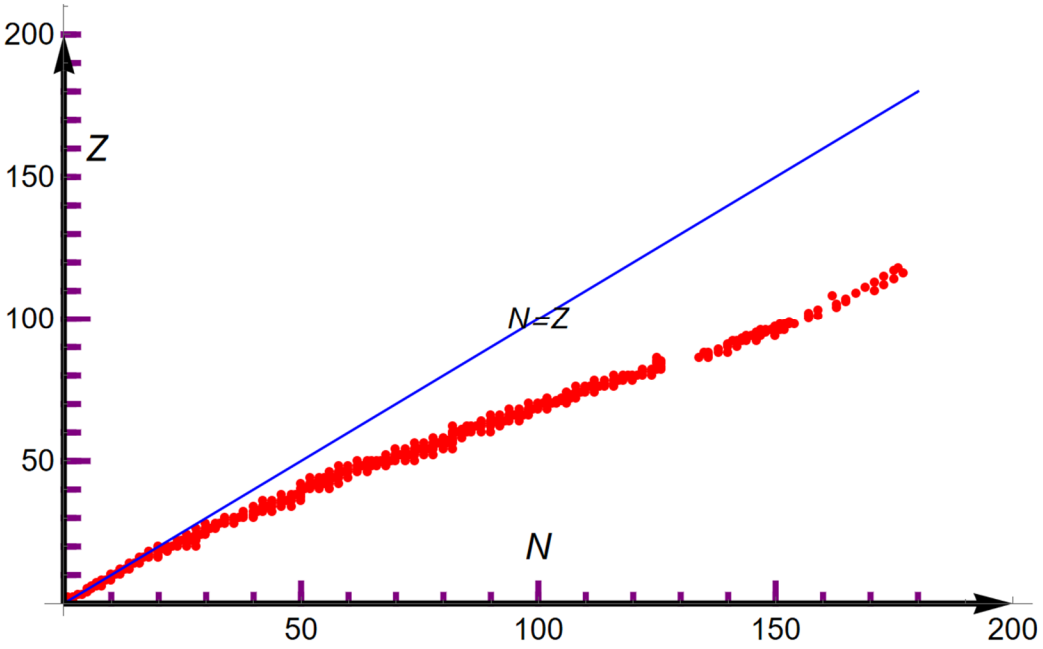


Fig.1 ParametricPlot (Mathematica) of Z vs N. Experimental results (denoted by red dots) based on NIST data. Theoretical curve (the liquid drop model, denoted by black line with $\frac{a_c}{a_{sy}} = \frac{0.714}{23.2} = 0.0307758$).

As is expected, the experimental data of N vs A falls well on the theoretical curve denoted by black line. The curve is well described by the linear relation ($N = Z$) only for small Z (typically, $Z<15$). For $Z>160$, the experimental curve deviates from the theoretical curve.

We consider the value of Z and N the limit of large A.

$$Z = \frac{A}{2}\frac{1+\frac{a_c}{4a_{sy}}\frac{1}{A^{1/3}}}{1+\frac{a_c}{4a_{sy}}A^{2/3}} \simeq \frac{A}{2}\frac{1}{1+\frac{a_c}{4a_{sy}}A^{2/3}},$$

or

$$Z = \frac{A}{2}\left(\frac{1}{1+0.008A^{2/3}}\right),$$

and

$$N = A - Z = A - \frac{A}{2}\left(\frac{1}{1+0.008A^{2/3}}\right) > \frac{A}{2}.$$

13. Binding energy of elements as a function of atomic number A: comparison between experimental values and theoretical values

Using the second formula, we get the experimental values of the binding energy per nucleus. The data are denoted by red dots. We also show the theoretical prediction (semi-empirical mass formula). These are denoted by blue dots. The experimental results are in very good agreement with the semi-empirical mass formula in the region of large A (A>56)

Table of the data from NIST and binding energy per nucleus

$$\left(\frac{E_B}{A}\right)_{\exp} = \frac{1}{A}[Z\ m(^1H) + N\ m_n - M(atom)]c^2, \qquad \text{(in units of MeV)}$$

and

$$\left(\frac{E_B}{A}\right)_{theory} = \frac{1}{A}[a_V A - a_S A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_{SYM}\frac{1}{A}(Z-N)^2 - \frac{1}{A^{1/2}}\delta(N,Z)],$$

(in units of MeV)

A: atomic mass number
Z; atomic number (nuclear charge number)
N: number of neutrons
M: atomic mass

$$A = Z + N$$

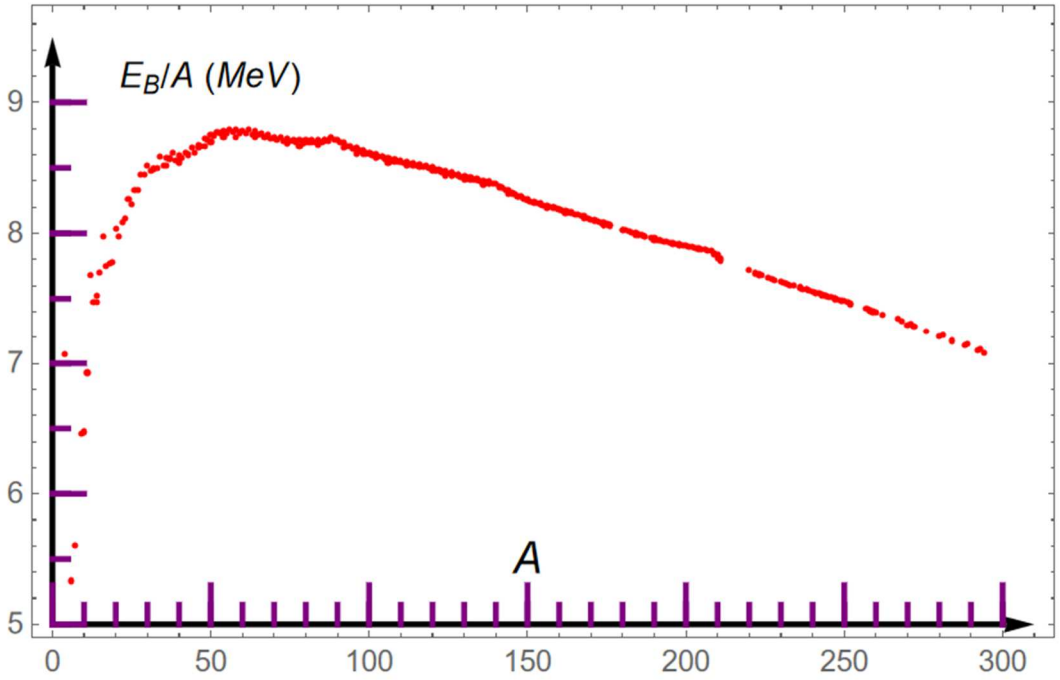


Fig.2 Binding energy/A as a function of the mass number A for all elements with $1 \le A \le 300$. $A = Z + N$. N: the number of neutrons. Z: the atomic number (the number of protons). The data of A, Z, and N for each element are obtained from those collected by NIST Physical Measurement Laboratory. The binding energy /A (experimental results, red).

The most obvious feature is that the plot of the binding energy E_B / A vs A is close to 8 MeV for almost all nuclei, with the maximum value of 8.790 MeV for iron (^{56}Fe). Beyond iron, E_B / A slopes gently down to about 7.570 MeV for ^{238}U; this decrease is due to manly to the increasing importance of the Coulomb repulsion of the protons. On the other side of ^{56}Fe, when A decreases below about 20. of The binding energy E_B / A falls rapidly to zero for ^{1}H (which has no binding energy); this decrease occurs because almost all nucleus in a small nucleus are close to the surface and the negative surface correction is proportionately large.

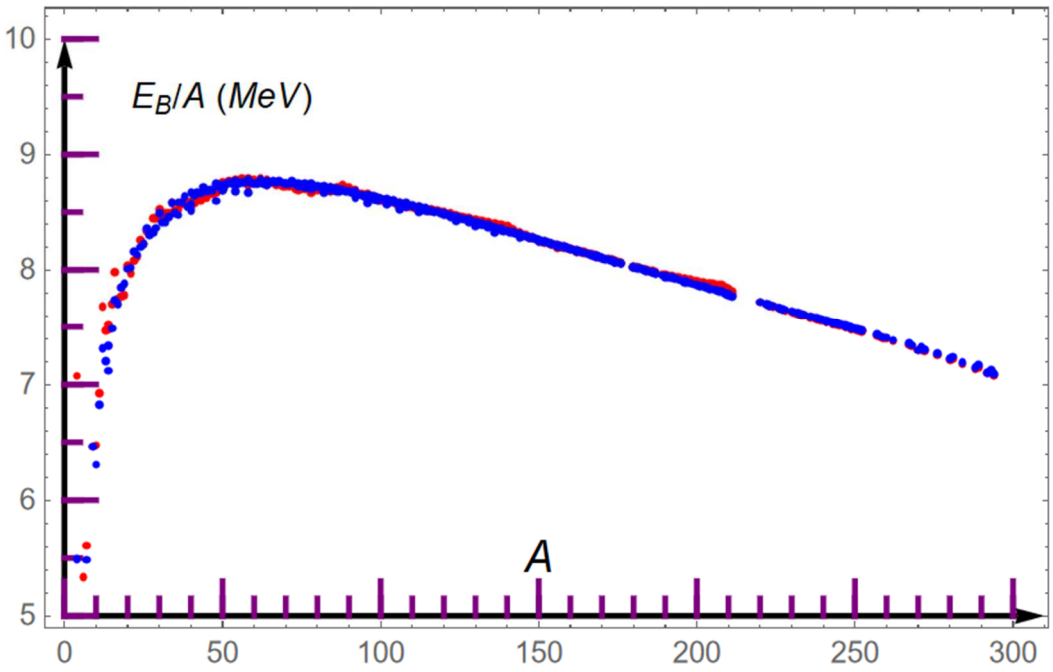


Fig.3 Comparison between experimental results and theoretical prediction from the liquid droplet model. Binding energy/A as a function of the mass number A for all elements $1 \le A \le 300$. $A = Z + N$. N: the number of neutrons. Z: the atomic number (the number of protons). The data of A, Z, and N for each element are obtained from those collected by NIST Physical Measurement Laboratory. The binding energy /A (experimental results, red) and the theoretical prediction (the liquid drop model, blue).

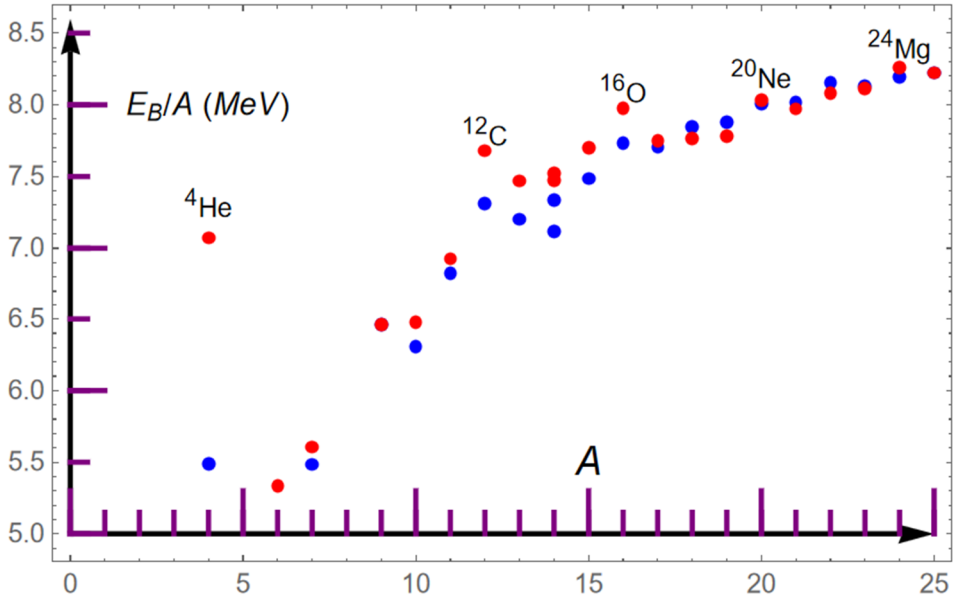


Fig.4 Binding energy per nucleon for the elements (E_B / A) with $1 \le A \le 25$. The binding energy per nucleus is noticeably large for ^{4}He, ^{12}C, ^{16}O, ^{20}Ne, and ^{24}Mg, indicating that these atoms are more stable among atoms. The liquid drop model (denoted by blue closed circles).

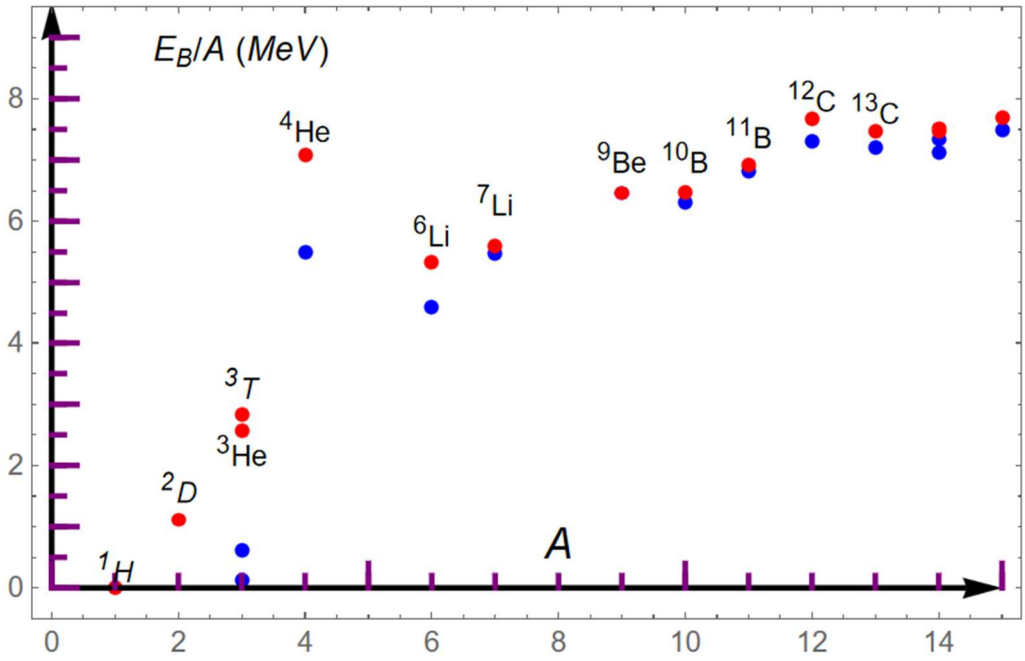


Fig.5 Binding energy per nucleon (E_B / A) vs A for the lightest elements for $1 \le A \le 15$. Note that E_B / A tends to increase with increasing A for $1 \le A \le 4$.

One of the most important features of a nucleus is its average binding energy per nucleon. The quantity is plotted as a function of A in **Figs.2 and 3.** The points are the data obtained from the measured masses in the manner just described. Note that E_B / A at first rises rapidly with increasing A, but very soon E_B / A *is roughly constant* at a value 8 MeV. If each nucleon in a nucleus exerted the same attraction on all the other nucleons, the binding energy per nucleon would continue to increase as more and more nucleons were added to the nucleus; that is, E_B / A would be proportional to A. The extremely important fact that E_B / A is not proportional to A is due, in part, to the short range of nuclear forces.

Note that E_B / A actually maximizes at about 8.7 MeV for $A \sim 60$, and then decreases slowly to about 7.6 MeV for $A \sim 240$. We shall find that the decrease is due to Coulomb repulsions between protons in the nucleus. One consequence is the phenomenon of *nuclear fission,* in which

a large A nucleus, such as ${}^{92}U_{238}$, splits into two intermediate A nuclei because the two intermediate A nuclei are more stable than the large A nucleus.

