Finite spherical square well potential: deuteron, with the use of Mathematica

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Here we discuss the bound states of deuteron in a three-dimensional (3D) spherical (attractive) square well potential with radius (a) and a potential depth (V_0). This problem is of interest because it is mathematically straightforward and approximate a number of real physical situation. This problem is so familiar and has been used as exercises in standard textbooks of quantum mechanics. In spite of this fact, we present our results of detailed calculations using the Mathematica (ContourPlot) in the this note. It may be useful to understanding the role of angular momentum and potential depth in this model.

The energy eigenvalues of the system are determined as a function of the potential depth and angular momentum $l = 0$, 1, 2, 3, 4…. The nature of the wave functions, which depend on the depth of potential well and the orbital angular momentum, is also examined. To this end, we use the ContourPlot (Mathematica) for the boundary condition of the wave functions at the radius a. The boundary condition is given by

$$\frac{x j_{l-1}(x)}{j_l(x)} = \frac{iy h_{l-1}^{(1)}(iy)}{h_l^{(1)}(iy)} \quad \text{with} \quad x = \sqrt{r_0^{\,2} - y^2}$$

for $l = 0$, 1, 2, 3,…. (**L.I. Schiff**). This condition for $l = 0$. reduces to a simple form

$$y = -x \cot x\,.$$

Notations of this boundary condition will be discussed below. For the first time, we found such a form of the boundary condition, in a book, Problems and Solutions of L.I. Schiff Quantum mechanics 3rd edition (Yoshioka, 1983) [in Japanese].

Typical result derived from this method, is shown below. These results are useful to the understanding of the phase shift analysis (of scattering) such as **Levinson**'s theorem.

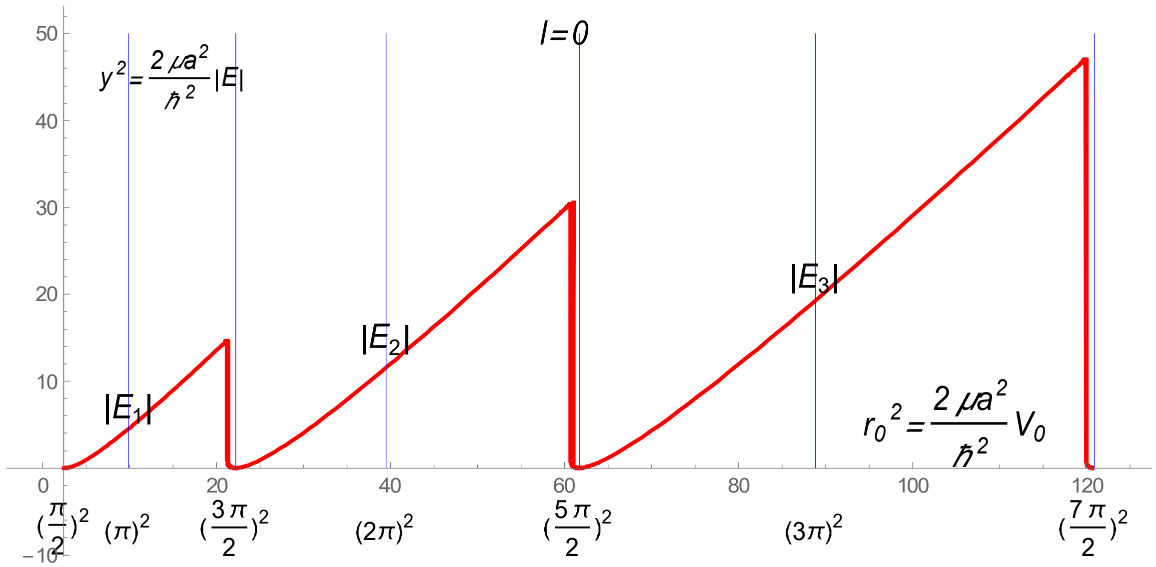


Fig.1 $l = 0$. Typical example. $|E|$: energy eigenvalue of the bound states which is the closest to $E = 0$ among bound states, but $E<0$. V_0 is a potential depth and a is a radius. The ground state with energy E_1. The first excited state with energy E_2. The second excited state with energy E_3.
$E_1 < E_2 < E_3 < 0$.

((**Levinson's theorem**))

Levinson's theorem is an important theorem in non-relativistic quantum scattering theory. It relates the number of bound states of a potential to the difference in phase of a scattered wave at zero and infinite energies. It was published by Norman Levinson in 1949.
https://en.wikipedia.org/wiki/Levinson%27s_theorem

((Deuteron))

Nucleus of deuterium (heavy hydrogen) that consists of one proton and one neutron (the notation: d or D). Deuterons are formed chiefly by ionizing deuterium (stripping the single electron away from the atom) and are used as projectiles to produce nuclear reactions after accumulating high energies in particle accelerators. A deuteron also results from the capture of a slow neutron by a proton, accompanied by the emission of a gamma photon.

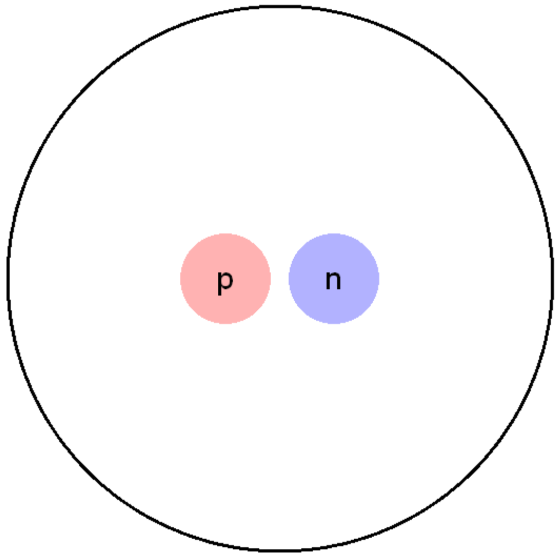


Fig.2 Deuteron consisting of one proton and one neutron in a nucleus.

The reduced mass of deuteron (neutron and proton) is

$$\mu = \frac{M_p M_n}{M_p + M_n} \approx \frac{M_P}{2}.$$

1. Schrödinger equation for the finite spherical square well

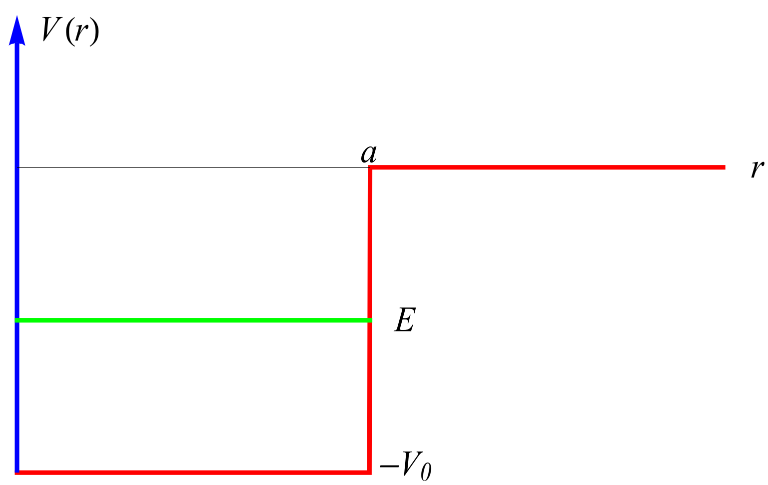


Fig.3 Spherical (attractive) square well potential $V(r)$ as a function of r. The potential depth V_0 and radius a. $-V_0 < 0$. $E = -|E| < 0$ (bound state).

The Hamiltonian for the deuteron in a finite spherical square well potential is given by

$$H = \frac{p_r^{\,2}}{2\mu} + \frac{\hbar^2 l(l+1)}{2\mu r^2} + V(r),$$

with the radial linear momentum,

$$p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r.$$

$V(r)$ is the potential energy for the spherical square well,

$$V(r) = \begin{cases} -V_0 & (r \le a) \\ 0 & (r > a) \end{cases},$$

where r is the distance between proton and neutron, and μ is the reduced mass,

$$\mu = \frac{M_n M_P}{M_n + M_P} \approx \frac{1}{2}M \qquad \text{(of proton or neutron).}$$

Then the Schrodinger equation can be written as

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}[r\psi(r)] + \frac{\hbar^2 l(l+1)}{2\mu r^2}\psi(r) + V(r)\psi(r) = E\psi(r),$$

where E is negative and numerically equal to the binding energy.