For lighter elements, the energy that can be released by assembling them from lighter elements decreases, and energy can be released when they fuse (fusion). This is true for nuclei lighter than iron/nickel. For heavier nuclei, more energy is needed to bind them, and that energy may be released by breaking them up into fragments (known as atomic fission). Nuclear power is generated at present by breaking up uranium nuclei in nuclear power reactors, and capturing the released energy as heat, which is converted to electricity. As a rule, very light elements can fuse comparatively easily, and very heavy elements can break up via fission very easily; elements in the middle are more stable and it is difficult to make them undergo either fusion or fission in an environment such as a laboratory. The reason the trend reverses after iron is the growing positive charge of the nuclei, which tends to force nuclei to break up. It is resisted by the strong nuclear interaction, which holds nucleons together. The electric force may be weaker than the strong nuclear force, but the strong force has a much more limited range: in an iron nucleus, each proton repels the other 25 protons, while the nuclear force only binds close neighbors. So, for larger nuclei, the electrostatic forces tend to dominate and the nucleus will tend over time to break up.

As nuclei grow bigger still, this disruptive effect becomes steadily more significant. By the time polonium is reached (84 protons), nuclei can no longer accommodate their large positive charge, but emit their excess protons quite rapidly in the process of alpha radioactivity—the emission of helium nuclei, each containing two protons and two neutrons. (Helium nuclei are an especially stable combination.) Because of this process, nuclei with more than 94 protons are not found naturally on Earth (see periodic table). The isotopes beyond uranium (atomic number 92) with the longest half-lives are plutonium-244 (80 million years) and curium-247 (16 million years).

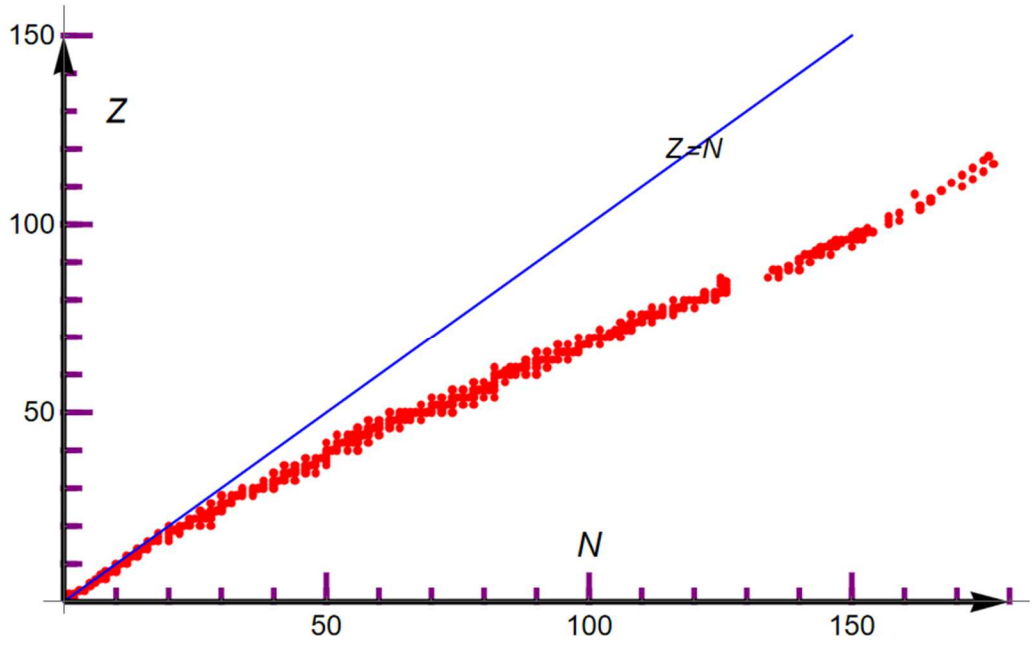


Fig.6 The relation between Z (atomic number, number of protons) vs N (the number of neutrons) in nuclei. The linear relation of $Z = N$ is denoted by blue line. It may be noticed that there is no element with number of neutrons between 126 and 133. $A = Z + N$.

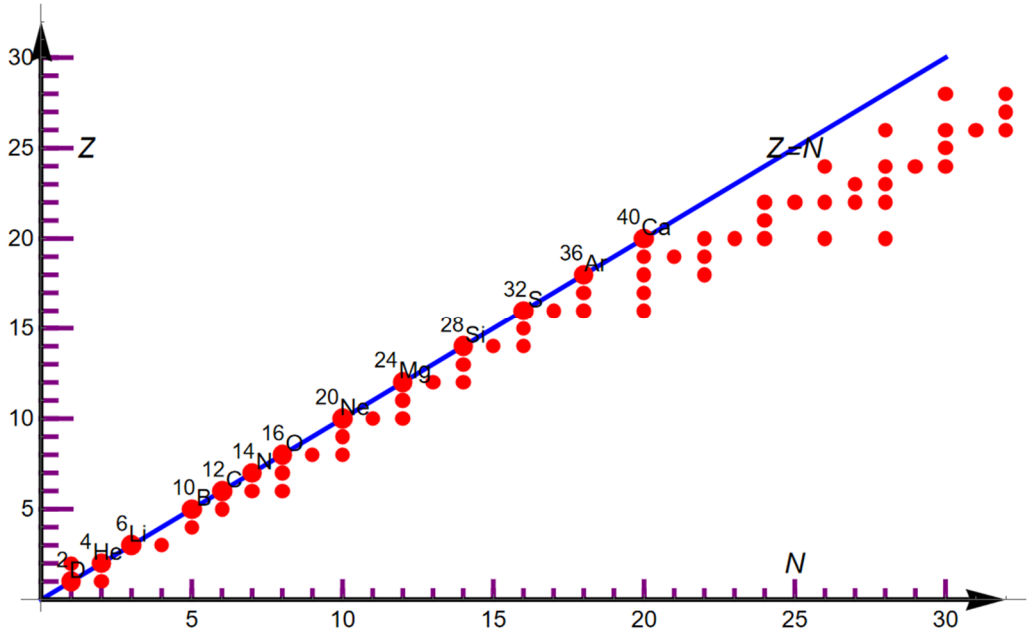


Fig.7 The relation between Z (atomic number, number of protons) vs N (the number of neutrons) for elements. The points for ^{2}D, ^{4}He, ^{6}Li, ^{10}B, ^{12}C, ^{14}N, ^{16}C, ^{20}Ne, ^{24}Mg, ^{28}Si, and ^{32}S, ^{36}Al, ^{40}Ca are located on the linear relation $Z = N$.

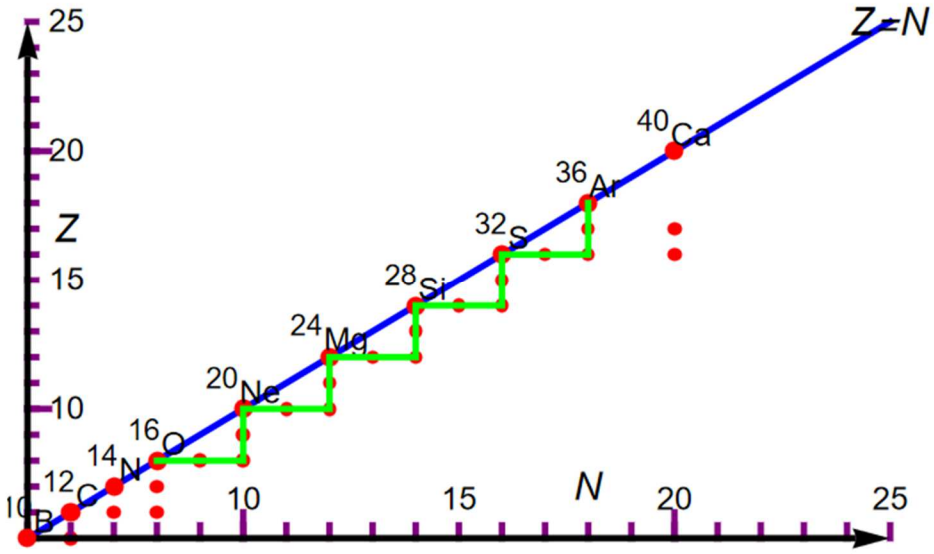


Fig.8 The relation between Z (atomic number, number of protons) vs N (the number of neutrons) for elements.

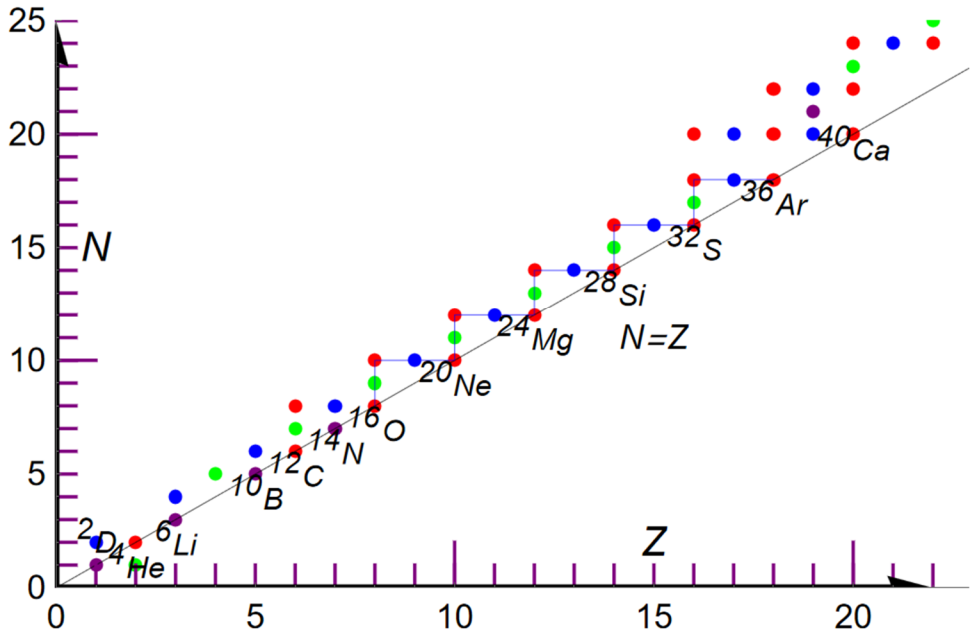


Fig.9 The plot of N vs Z for elements. The linear relation ($N = Z$) is valid for small Z.

((**Note**)) **Radioactive decay**

$$p \to n + e^{+} + \bar{\nu} \qquad \begin{matrix} Z \to Z-1 \\ N \to N+1 \end{matrix}$$

$$n \to p + e^{-} + \nu \qquad \begin{matrix} Z \to Z+1 \\ N \to N-1 \end{matrix}$$

14. Fission and Fusion ((Eisberg and Resnick))

The Earth's inner core is thought to be slowly growing as the liquid outer core at the boundary with the inner core cools and solidifies due to the gradual cooling of the Earth's interior (about 100 degrees Celsius per billion years). The iron crystallizes onto the inner core. The liquid just above it becomes enriched in oxygen, and therefore less dense than the rest of the outer core. This process creates convection currents in the outer core, which are thought to be the prime driver for the currents that create the Earth's magnetic field. The existence of the inner core also affects the dynamic motions of liquid in the outer core, and thus may help fix the magnetic field.

Unlike the mineral-rich crust and mantle, the core is made almost entirely of metal—specifically, iron and nickel. The shorthand used for the core's iron-nickel alloys is simply the elements' chemical symbols—NiFe. Elements that dissolve in iron, called siderophiles, are also found in the core.

So that, the nuclear fusion occurs for the elements with A<56, while the nuclear fission occurs for the element with A>56.

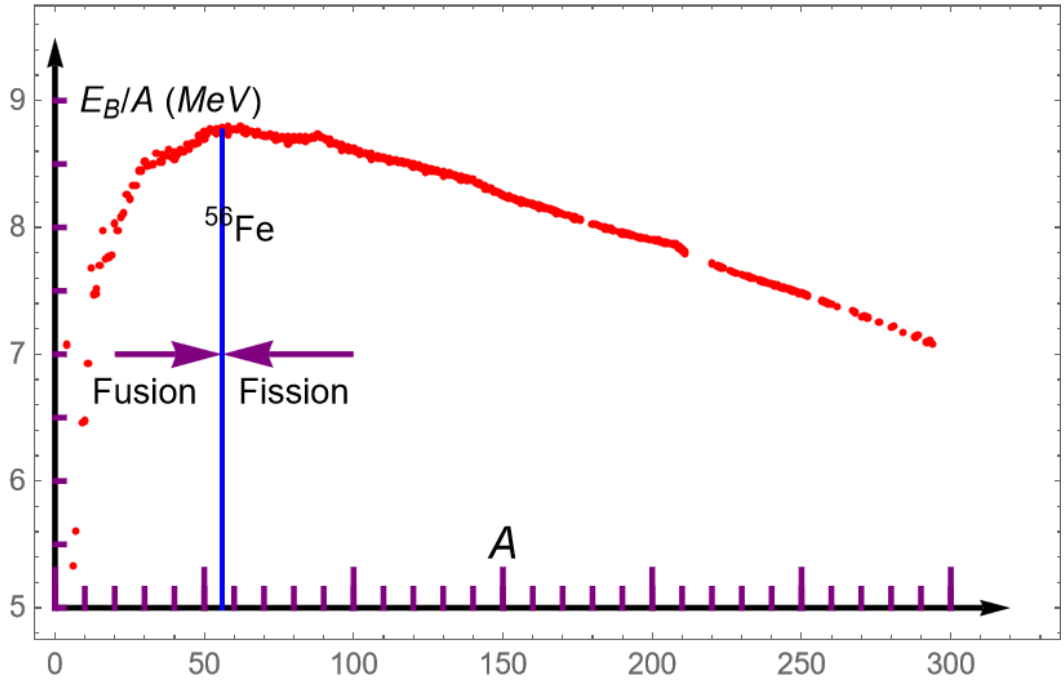


Fig.10 Plot of E_B / A versus A for elements. There are two regions of fission and fusion.

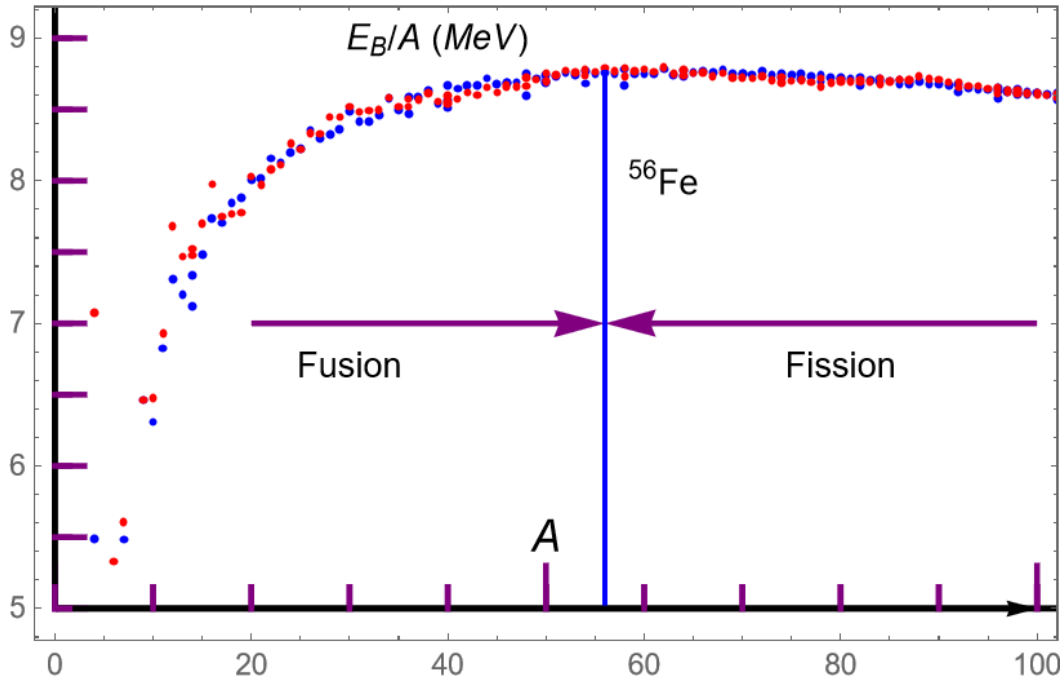


Fig.11 Plot of E_B / A versus A for elements. There are two regions of fission and fusion The maximum of the binding energy per nucleus is the greatest for nuclei with a mass near ^{56}Fe. Therefore, fusion of nuclei with mass numbers much less than that of Fe, and fission of nuclei with mass numbers greater than that of Fe, are exothermic processes.