$$E < 0,$$

and

$$p_r^{\,2}\psi(r) = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r(\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r} r)\psi(r) = -\hbar^2\frac{1}{r}\frac{\partial^2}{\partial r^2}[r\psi(r)].$$

(i) For $r<a$,

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial^2}{\partial r^2}[r\psi(r)] + \frac{\hbar^2 l(l+1)}{2\mu r^2}\psi(r) - V_0\psi(r) = E\psi(r),$$

or

$$-\frac{1}{r}\frac{\partial^2}{\partial r^2}[r\psi(r)] + \frac{l(l+1)}{r^2}\psi(r) - \frac{2\mu}{\hbar^2}(E + V_0)\psi(r) = 0.$$

We put

$$r\psi(r) = u(r),$$

or

$$\frac{d^2}{dr^2}u(r) - \frac{l(l+1)}{r^2}u(r) + \frac{2\mu}{\hbar^2}(E+V_0)u(r) = 0 \qquad \text{(for } r<a\text{).}$$

(ii) $r>a$,

$$\frac{d^2}{dr^2}u(r) - \frac{l(l+1)}{r^2}u(r) + \frac{2\mu}{\hbar^2}Eu(r) = 0 \qquad \text{(for } r>a\text{).}$$

The effective potential is defined as

$$V_{eff} = \begin{cases} -V_0 + \dfrac{\hbar^2 l(l+1)}{2\mu r^2} & (r \le a) \\ \\ \dfrac{\hbar^2 l(l+1)}{2\mu r^2} & (r > a) \end{cases}$$

The normalized effective potential $\tilde{V}_{eff}$ can be rewritten as

$$\begin{aligned} \tilde{V}_{eff} &= \frac{2\mu a^2 V_{eff}}{\hbar^2} \\ &= -\frac{2\mu a^2 V_0}{\hbar^2} + l(l+1)(\frac{a}{r})^2 \qquad \text{for } r \le a \\ &= -r_0^{\ 2} + l(l+1)(\frac{a}{r})^2 \end{aligned}$$

and

$$\begin{aligned} \tilde{V}_{eff} &= \frac{2\mu a^2 V_{eff}}{\hbar^2} \\ &= \frac{2\mu a^2 V}{\hbar^2} \qquad \text{for } r > a \\ &= l(l+1)(\frac{a}{r})^2 \end{aligned}$$

When $r/a = 1$,

$$\frac{2\mu a^2 V_{eff}}{\hbar^2}\Big|_{r/a=1} = -r_0^{\ 2} + l(l+1)$$

So that the bound state (negative energy eigenvalue) exists only when

$$r_0^{\ 2} > l(l+1).$$

We make a plot of $\tilde{V}_{eff} = \dfrac{2\mu a^2 V_{eff}}{\hbar^2}$ as a function of r/a for each l, where r_0 is changed as a parameter.

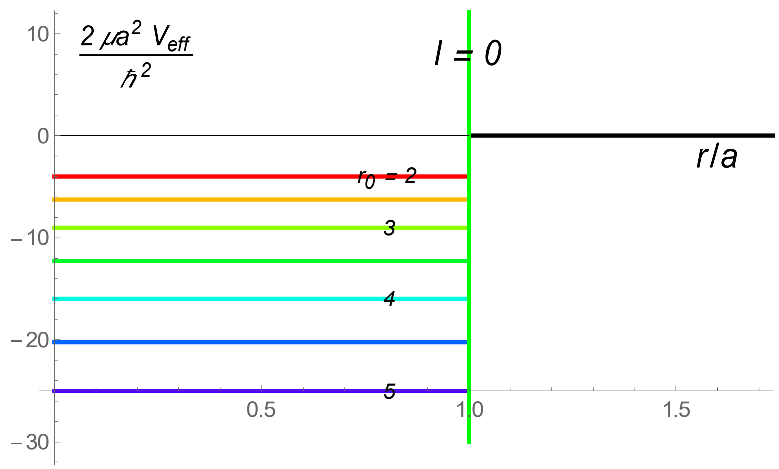


Fig.4 $l = 0$. Effective potential $\tilde{V}_{eff} = \dfrac{2\mu a^2 V_{eff}}{\hbar^2}$. $r_0 > 0$ for the bound state.