((Nuclear fission))

Use **Fig.10** to estimate the difference between the binding energy of a $^{92}U_{238}$ nucleus and the sum of the binding energies of the two nuclei produced if it fissions symmetrically. The figure shows that the average binding energy per nucleon for a nucleus of mass number around $A = 238$ is $E_B / A = 7.5697858$ MeV. So, the binding energy of the nucleus present before the fission is 238 x 7.5697858 MeV = 1801.6 MeV. The figure also shows that the average binding energy per nucleon for a nucleus of mass number around $A = 238/2 = 119$ (Sn) is $E_B / A \sim 8.499056$ MeV. So, each of the two nuclei present after the symmetrical fission has a binding energy of $E_B = \sim 119 \times 8.499056$ MeV=1011.4 MeV. The sum of their binding energies is ~ 2022.8 MeV. This sum is larger than the initial binding energy 1801.6 MeV by about 221.2 MeV. Thus, the final state (after the nucleus fissions) is more stable than the initial state (before the nucleus fissions), because the total binding energy is higher in the final state. When the total binding energy increases by about 221.2 MeV in the fission, energy in this amount is liberated. Most of it goes into the kinetic energy of the two nuclei produced in the fission. In a nuclear reactor this kinetic energy is degraded into thermal energy, which is the source of the power produced by the reactor.

((Nuclear fusion))

In *nuclear fusion* two or more nuclei of very small A combine to form a larger nucleus that has a higher average binding energy per nucleon because its value of A is nearer the value $A = 60$, at which E_B / A maximizes. It might seem that only a few nuclei near $A = 60$ would be stable. This is not true because there are other factors, to be discussed later, which inhibit fission and fusion. We conclude this section by considering the distribution of Z and A values of the stable nuclei, which is additional information obtained from the mass spectrometer measurements.

15. Fermi gas model

The Fermi gas model is a theoretical model used to describe the behavior of a gas of non-interacting fermions, such as protons and neutrons, under certain conditions. In the context of the liquid drop model of the atomic nucleus, the Fermi gas model is used to describe the distribution of protons and neutrons in the nucleus.

In the Fermi gas model, the protons and neutrons are treated as if they are independent particles that move around inside the nucleus, subject to the nuclear potential well that confines them. Because protons and neutrons are fermions, they are subject to the Pauli exclusion principle, which states that no two fermions can occupy the same quantum state simultaneously. This leads to the Fermi-Dirac statistics, describing the distribution of particles in a system of non-interacting fermions.

In the context of the nucleus, the Fermi gas model assumes that the protons and neutrons are distributed in energy levels that are determined by the nuclear potential well. The energy levels are filled according to the Fermi-Dirac distribution, which describes the probability of finding a particle in a given energy level. The model predicts that the distribution of protons and neutrons in the nucleus should be roughly uniform up to a certain energy level, known as the Fermi energy. Beyond this energy level, the distribution drops off rapidly.

The Fermi gas is a simplified model that neglects many important effects, such as the strong interaction between protons and neutrons and the deformation of the nuclear shape. However, it provides a useful starting point for understanding the behaviors of protons and neutrons in the nucleus. and it forms the basis for more advanced models of nuclear structure.

The Fermi energy in the liquid drop model of the atomic nucleus is a measure of the energy of the highest occupied state of the protons and neutrons in the nucleus. It is an important parameter in the Fermi gas model, which assumes that the protons and neutrons in the nucleus behave like a gas of non-interacting fermions.

The value of the Fermi energy depends on the size of the nucleus and the number of protons and neutrons it contains. In general, the Fermi energy is on the order of few MeV for most nuclei. However, it can vary depending on the specific nucleus and its properties.

It is important to note that the Fermi energy is a theoretical quantity and is not directly measurable. However, it is a useful parameter in the Fermi gas model and other models of nuclear structure. as it provides a way to describe the distribution of protons and neutrons in the nucleus.

In the nuclear shell model, a square well potential is often used to represent the potential energy of nucleons (protons and neutrons) inside the nucleus. The depth of the potential well is an important parameter that affects the behavior of nucleons inside the nucleus. The depth of the

potential well in the nuclear shell model can vary depending on the specific nucleus and its properties. However, typical values for the depth of the potential well are on the order of a few tens of MeV. For example, the depth of the potential well for a nucleus like oxygen-16 (which has 8 protons and 8 neutrons) is estimated to be around 40 MeV. This depth is much larger than the typical kinetic energies of nucleons inside the nucleus, which are on the order of a few MeV.

It is important to note that the square well potential is a simplification and does not capture all of the complexities of the nuclear potential. In reality, the potential energy of nucleons inside the nucleus is determined by the strong nuclear force, which is a complex and non-linear interaction between nucleons. However, the square well potential is a useful model that provides a way to describe the behavior of nucleons in the nucleus and to understand the properties of nuclear structure.

The radius of proton and neutron in a nucleus depends on the specific nucleus and its properties. In general, the radius of a proton or neutron in a nucleus is much smaller than the radius of the nucleus itself. The typical size of a nucleus is around 1 femtometer (1fm = 10^{-15} m), and the size of a proton or neutron is on the order of 0.8 - 0.9 fm. This means that the protons and neutrons are packed tightly together in the nucleus, with very little empty space between them.

It is important to note that the size of a proton or neutron in nucleus can vary depending on the specific nucleus and its properties. For example, in a nucleus with a large number of protons and neutrons, the nucleons (protons and neutrons) may be more spread out, leading to a slightly larger radius. Additionally, the size of a proton or neutron can also be affected by the nuclear force and other interactions between the nucleons in the nucleus.

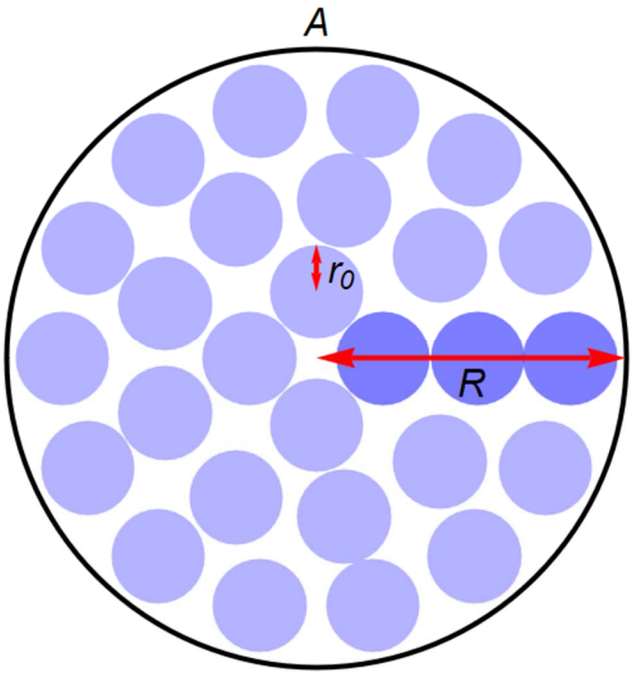


Fig.12 The packing of protons and neutrons in a nucleus. Nuclear shell model. Fermi gas model. $R = r_0 A^{1/3} = 1.2 A^{1/3}$ [fm].

$$N_F = \frac{2V}{(2\pi)^3}\frac{4\pi}{3}{k_F}^3 = \frac{V}{3\pi^2}{k_F}^3 .$$

with

$$V = \frac{4\pi}{3}{r_0}^3 A = \frac{4\pi}{3}R^3 .$$

So that, we have

$$N_F = \frac{1}{3\pi^2}{k_F}^3\frac{4\pi}{3}{r_0}^3 A = \frac{4}{9\pi}A(r_0 k_F)^3 .$$

Note that

$$R = r_0 A^{1/3} = 1.2 A^{1/3} \text{ [fm]}.$$

We assume that r_0 is on the order of $r_0 = 1.2$ fm. We also assume that

$$N_F = N = Z = \frac{A}{2},$$

or

$$N_F = \frac{A}{2} = \frac{4}{9\pi}A(r_0 k_F)^3 ,$$

or

$$r_0 k_F = \left(\frac{9\pi}{8}\right)^{1/3} .$$

The Fermi wave number is obtained as

$$k_F = \frac{1}{r_0}\left(\frac{9\pi}{8}\right)^{1/3} = \frac{1.52324}{1.2\ \text{fm}} = 1.26936\ (\text{fm})^{-1},$$

which is independent of A. Thus, the Fermi energy is

$$\begin{aligned}
E_F &= \frac{\hbar^2 k_F{}^2}{2m} \\
&= \frac{\hbar^2}{2m}(k_F r_0)^{2/3}\frac{1}{r_0{}^2} \\
&= \frac{\hbar^2}{2m}\left(\frac{9\pi}{8}\right)^{2/3}\frac{1}{r_0{}^2} \\
&= \frac{1}{2mc^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(\frac{\hbar c}{r_0}\right)^2 \\
&= \frac{1}{2\times 938.272\ (\text{MeV})}\left(\frac{197.327\ \text{MeV}}{1.2\ \text{fm}}\right)^2 (2.32025)^2 \\
&= 33.4339\ \text{MeV}
\end{aligned}$$

where

$Mc^2 = 938.272$ MeV (M: mass of proton)

$r_0 = 1.2$ fm (typical value of radius)

$$\frac{\hbar c}{r_0} = \frac{197.327\ \text{MeV}}{1.2\ \text{fm}} = 164.439\ \frac{\text{MeV}}{\text{fm}}$$

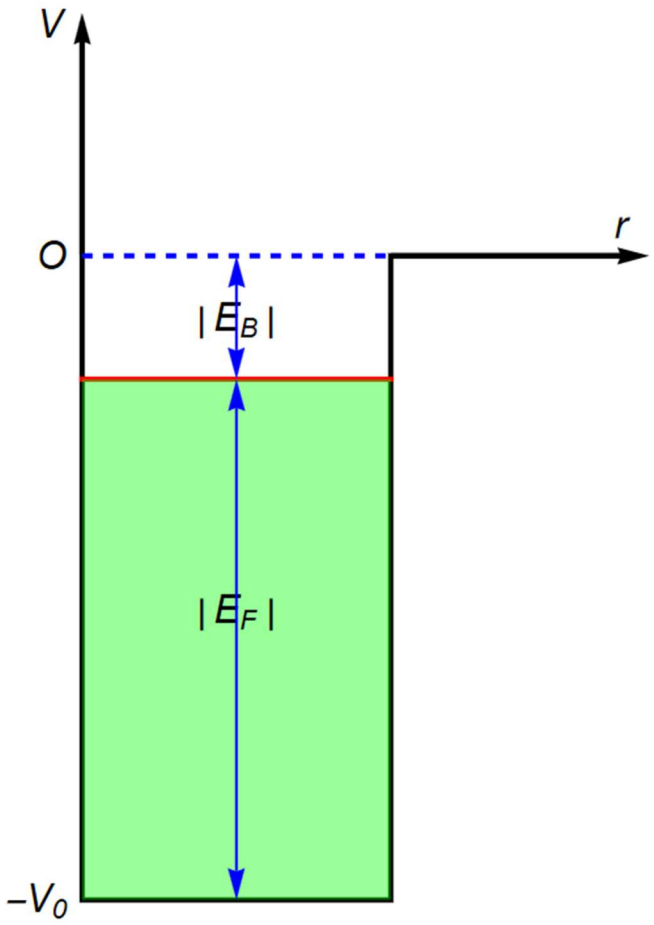


Fig.13 The square-well potential energy of the nucleus. $|E_F| = 33.4339$ MeV . . $|E_B| = 8.0$ MeV . The potential depth is $V_0 = |E_E| + |B| = 41.4$ MeV

16. Evaluation of de Broglie wavelength

The de Broglie-wavelength evaluated as

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{m_p c} = \frac{2\pi\hbar c}{m_p c^2} = 1.32141 \text{ fm}$$

The energy of proton rest energy:

$$m_p c^2 = 938.272 \text{ MeV}, \qquad \hbar c = 197.327 \text{ MeV} .$$

where 1 fm is a femtometer (10^{-15} m) or one fermi (after Enrico Fermi). This wavelength is within the typical range of nuclear dimensions, and is therefore reasonable to expect to localize nucleons of such energies within the nucleus.

17. Woods-Saxon potential

https://en.wikipedia.org/wiki/Woods%E2%80%93Saxon_potential

The Woods–Saxon potential is a mean field potential for the nucleons (protons and neutrons) inside the atomic nucleus, which is used to describe approximately the forces applied on each nucleon, in the nuclear shell model for the structure of the nucleus. The potential is named after **Roger D. Woods** and **David S. Saxon**. The form of the potential, in terms of the distance r from the center of nucleus, is:

$$V(r) = -\frac{V_0}{1+\exp(\frac{r-R}{a})},$$

where V_0 (having dimension of energy) represents the potential well depth, a is a length representing the "surface thickness" of the nucleus, and $R = r_0 A^{1/3}$ is the nuclear radius where r_0 = 1.25 fm and A is the mass number. Typical values for the parameters are: $V_0 = 50$ MeV, $a \approx 0.5$ fm. For large atomic number A this potential is similar to a potential well.

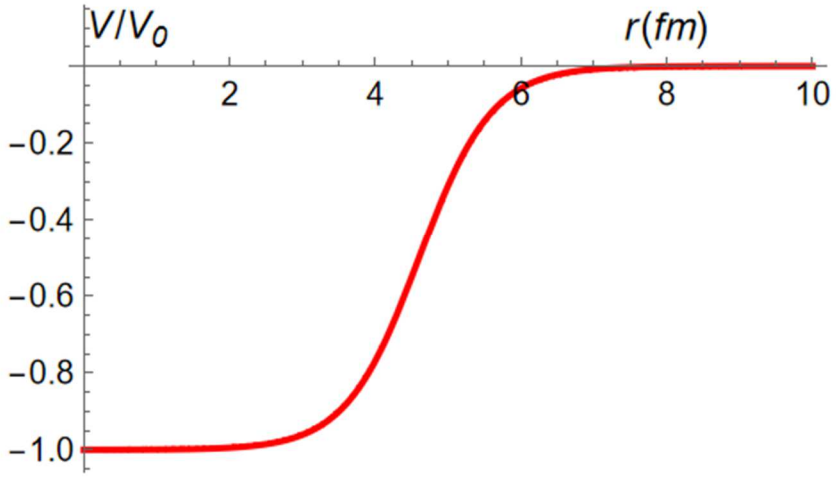


Fig.14 Woods-Saxon potential for A = 50, relative to V_0 (= 50 MeV) with a = 0.5 fm and R = 4.6 fm. $R = r_0 A^{1/3}$ with r_0 = 1.25 fm.

18. Liquid droplet model: (prediction of binding energy of nucleus

The liquid drop model of the atomic nucleus is a theoretical model that describes the properties of atomic nuclei in terms of a droplet of incompressible nuclear matter. The model includes several parameters that are used to calculate the binding energy of the nucleus, which is energy required to break the nucleus into its constituent protons and neutrons.

The standard values of the parameters in the liquid drop model depend on the specific version of the model being used. However, here are some typical values for the parameters:

1. Volume energy coefficient (a_V): This parameter represents the contribution of the volume of the nucleus to its binding energy. Typical values of a_V are around 15-17 MeV.
2. Surface energy coefficient (a_s): This parameter represents the contribution of the surface of the nucleus to its binding energy. Typical values for a_s are around 16-19 MeV.
3. Coulomb energy coefficient (a_c): This parameter represents the contribution of the electrostatic repulsion between protons in the nucleus to its binding energy. Typical values for a_c are around 0.7 - 1.2 MeV.
4. Asymmetry energy coefficient (a_{sy}): This parameter represents the contribution of the difference in proton and neutron numbers to the binding energy. Typical values for a_{sy} are around 22-24 MeV.
5. Pairing energy (a_p): This parameter represents the contribution of the pairing of nucleons with opposite spins to the binding energy. Typica; values for a_p are around -11 to -12 MeV.

It is important to note that these values are not fixed and can vary depending on the specific version of the liquid drop model being used and the properties of the nucleus being studied. However, these typical values provide a starting point for understanding the parameters in the liquid drop model and their contributions to the binding energy of atomic nuclei.

The formula represents the liquid drop model proposed by **George Gamow**, which can account for most of the terms in the formula and gives rough estimates for the values of the coefficients. It was first formulated in 1935 by German physicist Carl Friedrich von Weizsäcker and although refinements have been made to the coefficients over the years, the structure of the formula remains the same today.
https://en.wikipedia.org/wiki/George_Gamow

The liquid drop model is a model in nuclear physics which treats the nucleus as a drop of incompressible nuclear fluid first proposed by **George Gamow** and developed by **Niels Bohr** and **John Archibald Wheeler**. The fluid is made of nucleons (protons and neutrons), which are held together by the strong nuclear force. This is a crude model that does not explain all the properties of the nucleus, but It does explain the spherical shape of most nuclei. It also helps to predict the binding energy of the nucleus. The liquid drop model approximates the nucleus as a sphere with a uniform interior density, that abruptly drops to zero at its surface. The radius is proportional to $A^{1/3}$; the surface area is proportional to $A^{2/3}$; and the volume is proportional to A. Since the mass is also proportional to A , which is the number of nucleons in the nucleus

The volume term

This accounts for a binding energy proportional to the nuclear mass, or volume. The term describes the tendency to have the binding energy per nucleon a constant. Such a term would be present for a classical liquid drop. Because it is negative, it reduces the mass, and therefore increases the binding energy.

The surface term

It is a correction proportional to the surface area of the nucleus. Since the term is positive, it increases the mass and consequently reduces the binding energy. In a classical drop of liquid, this term would represent the effect of the surface tension energy. It would arise from the fact that a molecule at the surface of the drop feels attractive forces only from one side, so its binding energy is less than the binding energy of a molecule in the interior which feels attractive forces from all sides. Therefore, simply setting the total binding energy proportional to the volume of the drop overestimates the binding energy of the surface molecules, and a correction proportional to the number of such molecules, or to the surface area, must be made to reduce the binding energy. The same thing happens in a nucleus.

The Coulomb term

It accounts for the positive Coulomb energy of the charged nucleus, which is assumed to have a uniform charge distribution of radius proportional to $A^{1/3}$. This effect of the Coulomb repulsions between the protons increases the mass and reduces the binding energy. A similar term would be present for a charged drop of a classical liquid.

The asymmetry term

It accounts for the observed tendency to have $Z = N$. Note that it is zero for $Z = N = (A - Z)$, or $2Z = A$, but is otherwise positive and increases with increasing departures from that condition. That is, the greater the departure from $Z = N$, the larger the mass or the smaller the binding energy.

The pairing term

The tendency of nuclei to have even Z and even N is accounted for by the *pairing* term. It decreases the mass if both Z and N are even, and increases it if both Z and N are odd. Thus, it maximizes the binding energy if both Z and N are even. A qualitative explanation of the origin of this term involves the quantum mechanical properties of indistinguishability of identical particles.

Here we show the liquid drop model. This model was proposed by Gamov, Bohr, and Wheeler. The binding energy is predicted as a function of A, Z and N by a semiempirical mass formula

$$E_B = a_V A - a_S A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_{SYM} \frac{1}{A}(Z-N)^2 - \frac{1}{A^{1/2}}\delta ,$$

or

$$\frac{E_B}{A} = \frac{1}{A}[a_V A - a_S A^{2/3} - a_c \frac{1}{A^{1/3}} Z(Z-1) - a_{sy} \frac{1}{A}(Z-N)^2 - \frac{1}{A^{1/2}}\delta], \qquad (3)$$

with

$a_V = 15.75$ MeV, $a_S = 17.8$ MeV,

$a_C = 0.711$ MeV, $a_{sy} = 23.7$ MeV, $\delta = 11.2$ MeV.

The value of δ is defined by

$$\delta = \begin{cases} -11.2 \text{ MeV} & \text{for even } Z \text{ and even } N \\ 0 & \text{for even } Z \text{ and odd } N, \text{ odd } Z \text{ and even } N \\ 11.2 \text{ MeV} & \text{for odd } Z \text{ and odd } N \end{cases}$$

where

$a_V A$: Volume term (dominant term),

$-a_S A^{2/3}$: Surface term (the interaction of pairs),

$-a_c \frac{1}{A^{1/3}} Z(Z-1)$: Coulomb term,

$-a_{sy} \frac{1}{A} (Z-N)^2$: Asymmetry term,

$-\frac{1}{A^{1/2}} \delta$: Pairing term.

((**Note**))

$a_V = 16.8$ MeV, $a_s = 18.3$ MeV
$a_c = 0.714$ MeV, $a_{sy} = 23.2$ MeV
$\delta = 33.5$ MeV (e-e)
$\delta > 0$ for even-even
$\delta < 0$ for odd-odd
$\delta = 0$ for odd-even, or even-odd

((**Das**))

$a_V = 15.6$ MeV, $a_s = 16.8$ MeV

$a_c = 0.72$ MeV, $a_{sy} = 23.3$ MeV

$\delta = 34.0$ MeV (*e*-*e*)

$\delta > 0$ for even-even

$\delta < 0$ for odd-odd

$\delta = 0$ for odd-even, or even-odd

((Evaluation of the energy separation))

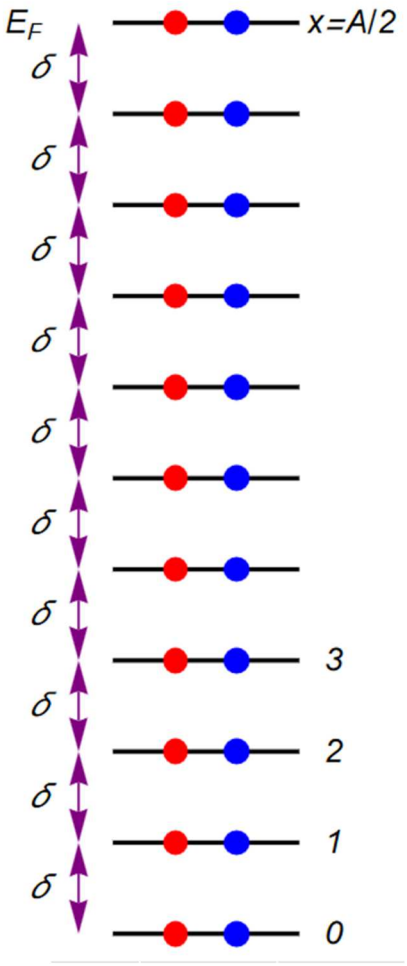


Fig.15 Spin-up state (red) and spin-down state (blue) for both protons and neutrons. $Z = N = \frac{A}{2}$. $x = \frac{A}{2}$ (Pauli's exclusion principle).

Suppose that the number of neutrons is equal to that of proton in a nucleus; $A/2$. The Fermi energy of this system is roughly evaluated as

$$E_F = x2\delta = 2\delta\frac{A}{2} = A\delta, \qquad \text{for } x = \frac{A}{2},$$

or

$$\delta = \frac{E_F}{A}.$$

((**Asymmetric term**))

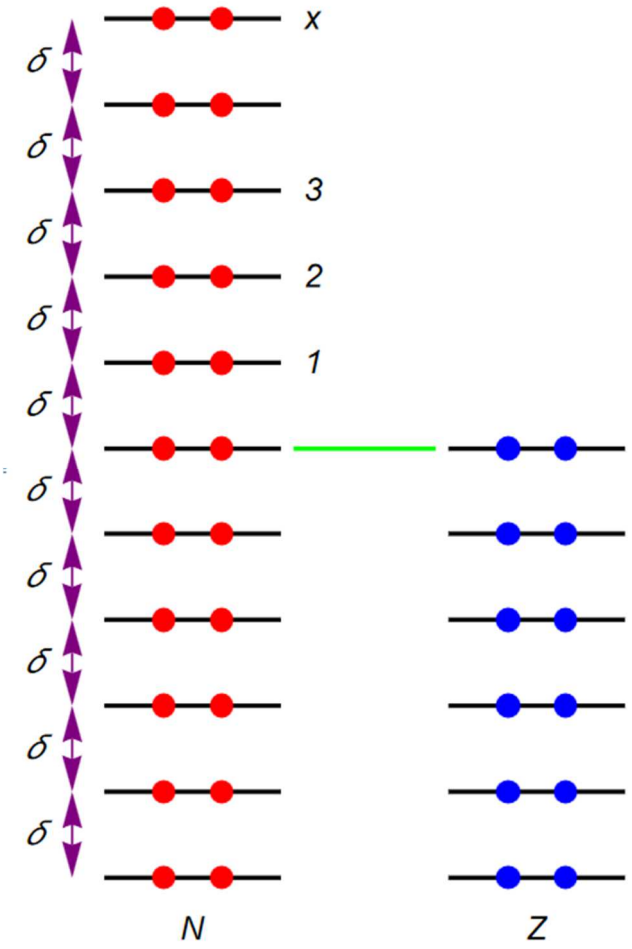


Fig.16 Antisymmetric effect. $N > Z$. There are two states for each state (Pauli exclusion principle).

$$
\begin{aligned}
\delta E_{tot} &= 1(2\delta) + 2(2\delta) + 3(2\delta) + ... + x(2\delta) \\
&= 2\delta(1 + 2 + 3 + ... + x) \\
&= 2\delta \frac{1}{2} x(x+1) \\
&= \delta x(x+1) \\
&\cong \delta x^2 \\
&= \frac{\delta}{4}(N-Z)^2
\end{aligned}
$$

where $x = \frac{1}{2}(N-Z)$.

So that, we have

$$\Delta E_{tot} = \frac{\delta}{4}(N-Z)^2 = \frac{E_F}{4A}(N-Z)^2 \approx \frac{1}{A}(N-Z)^2$$

((**Nuclear force**))

For the atomic mass ($A = N + Z$), the number of pairs for nucleus (neutron and proton) is evaluated as

$$_A P_2 = \frac{1}{2} A(A-1).$$

If the nuclear forces between any two nucleus were independently of the presence of other nucleus, the binding energy per nucleus is estimated as

$$E_B \propto \frac{A^2 - A}{2A} \approx A$$

((**Surface term**))

The surface term is assumed to be proportional to the surface area ($4\pi R^2$). Note that the nuclear radius R is approximated as

$$R = R_0 A^{1/3},$$

where $R_0 = 1.25$ fm. So that, the surface term is obtained as

$$4\pi R^2 = 4\pi R_0{}^2 A^{2/3}$$

((**Repulsive Coulomb energy from protons**))

The potential energy from the repulsive Coulomb interactions between protons is given by

$$\phi \simeq \frac{q^2}{4\pi\varepsilon_0 R} = \frac{K}{A^{1/3}}.$$

Noting the number of pairs from protons, the total potential energy is

$$\frac{Z(Z-1)}{2}\phi \simeq \frac{Z(Z-1)}{2}\frac{K}{A^{1/3}}.$$

19. The pairing effect. The linear relation $Z = N$: The Fermi gas model ((Townsend))

The energy-level diagram for ^{12}C shows why it is natural to have the number of protons equal to the number of neutrons in the nucleus, at least for small Z. The fourth term in the semiempirical mass formula, the asymmetry term, is a measure of the reduced binding energy as the nucleus moves away from the $Z = N$ condition. To see how the dependence of this term on Z and A arises, we can apply the Fermi gas that we used in treating elements in a conductor. We see that the Fermi energy is given by

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3},$$

where N is the number of identical fermions in the box of volume V. The total energy of the fermion is given by

$$E_{total} = \frac{3}{5}NE_F.$$

Since in the nucleus there are two fermion gases, one composed of protons and the other composed of neutrons, the total energy is given by

$$E_{total} = \frac{3}{5}Z\frac{\hbar^2}{2m}\left(\frac{3\pi^2 Z}{V}\right)^{2/3} + \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$
$$\propto Z\left(\frac{Z}{V}\right)^{2/3} + N\left(\frac{N}{V}\right)^{2/3}$$

where Z is the number of protons and here N is the number of neutrons. We have taken the mass m to be the same for the neutron and the proton. Since $V \sim A$ and $N = A - Z$, we see that

$$E_{total} \sim \frac{1}{A^{2/3}}[Z^{5/3} + (A-Z)^{5/3}].$$

For a fixed number of nucleons, that is, for fixed A, we can find the value of Z (and therefore N) that minimizes the overall energy by setting the derivative of E_{total} with respect to Z equal to zero.

$$\frac{\partial}{\partial Z}E_{total} \propto \frac{1}{A^{2/3}}[Z^{2/3} - (A-Z)^{2/3}] = 0,$$

which requires $Z = A - Z = N$, that is, the number of protons in the nucleus equal the number of neutrons for the minimum energy state.

Suppose that $Z = \frac{A}{2} + x$. So that, $A - Z = \frac{A}{2} - x$. The total energy can be described as a function of x as

$$\begin{aligned}
\delta E_{total} &\sim \frac{A}{2^{5/3}}[(1+\frac{2x}{A})^{5/3} + (1-\frac{2x}{A})^{5/3}] - \frac{A}{2^{5/3}}2 \\
&= \frac{A}{2^{5/3}}(\frac{40x^2}{9A^2} + \frac{160x^4}{243A} + ...) \\
&\simeq 1.3991\frac{1}{A}(Z - \frac{A}{2})^2 \\
&= 1.3991\frac{1}{4A}(Z-N)^2
\end{aligned}$$

which is the fourth term in the semi empirical mass formula.

20. Evidence of magic number from binding energy vs *Z* (or *N*)

((**Wikipedia**)) **Magic number**

https://www.energy.gov/science/np/articles/magic-gone-neutron-number-32

This theory of a nuclear shell model originates in the 1930's, but it was not until 1949 that German physicists Maria Goeppert Mayer and Johannes Hans Daniel Jensen et al. independently devised the correct formulation. The numbers of nucleons for which shells are filled are called magic numbers. Magic numbers of 2, 8, 20, 28, 50, 82 and 126 have been observed for neutrons, and the next number is predicted to be 184. Protons share the first six of these magic numbers, and 126 has been predicted as a magic proton number since the 1940s. Nuclides with a magic number of each—such as ^{16}O (Z = 8, N = 8), ^{132}Sn (Z = 50, N = 82), and ^{208}Pb (Z = 82, N = 126)—are

referred to as "doubly magic" and are more stable than nearby nuclides as a result of greater binding energies.

Here we consider the experimental results $E_B(N)/A$ derived from the NIST Data and the theoretical prediction from liquid drop model,

$$\frac{E_B(N)}{A} = \frac{1}{A}[Z\ m(^1H) + N\ m_n - M(atom)]c^2$$

(in units of MeV)

and

$$\frac{E_B}{A} = \frac{1}{N}(a_V A - a_S A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_{SYM}\frac{1}{A}(Z-N)^2 - \frac{1}{A^{1/2}}\delta(N,Z))$$

(in units of MeV)

The difference is defined by

$$\delta(\frac{E_B}{A}) = \left(\frac{E_B}{A}\right)_{\exp} - \left(\frac{E_B}{A}\right)_{theory}$$

with

$a_V = 15.75$ MeV, $a_S = 17.8$ MeV,

$a_C = 0.711$ MeV, $a_{SYM} = 23.7$ MeV, $\delta = 11.2$ MeV.

$$\delta = \begin{cases} -11.2 \text{ MeV} & \text{for even } Z \text{ and even } N \\ 0 & \text{for even } Z \text{ and odd } N, \text{ odd } Z \text{ and even } N \\ 11.2 \text{ MeV} & \text{for odd } Z \text{ and odd } N \end{cases}$$

(i) Plot of $\delta(\frac{E_B}{A}) = \left(\frac{E_B}{A}\right)_{\exp} - \left(\frac{E_B}{A}\right)_{theory}$ as a function of N.