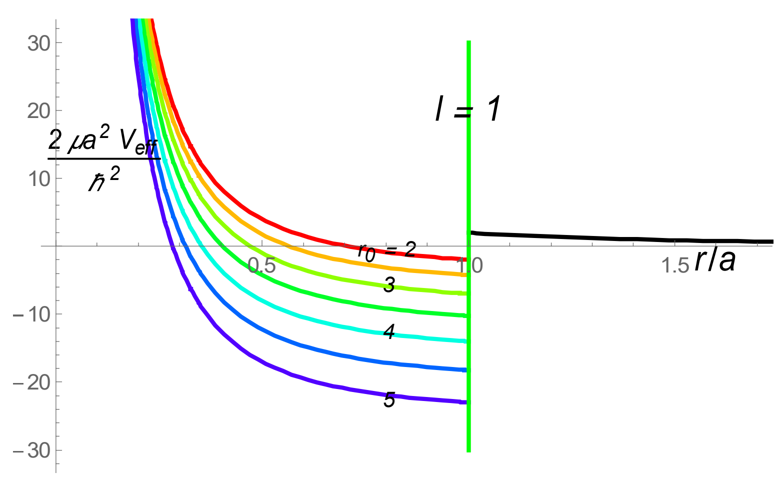


Fig.5 $l = 1$. Effective potential $\tilde{V}_{eff} = \dfrac{2\mu a^2 V_{eff}}{\hbar^2}$. $r_0 > \sqrt{l(l+1)} = \sqrt{2} = 1.4142$ for the bound state. r_0 is changed as a parameter; $r_0 = 2 - 5$. $\Delta r_0 = 0.5$.

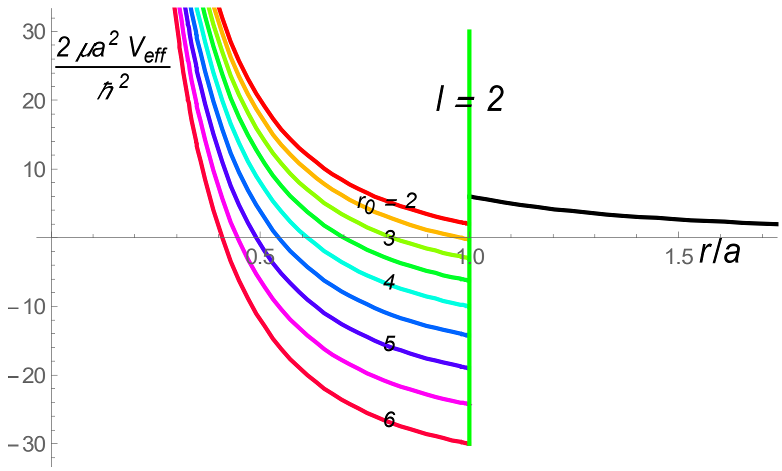


Fig.6 $l = 2$. Effective potential $\tilde{V}_{eff} = \dfrac{2\mu a^2 V_{eff}}{\hbar^2}$. $r_0 > \sqrt{l(l+1)} = \sqrt{6} = 2.4495$ for the bound state. r_0 is changed as a parameter; $r_0 = 2 - 5$. $\Delta r_0 = 0.5$.

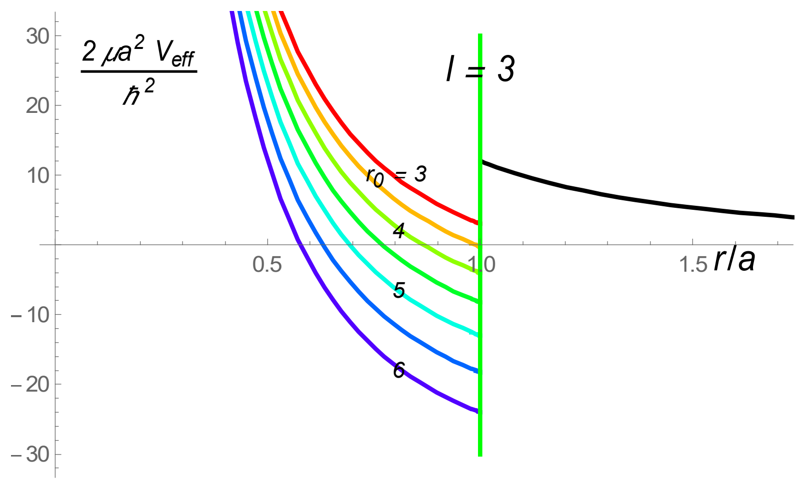


Fig.7 $l = 3$. Effective potential $\tilde{V}_{eff} = \dfrac{2\mu a^2 V_{eff}}{\hbar^2}$. $r_0 > \sqrt{l(l+1)} = 2\sqrt{3} = 3.4641$ for the bound state. r_0 is changed as a parameter; $r_0 = 3 - 6$. $\Delta r_0 = 0.5$.