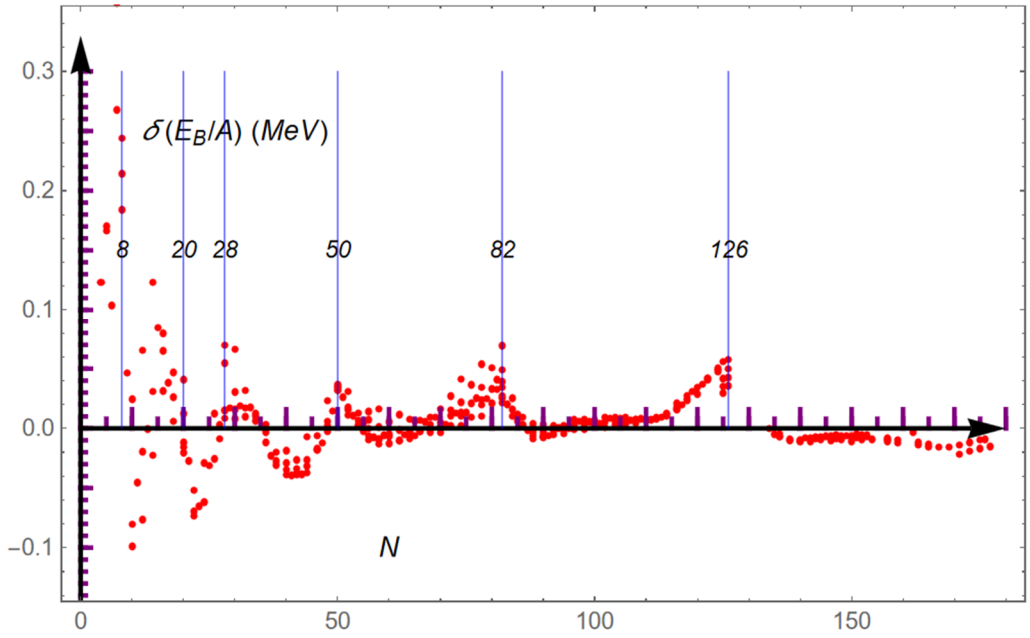


Fig.17 The difference between the binding energy of the elements and the prediction of the semi-empirical mass formula, as a function of the number of neutrons N in the nucleus. These data provide clear, evidence for the magic numbers 8, 28, 50, 82, and 126, for neutrons. Similar evidence shows that 20, 28, 50, and 82 are also magic numbers for protons. But there is no concrete evidence, pro or con, concerning 126 for protons since nuclei with such large Z values have not yet been detected. Local maxima appear at Z = 7, 14, 28, 50, 82, 126 (magic numbers are marked by red).

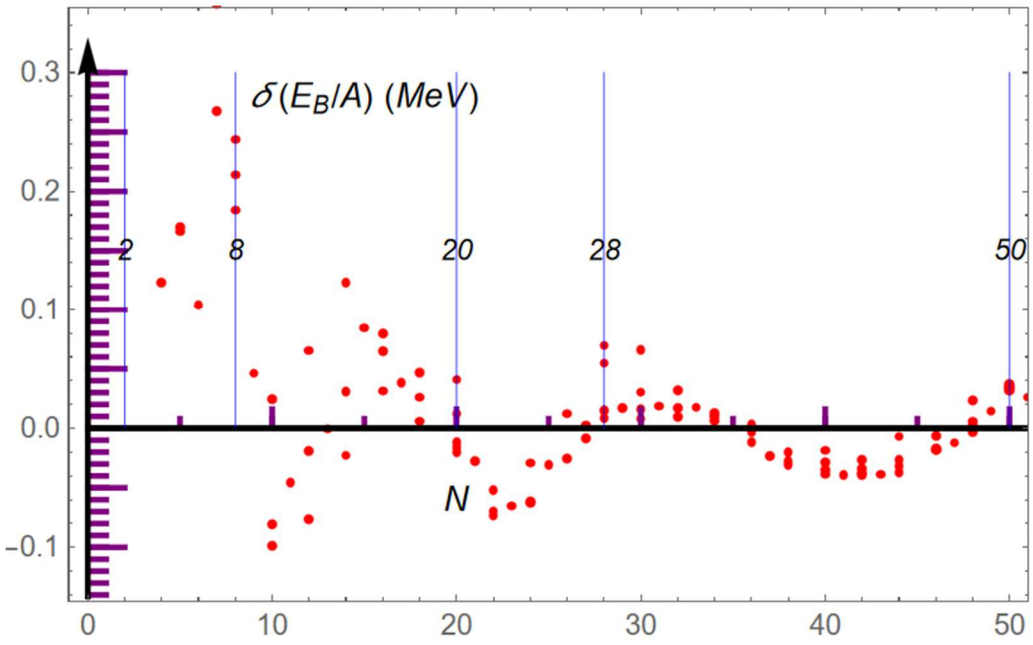


Fig.18 Plot of $\delta(\frac{E_B}{A})$ vs N. Local maxima appear at N = 6, 14, 20, 28, 54, 82. (magic numbers are marked by red).

(ii) Plot of $\delta(\frac{E_B}{N}) = \left(\frac{E_B}{N}\right)_{\exp} - \left(\frac{E_B}{N}\right)_{theory}$ as a function of Z.

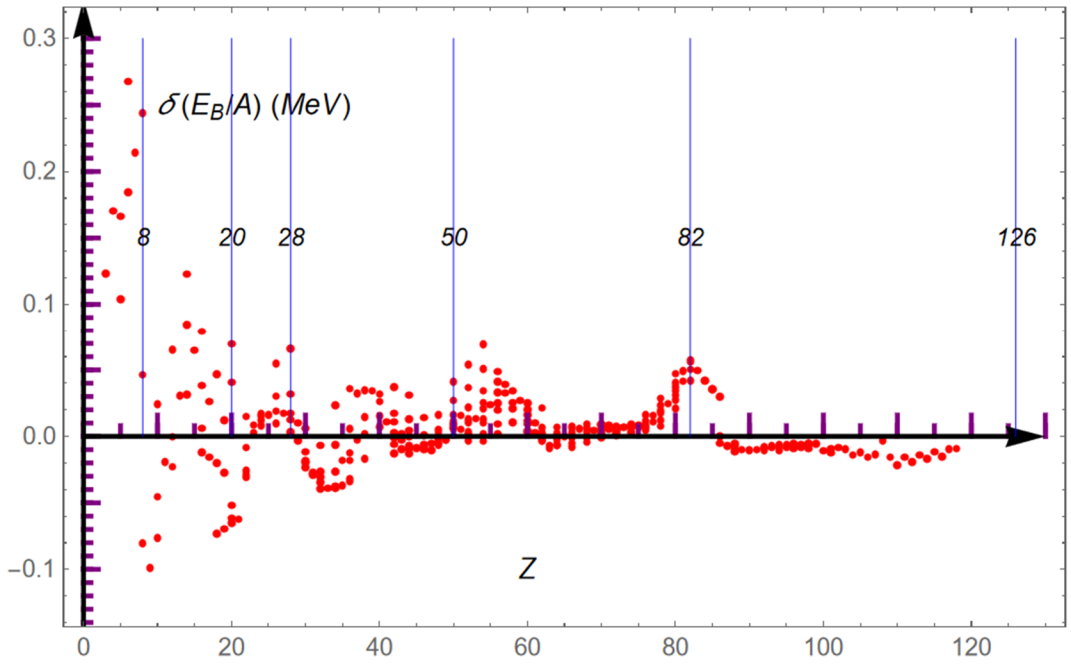


Fig.19 Plot of $\delta(\frac{E_B}{A})$ vs Z. Local maxima appear at Z = 6, 14, 20, 28, 54, 82. (magic numbers are marked by red).

21. Analysis based on the data from NIST

Table

Z	N	Number of elements
Even	Even:	188
Even	Odd	77
Odd	Even	65
Odd	Odd	23
Total		353

((**Note**))
Odd-Odd cases (23)

${}^{z}_{1}D_{1}$	${}^{6}_{3}Li_{3}$	${}^{10}_{5}B_{5}$	${}^{14}_{7}N_{7}$	${}^{40}_{19}K_{21}$	${}^{50}_{23}V_{27}$
${}^{98}_{43}Tc_{55}$	${}^{138}_{57}La_{81}$	${}^{176}_{71}Lu_{105}$	${}^{181}_{73}Ta_{108}$	${}^{210}_{85}At_{125}$	${}^{236}_{93}Np_{143}$
${}^{252}_{99}Es_{153}$	${}^{258}_{101}Md_{157}$	${}^{260}_{101}Md_{159}$	${}^{262}_{103}Lr_{159}$	${}^{268}_{105}Db_{163}$	${}^{272}_{107}Bh_{165}$

$^{276}_{109}\mathrm{Mt}_{167}$ $^{280}_{111}\mathrm{Rg}_{169}$ $^{284}_{113}\mathrm{Nh}_{171}$ $^{288}_{115}\mathrm{Mc}_{173}$ $^{292}_{117}\mathrm{Ts}_{175}$

22. Energy eigenvalue of particle in an infinite spherical well (spherical quantum dot)

In order to obtain more precise information on the nucleon energy levels, we need to solve the Schrodinger equation. Here we discuss the wave function of a particle in an infinite spherical well in three dimensions, with taking no account of spin orbit interaction. Note that the wave function is given by

$$\psi_{k\ell m}(r,\theta,\varphi) = \langle r,\theta,\varphi | k,l,m \rangle = \sqrt{\frac{2k^2}{\pi}} j_\ell(kr) Y_{\ell m}(\theta,\varphi),$$

with

$$j_\ell(ka) = 0,$$

where $E_k = \dfrac{\hbar^2 k^2}{2m}$ and $x = ka$.

The energy eigenvalue is dependent on the value of l. Suppose that $x(l, n_r)$ is the n_r-th zero points where the spherical Bessel function $j_l(x)$ becomes zero, where $n_r = 1, 2, 3, ...$ (integer). The energy eigenvalue is

$$E(l,n_r) = \frac{\hbar^2}{2m} \frac{[x(l,n_r)]^2}{a^2},$$

or

$$E_r = \frac{2m}{\hbar^2} a^2 E(l,n_r) = [x(l,n_r)]^2.$$

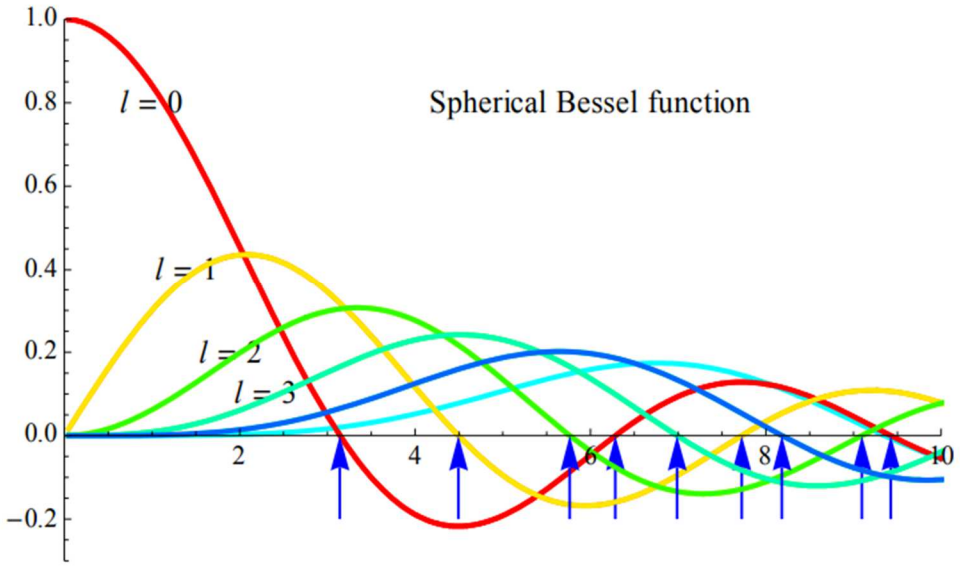


Fig.20 The plot of $j_\ell(x)$ as a function of x. The values of x when $j_\ell(x) = 0$ are denoted by the blue arrows. $x = ka$.

The energy levels of the infinite spherical well is shown for each l (= 0, 1, 2, 3, 4,...)

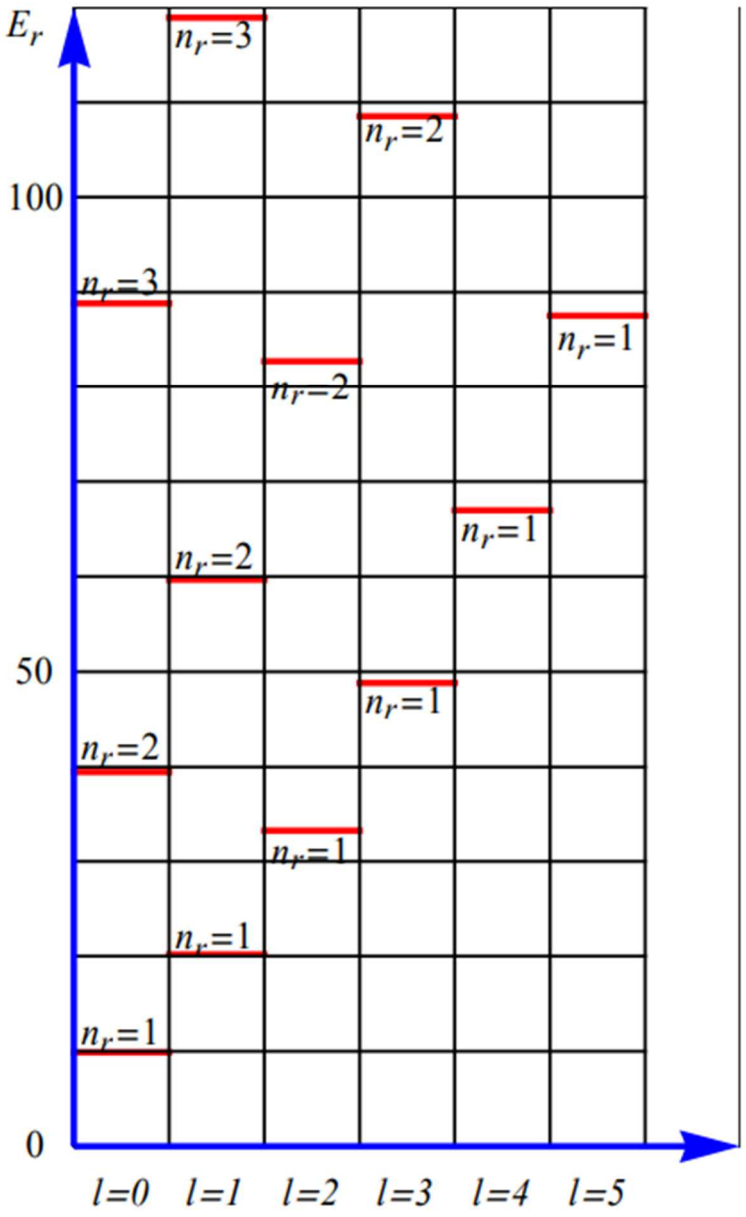


Fig.23 The energy levels of the infinite spherical well. $E_r = E(l,n_r)\frac{2m}{\hbar^2}a^2 = [x(l,n_r)]^2$.

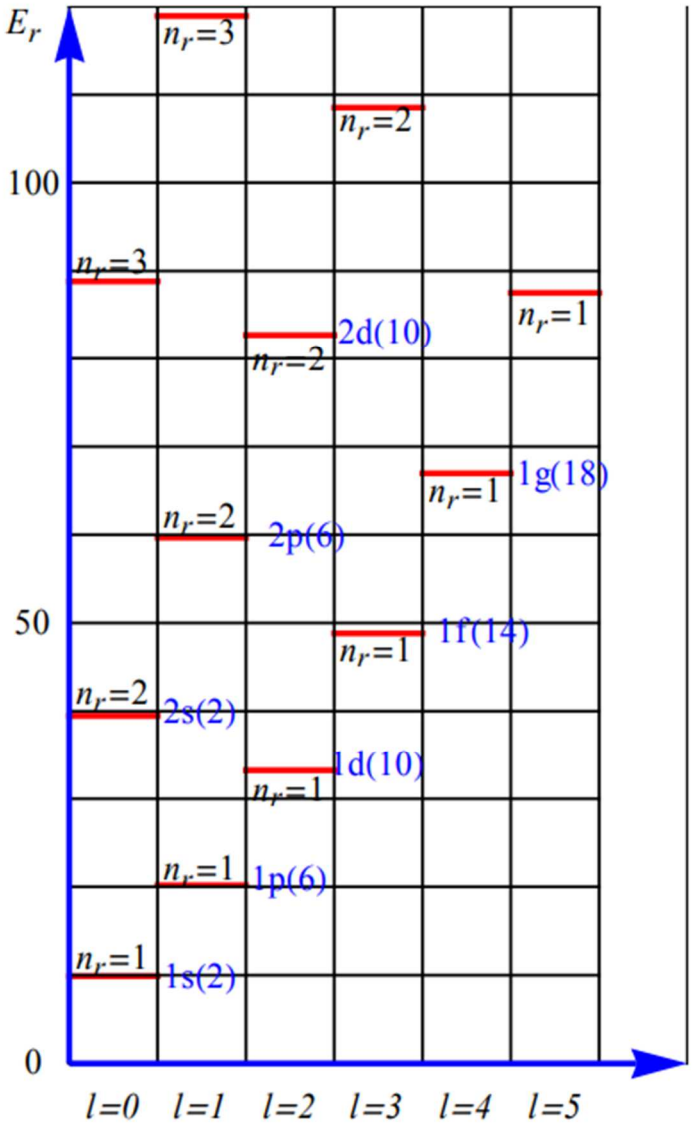


Fig.22 The energy levels of the infinite spherical well. $E_r = E(l,n_r)\frac{2m}{\hbar^2}a^2 = [x(l,n_r)]^2$.
In this mode; the magic number is obtained as 2, 8, 18, 20, 34, 40, 58, 68, ….

23. Effect of the spin-orbit interaction

Filling order of the levels through Z or for a single nucleon in a nucleus, including the spin-orbit energy proposed by Goeppert- Mayer and Jensen. The levels on the left are the corresponding levels in the absence of the spin-orbit energy. The numbers to the right of each level are the level's degeneracy and (in parentheses) the running total of protons or neutrons needed to fill through that level. On the far right are the closed-shell numbers, which agree perfectly with the observed magic numbers. The ordering of certain nearby levels is ambiguous (just as it is in atoms) and can be

different for protons and neutrons. Beyond 82, where the proton well is strongly distorted by Coulomb repulsion, the level orderings for protons and neutrons are significantly different.

(a) Schematic energy level diagram (Taylor et al.)

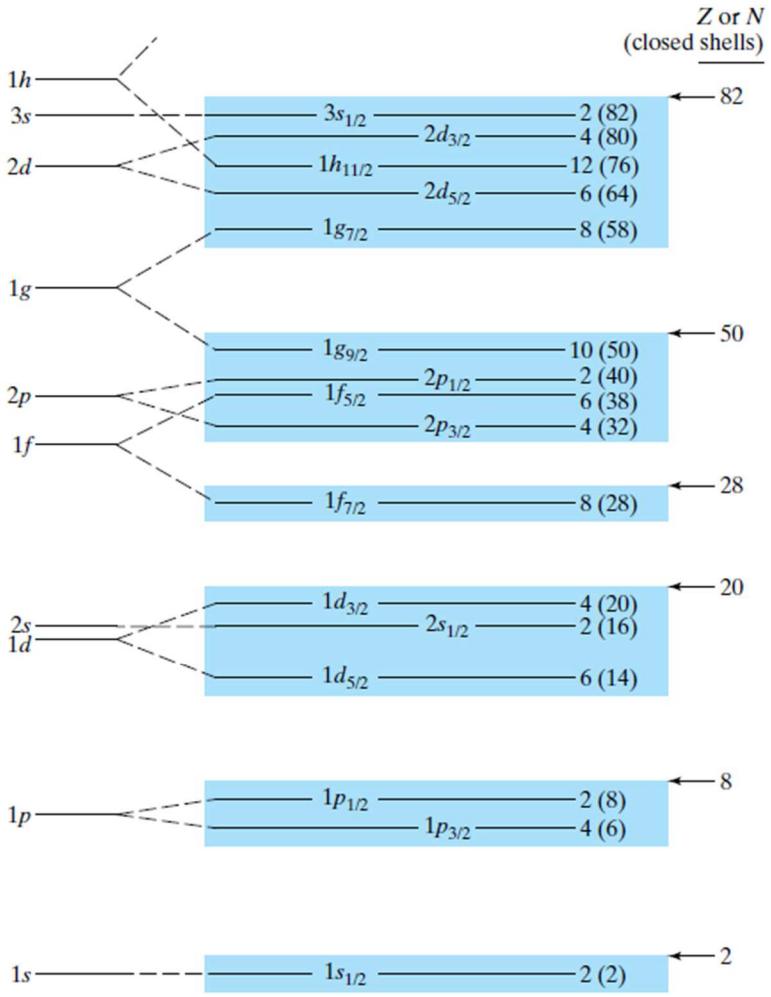


Fig.23 Filling order of the levels through Z or for a single nucleon in a nucleus, including the spin-orbit energy proposed by Goeppert- Mayer and Jensen. The levels on the left are the corresponding levels in the absence of the spin-orbit energy. The numbers to the right of each level are the level's degeneracy and (in parentheses) the running total of protons or neutrons needed to fill through that level. On the far right are the closed-shell numbers, which agree perfectly with the observed magic numbers. The ordering of certain nearby levels is ambiguous (just as it is in atoms) and can be different for protons and neutrons. Beyond 82, where the proton well is

strongly distorted by Coulomb repulsion, the level orderings for protons and neutrons are significantly different (**Taylor et al.**)

(b) Schematic energy level diagram (Sakurai and Napolitano)

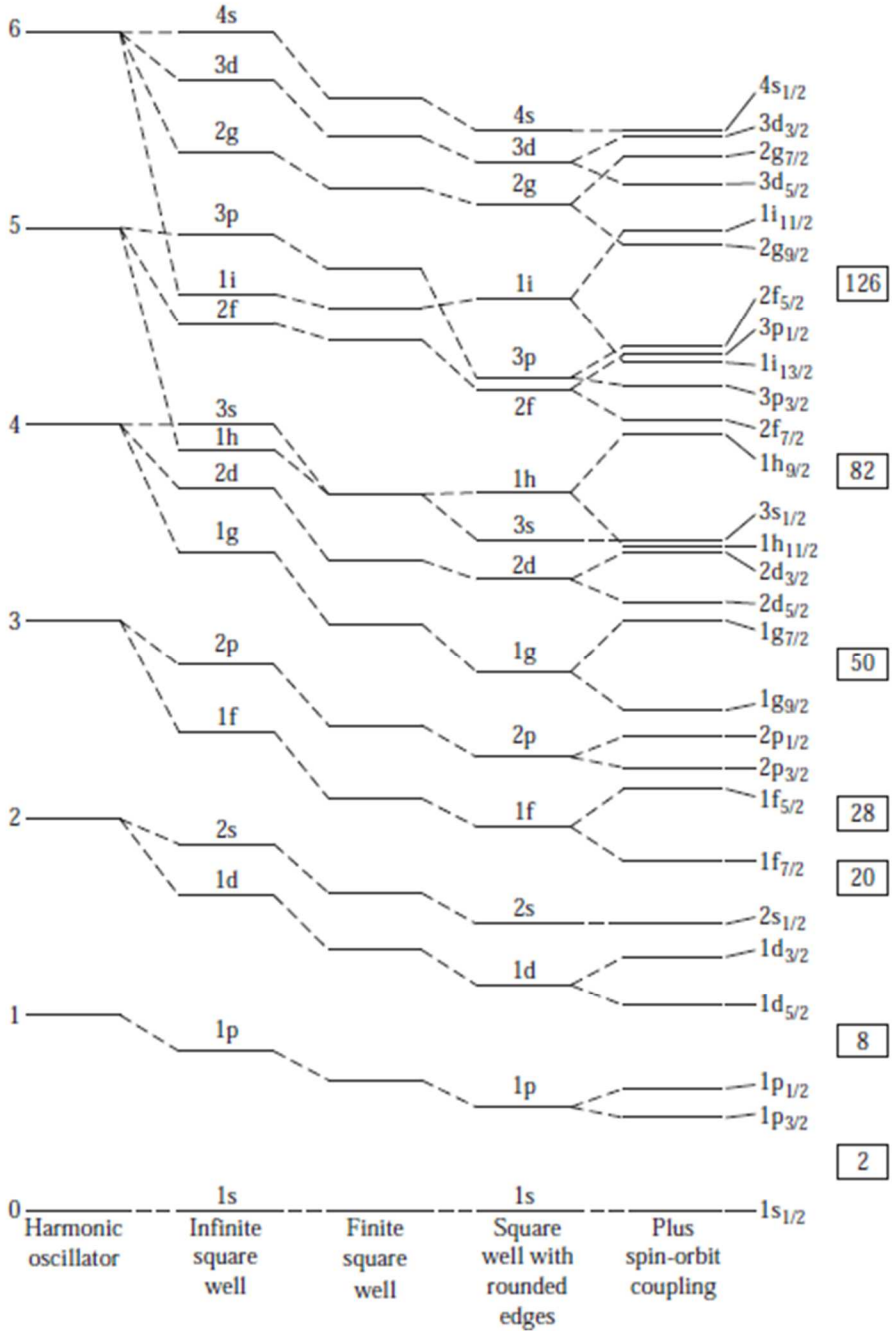


Fig.24 From Sakurai and Napolitano (Energy levels in the nuclear shell model (Haxerl et al. *Z.Phys*.**128**, 295 (1950).

24. Spin-orbit interaction: Clebsch-Gordan coefficients

The nuclear shell model for the magic number was first explained by Maria Goeppert Mayer and Johannes Hans Daniel Jensen et al. independently.

((**Model**)) Spin-orbit interaction

Let us now try making a nucleus by filling the energy levels with protons and neutrons. Since protons and neutrons are both spin 1/2 particles. If we neglect the Coulomb repulsion between the protons as a first approximation, the energy levels for protons and neutrons are the same. Suppose that we fill the energy levels with neutrons, we can put two neutrons in the $n = 1$, $l = 0$ (s) ground state. Here we use the Clebsch-Gordan co-efficient for the addition of angular momentum.

(i) 1s (j = ½) state

$$D_{l=0} \times D_{s=1/2} = D_{1/2} \qquad (j = 1/2 \text{ for 2 states})$$

with

$D_{1/2}$ ($j = \frac{1}{2}$, 2 states)

$$\left| j = \frac{1}{2}, m = \frac{1}{2} \right\rangle = \left| l = 0, m_l = 0 \right\rangle \otimes \left| +z \right\rangle,$$

$$\left| j = \frac{1}{2}, m = -\frac{1}{2} \right\rangle = \left| l = 0, m_l = 0 \right\rangle \otimes \left| -z \right\rangle.$$

(ii) 2p (j = 3/2) and 2p (j = ½)

$$D_{l=1} \times D_{s=1/2} = D_{3/2} + D_{1/2} \qquad (j = 3/2 \text{ for 4 states and } j = 1/2 \text{ for 2 states})$$

with

$D_{3/2}$ ($j = \frac{3}{2}$, 4 states)

$$\left| j = \frac{3}{2}, m = \frac{3}{2} \right\rangle = \left| l = 1, m_l = 1 \right\rangle \otimes \left| +z \right\rangle_s,$$

$$\left| j=\frac{3}{2}, m=\frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} \left| l=1, m_l=1 \right\rangle \otimes \left| -z \right\rangle_s + \sqrt{\frac{2}{3}} \left| l=1, m_l=0 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{3}{2}, m=-\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \left| l=1, m_l=0 \right\rangle \otimes \left| -z \right\rangle_s + \frac{1}{\sqrt{3}} \left| l=1, m_l=-1 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{3}{2}, m=-\frac{3}{2} \right\rangle = \left| l=1, m_l=-1 \right\rangle \otimes \left| -z \right\rangle_s .$$

$D_{1/2}$

$$\left| j=\frac{1}{2}, m=\frac{1}{2} \right\rangle = \sqrt{\frac{2}{3}} \left| l=1, m_l=1 \right\rangle \otimes \left| -z \right\rangle_s - \frac{1}{\sqrt{3}} \left| l=1, m_l=0 \right\rangle \otimes \left| +z \right\rangle_s .$$

$$\left| j=\frac{1}{2}, m=-\frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} \left| l=1, m_l=0 \right\rangle \otimes \left| -z \right\rangle_s - \sqrt{\frac{2}{3}} \left| l=1, m_l=-1 \right\rangle \otimes \left| +z \right\rangle_s .$$

(iii) 3d (j = 5/2) and 3d (j = 3/2)

$D_{l=2} \times D_{s=1/2} = D_{5/2} + D_{3/2}$ (j = 5/2 for 6 states, j = 3/2 for 4 states)

with

$D_{5/2}$ ($j = \frac{5}{2}$, 6 states)

$$\left| j=\frac{5}{2}, m=\frac{5}{2} \right\rangle = \left| l=2, m_l=2 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{5}{2}, m=\frac{3}{2} \right\rangle = \frac{1}{\sqrt{5}} \left| l=2, m_l=2 \right\rangle \otimes \left| -z \right\rangle_s + \frac{2}{\sqrt{5}} \left| l=2, m_l=1 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{5}{2}, m=\frac{1}{2} \right\rangle = \sqrt{\frac{2}{5}} \left| l=2, m_l=1 \right\rangle \otimes \left| -z \right\rangle_s + \sqrt{\frac{3}{5}} \left| l=2, m_l=0 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{5}{2}, m=-\frac{1}{2} \right\rangle = \sqrt{\frac{3}{5}} \left| l=2, m_l=0 \right\rangle \otimes \left| -z \right\rangle_s + \sqrt{\frac{2}{5}} \left| l=2, m_l=-1 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{5}{2}, m=-\frac{3}{2} \right\rangle = \frac{2}{\sqrt{5}} \left| l=2, m_l=-1 \right\rangle \otimes \left| -z \right\rangle_s + \frac{1}{\sqrt{5}} \left| l=2, m_l=-2 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{5}{2}, m=-\frac{5}{2} \right\rangle = \left| l=2, m_l=-2 \right\rangle \otimes \left| -z \right\rangle_s .$$

$D_{3/2}$ ($j=\frac{3}{2}$, 4 states)

$$\left| j=\frac{3}{2}, m=\frac{3}{2} \right\rangle = \frac{2}{\sqrt{5}} \left| l=2, m_l=2 \right\rangle \otimes \left| -z \right\rangle_s - \frac{1}{\sqrt{5}} \left| l=2, m_l=1 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{3}{2}, m=\frac{1}{2} \right\rangle = \sqrt{\frac{3}{5}} \left| l=2, m_l=1 \right\rangle \otimes \left| -z \right\rangle_s - \sqrt{\frac{2}{5}} \left| l=2, m_l=0 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$\left| j=\frac{3}{2}, m=-\frac{1}{2} \right\rangle = \sqrt{\frac{2}{5}} \left| l=2, m_l=0 \right\rangle \otimes \left| -z \right\rangle_s - \sqrt{\frac{3}{5}} \left| l=2, m_l=-1 \right\rangle \otimes \left| +z \right\rangle_s ,$$

$$, \left| j=\frac{3}{2}, m=-\frac{3}{2} \right\rangle = \frac{1}{\sqrt{5}} \left| l=2, m_l=-1 \right\rangle \otimes \left| -z \right\rangle_s - \frac{2}{\sqrt{5}} \left| l=2, m_l=-2 \right\rangle \otimes \left| +z \right\rangle_s$$

where

$$\left| +z \right\rangle_s = \left| s=\frac{1}{2}, m_s=\frac{1}{2} \right\rangle \qquad \text{(spin-up state for spin 1/2}$$

$$\left| -z \right\rangle_s = \left| s=\frac{1}{2}, m_s=-\frac{1}{2} \right\rangle \qquad \text{(spin-down state for spin 1/2)}$$

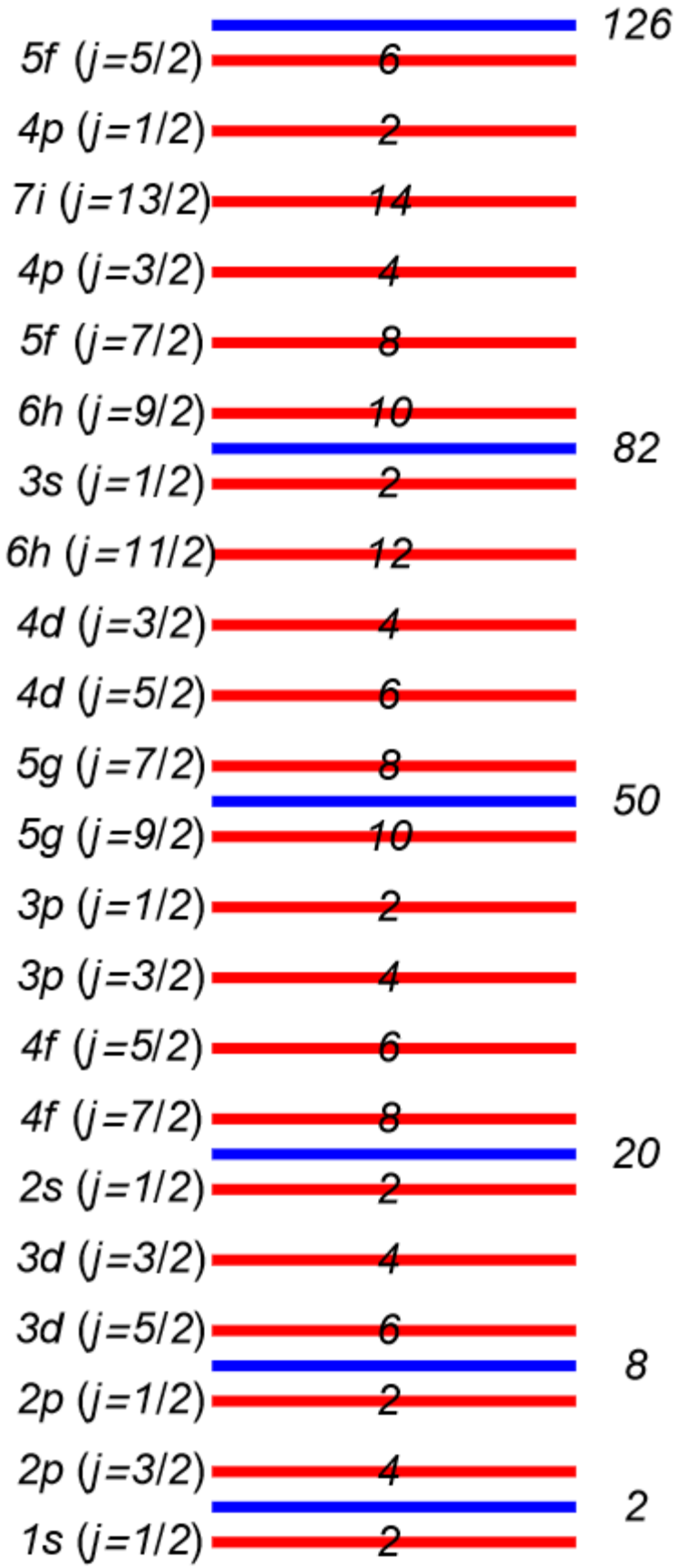

126
5f (j=5/2) 6
4p (j=1/2) 2
7i (j=13/2) 14
4p (j=3/2) 4
5f (j=7/2) 8
6h (j=9/2) 10
82
3s (j=1/2) 2
6h (j=11/2) 12
4d (j=3/2) 4
4d (j=5/2) 6
5g (j=7/2) 8
50
5g (j=9/2) 10
3p (j=1/2) 2
3p (j=3/2) 4
4f (j=5/2) 6
4f (j=7/2) 8
20
2s (j=1/2) 2
3d (j=3/2) 4
3d (j=5/2) 6
8
2p (j=1/2) 2
2p (j=3/2) 4
2
1s (j=1/2) 2

Fig.25 The schematic ordering of the energy levels in a variety of potential energy wells. (Adapted from B.T. Feld, Ann. Rev. Nuclear Sci. 2, 239 (1953), as produced by R.B. Leighton, Principles of Modern Physics, McGraw-Hill (New York, 1959). 2, 8, 20, 28, 50, 82, 126. Numbers such as 2, 4, 6, 8. 10 are degeneracy in states. The blue line does not denote the energy levels. It show the number if states below the corresponding to the blue line.

25. Energy levels of the ground state

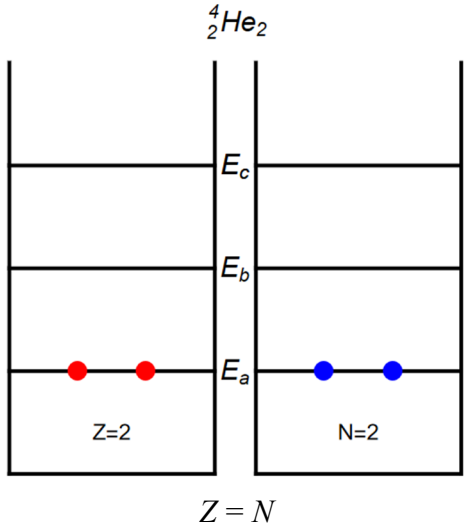


Fig.26 The energy levels of the ground state for $^4_2He_2$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2 and neutrons in the energy level 1s (j = 1/2) with degeneracy 2.

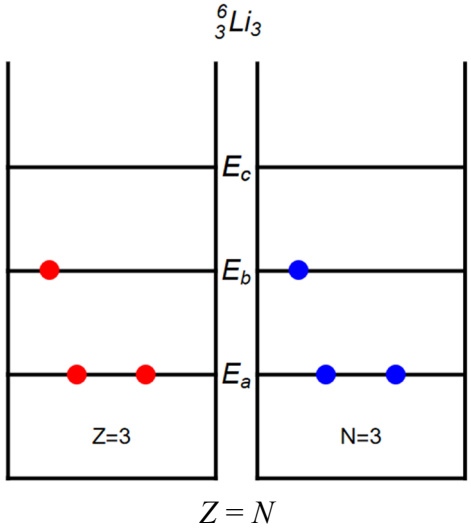


Fig.27 The energy levels of the ground state for ${}^{6}_{3}\mathrm{Li}_{3}$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2 and one proton in the energy level 2p (j = 3/2) with degeneracy 4. Two neutrons in the energy level 1s (j = 1/2) with degeneracy 2 and one neutron in the energy level 2p (j = 3/2) with degeneracy 4.

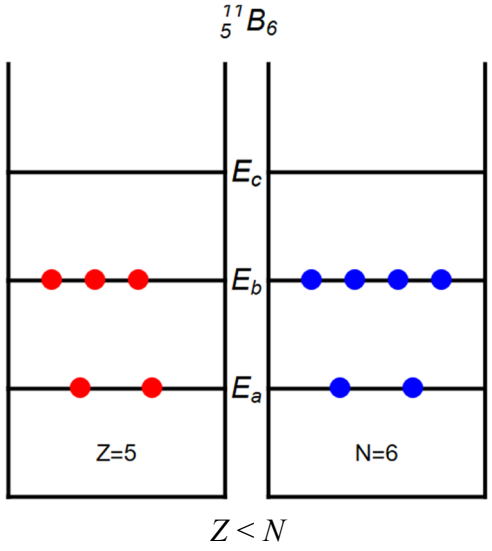


Fig.28 energy levels of the ground state for ${}^{11}_{5}\mathrm{B}_{6}$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2 and three proton in the energy level 2p (j = 3/2) with degeneracy 4. Two neutrons in the energy level 1s (j = 1/2) with degeneracy 2 and four neutrons in the energy level 2p (j = 3/2) with degeneracy 4.

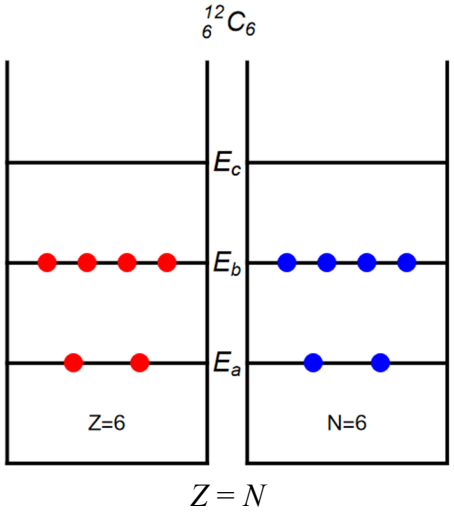
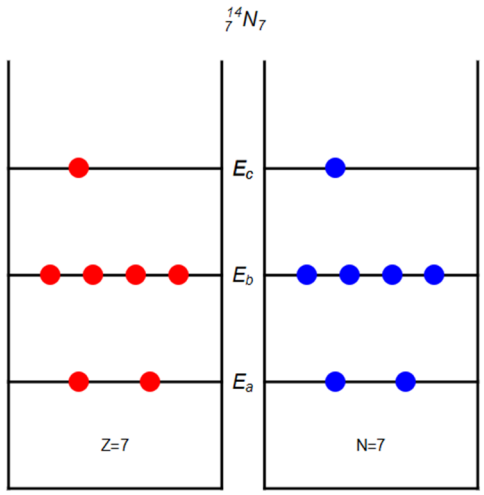
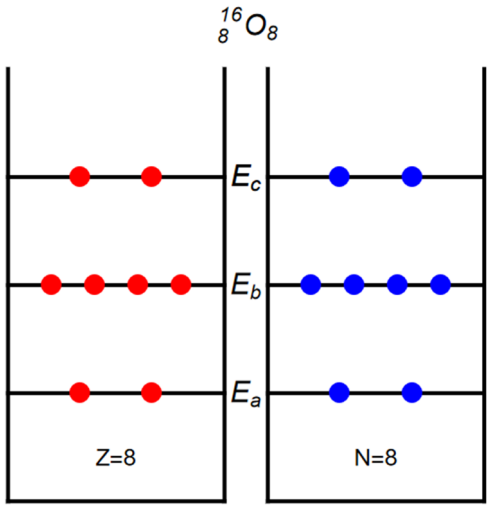


Fig.29 The energy levels of the ground state for $^{12}_{6}C_{6}$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2 and four proton in the energy level 2p (j = 3/2) with degeneracy 4. Two neutrons in the energy level 1s (j = 1/2) with degeneracy 2 and four neutrons in the energy level 2p (j = 3/2) with degeneracy 4.



$Z = N$

Fig.30 The energy levels of the ground state for $^{14}_{7}N_{7}$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2, four protons in the energy level 2p (j = 3/2) with degeneracy 4, and one proton in the energy level 2p (j = 1/2) with degeneracy 2. Two neutrons in the energy level 1s (j = 1/2) with degeneracy 2 and four neutrons in the energy level 2p (j = 3/2) with degeneracy 4, and one proton in the energy level 2p (j = 1/2) with degeneracy 2.



$Z = N$

Fig.31 The energy levels of the ground state for $^{16}_{8}O_{8}$. Two protons in the energy level 1s (j = 1/2) with degeneracy 2, four protons in the energy level 2p (j = 3/2) with degeneracy 4, and two proton in the energy level 2p (j = 1/2) with degeneracy 2. Two neutrons in the energy level 1s (j = 1/2) with degeneracy 2 and four neutrons in the energy level 2p (j = 3/2) with degeneracy 4, and two proton in the energy level 2p (j = 1/2) with degeneracy 2.

26. Some characteristic properties

The magic number is the number of neutrons (or protons) in a full shell with numbers 2, 8, 20, 28, 50, 82, and 126. Nuclei which have neutron number and proton numbers each equal to one of the magic numbers are called "doubly magic", and are especially stable against decay. The known doubly magic isotopes are,

He-4	A=4,	Z = 2,	N = 2
O-16	A = 16,	Z = 8,	N = 8
Ca-40	A = 40,	Z = 20,	N = 20
Ti-50	A = 50,	Z = 22,	N = 28
Sn-132	A = 132	Z = 50,	N = 82
Pb-208	A = 208,	Z = 82	N = 126

27. Conclusion

In spite of so many existing calculations in binding energy of atoms, here we calculate the binding energy of all the atoms based on the NIST atomic data by using Mathematica. It has been clearly confirmed from this work that the difference between the binding energy of atoms and the smoothed background derived from the liquid drop model, shows a drastic sharp increase at the magic numbers of either protons or neutrons. The origin of magic number anomaly is discussed in terms of Fermi gas model and nuclear shell model with spin-orbit interaction.

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APPENDIX

Table-1
Atomic weight and isotopic compositions for all elements

List of A, Z, N, M, $\left(\frac{E_B}{A}\right)_{\exp}$ and $\left(\frac{E_B}{A}\right)_{theory}$ for each element

Note that element "i" means I.

Element	A	Z	N	M	EB(ex)/A(MeV)	EB(th)/A(MeV)
H	1	1	0	1.00783	0.	-26.461
D	2	1	1	2.0141	1.11228	-2.61983
T	3	1	2	3.01605	2.82727	0.610509
He	3	2	1	3.01603	2.57268	0.117529
He	4	2	2	4.0026	7.07392	5.4888
Li	6	3	3	6.01512	5.33233	4.6053
Li	7	3	4	7.016	5.60644	5.48336
Be	9	4	5	9.01218	6.46267	6.29239
B	10	5	5	10.0129	6.47508	6.30875
B	11	5	6	11.0093	6.92772	6.82388
C	12	6	6	12.	7.68015	7.31289
C	13	6	7	13.0034	7.46985	7.20223
C	14	6	8	14.0032	7.52032	7.3361
N	14	7	7	14.0031	7.47562	7.11822
N	15	7	8	15.0001	7.69946	7.48535
O	16	8	8	15.9949	7.97621	7.73242
O	17	8	9	16.9991	7.75073	7.70437
O	18	8	10	17.9992	7.7671	7.84748
F	19	9	10	18.9984	7.77902	7.87776
Ne	20	10	10	19.9924	8.03224	8.00797
Ne	21	10	11	20.9938	7.97171	8.01728
Ne	22	10	12	21.9914	8.08047	8.15677

Na	23	11	12	22.9898	8.11149	8.13084
Mg	24	12	12	23.985	8.26071	8.1954
Mg	25	12	13	24.9858	8.2235	8.22397
Mg	26	12	14	25.9826	8.33387	8.35656
Al	27	13	14	26.9815	8.33155	8.30071
Si	28	14	14	27.9769	8.44775	8.32474
Si	29	14	15	28.9765	8.44864	8.36406
Si	30	14	16	29.9738	8.52066	8.48929
P	31	15	16	30.9738	8.48117	8.4163
S	32	16	16	31.9721	8.49313	8.41362
S	33	16	17	32.9715	8.49763	8.45922
S	34	16	18	33.9679	8.5835	8.57752
S	36	16	20	35.9671	8.57539	8.58724
Cl	35	17	18	34.9689	8.52028	8.49423
Cl	37	17	20	36.9659	8.57028	8.58581
Ar	36	18	18	35.9675	8.51991	8.4731
Ar	38	18	20	37.9627	8.61428	8.63449
Ar	40	18	22	39.9624	8.59526	8.66855
K	39	19	20	38.9637	8.55703	8.54486
K	40	19	21	39.964	8.53809	8.56545
K	41	19	22	40.9618	8.57607	8.64551
Ca	40	20	20	39.9626	8.55131	8.51054

Ca	40	20	20	39.9626	8.55131	8.51054
Ca	42	20	22	41.9586	8.61656	8.66858
Ca	43	20	23	42.9588	8.60067	8.66599
Ca	44	20	24	43.9555	8.65818	8.71963
Ca	46	20	26	45.9537	8.66896	8.68933
Ca	48	20	28	47.9525	8.66669	8.59716
Sc	45	21	24	44.9559	8.61892	8.68134
Ti	46	22	24	45.9526	8.65644	8.68538
Ti	47	22	25	46.9518	8.66121	8.69223
Ti	48	22	26	47.9479	8.72299	8.74854
Ti	49	22	27	48.9479	8.71114	8.71972
V	50	22	28	49.9448	8.7557	8.74054
V	50	23	27	49.9472	8.69592	8.69309
V	51	23	28	50.944	8.7421	8.73365
Cr	50	24	26	49.946	8.70103	8.68879
Cr	52	24	28	51.9405	8.77597	8.7607
Cr	53	24	29	52.9406	8.76018	8.74324
Cr	54	24	30	53.9389	8.77794	8.76987
Mn	55	25	30	54.938	8.76501	8.74901
Fe	54	26	28	53.9396	8.73637	8.6816
Fe	56	26	30	55.9349	8.79034	8.7599
Fe	57	26	31	56.9354	8.77027	8.75142
Fe	58	26	32	57.9333	8.79224	8.78247

Co	59	27	32	58.9332	8.76803	8.75084
Ni	58	28	30	57.9353	8.73205	8.6659
Ni	60	28	32	59.9308	8.78077	8.74889
Ni	61	28	33	60.9311	8.76502	8.7476
Ni	62	28	34	61.9283	8.79455	8.78205
Ni	64	28	36	63.928	8.77746	8.77413
Cu	63	29	34	62.9296	8.75213	8.74199
Cu	65	29	36	64.9278	8.75709	8.76012
Zn	64	30	34	63.9291	8.7359	8.72969
Zn	66	30	36	65.926	8.75963	8.7713
Zn	67	30	37	66.9271	8.73415	8.75723
Zn	68	30	38	67.9248	8.75568	8.77546
Zn	70	30	40	69.9253	8.72981	8.74831
Ga	69	31	38	68.9256	8.72458	8.75196
Ga	71	31	40	70.9247	8.71761	8.74645
Ge	70	32	38	69.9242	8.7217	8.75225
Ge	72	32	40	71.9221	8.73175	8.76639
Ge	73	32	41	72.9235	8.70505	8.74434
Ge	74	32	42	73.9212	8.7252	8.75155
Ge	76	32	44	75.9214	8.70524	8.71218
As	75	33	42	74.9216	8.70088	8.73985
Se	74	34	40	73.9225	8.68772	8.72645

Se	76	34	42	75.9192	8.71148	8.7489
Se	77	34	43	76.9199	8.69469	8.7332
Se	78	34	44	77.9173	8.71781	8.74445
Se	80	34	46	79.9165	8.71081	8.71702
Se	82	34	48	81.9167	8.6932	8.66994
Br	79	35	44	78.9183	8.6876	8.72461
Br	81	35	46	80.9163	8.69593	8.71379
Kr	78	36	42	77.9204	8.66126	8.69508
Kr	80	36	44	79.9164	8.69293	8.7245
Kr	82	36	46	81.9135	8.71066	8.7289
Kr	83	36	47	82.9141	8.69572	8.708
Kr	84	36	48	83.9115	8.71745	8.71173
Kr	86	36	50	85.9106	8.71203	8.67594
Rb	85	37	48	84.9118	8.69744	8.70027
Rb	87	37	50	86.9092	8.71099	8.6788
Sr	84	38	46	83.9134	8.67751	8.69439
Sr	86	38	48	85.9093	8.70846	8.70637
Sr	87	38	49	86.9089	8.70524	8.69077
Sr	88	38	50	87.9056	8.73259	8.69806
Y	89	39	50	88.9058	8.71399	8.67965
Zr	90	40	50	89.9047	8.70998	8.67801
Zr	91	40	51	90.9056	8.69332	8.66704
Zr	92	40	52	91.905	8.69269	8.67742

Zr	94	40	54	93.9063	8.66682	8.66011
Zr	96	40	56	95.9083	8.63539	8.62816
Nb	93	41	52	92.9064	8.66419	8.65307
Mo	92	42	50	91.9068	8.65772	8.62067
Mo	94	42	52	93.9051	8.66232	8.64478
Mo	95	42	53	94.9058	8.64871	8.63786
Mo	96	42	54	95.9047	8.65398	8.65091
Mo	97	42	55	96.906	8.63508	8.63619
Mo	98	42	56	97.9054	8.63516	8.64119
Mo	100	42	58	99.9075	8.60463	8.61749
Tc	97	43	54	96.9064	8.62367	8.62149
Tc	98	43	55	97.9072	8.60999	8.61266
Tc	99	43	56	98.9063	8.6136	8.62306
Ru	96	44	52	95.9076	8.6094	8.57843
Ru	98	44	54	97.9053	8.62031	8.60743
Ru	99	44	55	98.9059	8.60868	8.60407
Ru	100	44	56	99.9042	8.61932	8.61946
Ru	101	44	57	100.906	8.60133	8.60868
Ru	102	44	58	101.904	8.60739	8.61643
Rh	104	44	60	103.905	8.58738	8.60006
Rh	103	45	58	102.905	8.58416	8.59314
Pd	102	46	56	101.906	8.58056	8.56658
Pd	104	46	58	103.904	8.58485	8.58379

Pd	105	46	59	104.905	8.57065	8.57653
Pd	106	46	60	105.903	8.57999	8.58671
Pd	108	46	62	107.904	8.56703	8.57686
Pd	110	46	64	109.905	8.54717	8.55563
Ag	107	47	60	106.905	8.5539	8.55892
Ag	109	47	62	108.905	8.54792	8.55842
Cd	106	48	58	105.906	8.53905	8.52275
Cd	108	48	60	107.904	8.55002	8.54454
Cd	110	48	62	109.903	8.55128	8.55273
Cd	111	48	63	110.904	8.53709	8.5426
Cd	112	48	64	111.903	8.54474	8.54873
Cd	113	48	65	112.904	8.52699	8.53323
Cd	114	48	66	113.903	8.53151	8.53379
Cd	116	48	68	115.905	8.51235	8.50904
In	113	49	64	112.904	8.52292	8.52577
In	115	49	66	114.904	8.51655	8.5192
Sn	112	50	62	111.905	8.51363	8.50221
Sn	114	50	64	113.903	8.52255	8.51511
Sn	115	50	65	114.903	8.51407	8.50802
Sn	116	50	66	115.902	8.52312	8.51634
Sn	117	50	67	116.903	8.50961	8.5041
Sn	118	50	68	117.902	8.51654	8.50707
Sn	119	50	69	118.903	8.49945	8.49019

Sn	120	50	70	119.902	8.5045	8.48833
Sn	122	50	72	121.903	8.48791	8.46106
Sn	124	50	74	123.905	8.46742	8.42612
Sb	121	51	70	120.904	8.48205	8.4768
Sb	123	51	72	122.904	8.47234	8.45675
Te	120	52	68	119.904	8.47704	8.48027
Te	122	52	70	121.903	8.47814	8.47612
Te	123	52	71	122.904	8.46555	8.46235
Te	124	52	72	123.903	8.47328	8.46283
Te	125	52	73	124.904	8.45805	8.44502
Te	126	52	74	125.903	8.46325	8.44127
Te	128	52	76	127.904	8.44875	8.41224
Te	130	52	78	129.906	8.43033	8.37645
I	127	53	74	126.904	8.44549	8.43218
Xe	124	54	70	123.906	8.43756	8.44102
Xe	126	54	72	125.904	8.44353	8.4415
Xe	128	54	74	127.904	8.4433	8.43317
Xe	129	54	75	128.905	8.43139	8.41831
Xe	130	54	76	129.904	8.43773	8.41682
Xe	131	54	77	130.905	8.42374	8.39841
Xe	132	54	78	131.904	8.42762	8.3932
Xe	134	54	80	133.905	8.41369	8.36298
Xe	136	54	82	135.907	8.39619	8.32676

Cs	133	55	78	132.905	8.40998	8.38604
Ba	130	56	74	129.906	8.40555	8.40368
Ba	132	56	76	131.905	8.40938	8.39986
Ba	134	56	78	133.905	8.40817	8.38828
Ba	135	56	79	134.906	8.39754	8.37266
Ba	136	56	80	135.905	8.40276	8.36963
Ba	137	56	81	136.906	8.39183	8.35086
Ba	138	56	82	137.905	8.39342	8.34452
La	138	57	81	137.907	8.37515	8.34249
La	139	57	82	138.906	8.37804	8.33889
Ce	136	58	78	135.907	8.37376	8.36336
Ce	138	58	80	137.906	8.37706	8.35615
Ce	140	58	82	139.905	8.37634	8.34205
Ce	142	58	84	141.909	8.34707	8.32167
Pr	141	59	82	140.908	8.354	8.32701
Nd	142	60	82	141.908	8.34603	8.32084
Nd	143	60	83	142.91	8.33049	8.31014
Nd	144	60	84	143.91	8.32692	8.31095
Nd	145	60	85	144.913	8.30919	8.29725
Nd	146	60	86	145.913	8.30409	8.29492
Nd	148	60	88	147.917	8.27718	8.27326
Nd	150	60	90	149.921	8.24957	8.24644
Pm	145	61	84	144.913	8.30266	8.29253
Pm	147	61	86	146.915	8.28437	8.28165

Sm	144	62	82	143.912	8.30368	8.28225
Sm	147	62	85	146.915	8.28057	8.27422
Sm	148	62	86	147.915	8.27963	8.27671
Sm	149	62	87	148.917	8.26346	8.26527
Sm	150	62	88	149.917	8.26162	8.26471
Sm	152	62	90	151.92	8.24406	8.24721
Sm	154	62	92	153.922	8.22683	8.22464
Eu	151	63	88	150.92	8.23929	8.24816
Eu	153	63	90	152.921	8.22869	8.2354
Gd	152	64	88	151.92	8.2334	8.2397
Gd	154	64	90	153.921	8.22479	8.23143
Gd	155	64	91	154.923	8.21325	8.21945
Gd	156	64	92	155.922	8.21532	8.21779
Gd	157	64	93	156.924	8.2035	8.20338
Gd	158	64	94	157.924	8.20182	8.19918
Gd	160	64	96	159.927	8.18301	8.17598
Tb	159	65	94	158.925	8.1888	8.18856
Dy	156	66	90	155.924	8.19243	8.20022
Dy	158	66	92	157.924	8.19012	8.19541
Dy	160	66	94	159.925	8.18405	8.18535
Dy	161	66	95	160.927	8.1733	8.17299
Dy	162	66	96	161.927	8.17345	8.17042
Dy	163	66	97	162.929	8.16178	8.15587
Dy	164	66	98	163.929	8.15871	8.151

Er	162	68	94	161.929	8.15239	8.15694
Er	164	68	96	163.929	8.14901	8.15021
Er	166	68	98	165.93	8.14196	8.13872
Er	167	68	99	166.932	8.13174	8.12611
Er	168	68	100	167.932	8.1296	8.12282
Er	170	68	102	169.935	8.11196	8.10281
Tm	169	69	100	168.934	8.11448	8.11017
Yb	168	70	98	167.934	8.1119	8.11266
Yb	170	70	100	169.935	8.10662	8.10437
Yb	171	70	101	170.936	8.09789	8.09358
Yb	172	70	102	171.936	8.09743	8.09176
Yb	173	70	103	172.938	8.08743	8.079
Yb	174	70	104	173.939	8.08386	8.07512
Yb	176	70	106	175.943	8.06408	8.05473
Lu	175	71	104	174.941	8.06915	8.06337
Lu	176	71	105	175.943	8.05903	8.05061
Hf	174	72	102	173.94	8.06855	8.06765
Hf	176	72	104	175.941	8.06137	8.05811
Hf	177	72	105	176.943	8.05185	8.04709
Hf	178	72	106	177.944	8.04946	8.04462
Hf	179	72	107	178.946	8.03856	8.0318
Hf	180	72	108	179.947	8.03495	8.02744
Ta	180	73	107	179.947	8.0259	8.01914
Ta	181	73	108	180.948	8.02342	8.01643

W	180	74	106	179.947	8.02546	8.02213
W	182	74	108	181.948	8.01832	8.0116
W	183	74	109	182.95	8.00833	8.00043
W	184	74	110	183.951	8.00509	7.99743
W	186	74	112	185.954	7.98862	7.97988
Re	185	75	110	184.953	7.99103	7.98383
Re	187	75	112	186.956	7.97796	7.96947
Os	184	76	108	183.952	7.9887	7.98402
Os	186	76	110	185.954	7.98285	7.97627
Os	187	76	111	186.956	7.97379	7.96666
Os	189	76	113	188.958	7.96301	7.95371
Os	190	76	114	189.958	7.96211	7.95028
Os	192	76	116	191.961	7.94854	7.93249
Ir	191	77	114	190.961	7.94813	7.93737
Ir	193	77	116	192.963	7.93815	7.92257
Pt	190	78	112	189.96	7.94659	7.93887
Pt	192	78	114	191.961	7.94251	7.93022
Pt	194	78	116	193.963	7.93596	7.91828
Pt	195	78	117	194.965	7.92657	7.90703
Pt	196	78	118	195.965	7.92654	7.90325
Pt	198	78	120	197.968	7.91416	7.88532
Au	197	79	118	196.967	7.91566	7.89091
Hg	196	80	116	195.966	7.91437	7.89346

Hg	198	80	118	197.967	7.91156	7.88408
Hg	199	80	119	198.968	7.90528	7.87425
Hg	200	80	120	199.968	7.9059	7.87167
Hg	201	80	121	200.97	7.89756	7.86044
Hg	202	80	122	201.971	7.89685	7.85638
Hg	204	80	124	203.973	7.88555	7.83842
Tl	203	81	122	202.972	7.88605	7.84453
Tl	205	81	124	204.974	7.87839	7.82921
Pb	204	82	122	203.973	7.87993	7.83794
Pb	206	82	124	205.974	7.87536	7.82517
Pb	207	82	125	206.976	7.86987	7.81401
Pb	208	82	126	207.977	7.86745	7.80973
Bi	209	83	126	208.98	7.84799	7.79827
Po	209	84	125	208.982	7.83519	7.79369
Po	210	84	126	209.983	7.83435	7.79186
At	210	85	125	209.987	7.81166	7.77637
At	211	85	126	210.987	7.81135	7.77579
Rn	211	86	125	210.991	7.79394	7.76417
Rn	220	86	134	220.011	7.71725	7.71719
Rn	222	86	136	222.018	7.69449	7.6995
Fr	223	87	136	223.02	7.68366	7.69087
Rn	223	88	135	223.019	7.6853	7.69036

Ra	224	88	136	224.02	7.67992	7.6868
Ra	226	88	138	226.025	7.66196	7.67135
Ra	228	88	140	228.031	7.64242	7.65384
Ac	227	89	138	227.028	7.6507	7.66067
Th	230	90	140	230.033	7.63099	7.64116
Th	232	90	142	232.038	7.61503	7.6258
Pa	231	91	140	231.036	7.61842	7.62857
U	233	92	141	233.04	7.60395	7.61184
U	234	92	142	234.041	7.60071	7.60909
U	235	92	143	235.044	7.59091	7.59957
U	236	92	144	236.046	7.58648	7.59578
U	238	92	146	238.051	7.57012	7.58055
Np	236	93	143	236.047	7.57921	7.58622
Np	237	93	144	237.048	7.57498	7.58341
Pu	238	94	144	238.05	7.56835	7.57528
Pu	239	94	145	239.052	7.56031	7.56682
Pu	240	94	146	240.054	7.55602	7.56393
Pu	241	94	147	241.057	7.54643	7.55454
Pu	242	94	148	242.059	7.54132	7.55068
Pu	244	94	150	244.064	7.52481	7.53562
Am	241	95	146	241.057	7.54327	7.54982
Cm	243	95	148	243.061	7.53017	7.53849
Cm	243	96	147	243.061	7.52692	7.53242
Cm	244	96	148	244.063	7.52395	7.53039

Cm	245	96	149	245.065	7.51577	7.52202
Cm	246	96	150	246.067	7.51147	7.51902
Cm	247	96	151	247.07	7.50193	7.50978
Cm	248	96	152	248.072	7.49673	7.50587
Bk	247	97	150	247.07	7.49894	7.50514
Bk	249	97	152	249.075	7.48602	7.49382
Cf	249	98	151	249.075	7.48289	7.48788
Cf	250	98	152	250.076	7.47995	7.48572
Cf	251	98	153	251.08	7.4705	7.47745
Cf	252	98	154	252.082	7.46535	7.47438
Es	252	99	153	252.083	7.45724	7.46289
Fm	257	100	157	257.095	7.42219	7.43313
Md	258	101	157	258.098	7.40967	7.41877
Md	260	101	159	260.104	7.39606	7.4084
No	259	102	157	259.101	7.39986	7.40819
Lr	262	103	159	262.11	7.37406	7.38359
Rf	267	104	163	267.122	7.34169	7.35557
Db	268	105	163	268.126	7.32801	7.33993
Sg	271	106	165	271.134	7.30496	7.32068
Bh	272	107	165	272.138	7.29007	7.30367
Hs	270	108	162	270.134	7.29508	7.29827
Mt	276	109	167	276.152	7.25074	7.26618
Ds	281	110	171	281.165	7.21972	7.24129
Rg	280	111	169	280.165	7.21179	7.22755
Cn	285	112	173	285.177	7.18497	7.20415
Nh	284	113	171	284.179	7.17381	7.18784
FI	289	114	175	289.19	7.14896	7.16589
Mc	288	115	173	288.193	7.13553	7.14714
Lv	293	116	177	293.204	7.11148	7.12659
Ts	292	117	175	292.207	7.09604	7.1055
Og	294	118	176	294.214	7.07954	7.08879