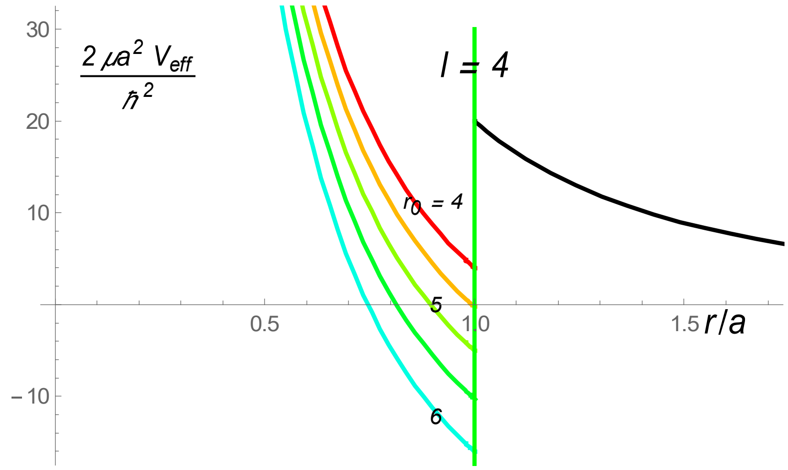


Fig.8 l = 4. Effective potential $\tilde{V}_{eff} = \frac{2\mu a^2 V_{eff}}{\hbar^2}$. $r_0 > \sqrt{l(l+1)} = 2\sqrt{5} = 4.4721$ for the bound state. r_0 is changed as a parameter; $r_0 = 4 - 6$. $\Delta r_0 = 0.5$.

3. Solution with *l* = 0 case

We consider the case when l = 0, for which there is no centrifugal barrier.

$$\frac{d^2}{dr^2}u(r) = -\frac{2\mu}{\hbar^2}(E+V_0)u(r) = -k_0^{\,2}u(r) \qquad (r<a)$$

and

$$\frac{d^2}{dr^2}u(r) = -\frac{2\mu}{\hbar^2}Eu(r) = q^2u(r) \qquad (r>a)$$

where

$$E+V_0 = \frac{\hbar^2 k_0^{\,2}}{2\mu}, \qquad E = -\frac{\hbar^2 q^2}{2\mu}.$$

with $E<0$ (bound state). This leads to the condition,

$$(k_0 a)^2 + (qa)^2 = \frac{2\mu}{\hbar^2}V_0 a^2 = r_0^{\,2}.$$

The solution of $u(r)$ is obtained as

$$u(r) = A\sin(k_0 r) \qquad (r<a)$$

$$u(r) = C\exp(-qr) \qquad (r>a)$$

The continuity of $u(r)$ and $u'(r)$ at $r = a$, leads to

$$A\sin(k_0 a) = Ce^{-qa}$$

$$Ak_0\cos(k_0 a) = C(-q)e^{-qa}$$

From these two equations, we have

$$qa = -k_0 a\cot(k_0 a) \qquad (2)$$

We assume that

$$qa = y, \qquad k_0 a = x$$

Then we have

$$y = -x\cot(x), \qquad (3)$$

with

$$x^2 + y^2 = \frac{2\mu}{\hbar^2}V_0 a^2 = r_0{}^2 \qquad (4)$$

where r_0 is the radius of the sphere. Figure shows a plot of Eqs.(3) and (4) in the *x*-*y* plane.

((**Note**))
For $l = 0$,

$$\frac{xj_{-1}(x)}{j_0(x)} = x\cot x, \qquad \frac{iyh_{-1}^{(1)}(iy)}{h_0^{(1)}(iy)} = -y$$

leading to the relation

$$y = -x\cot x$$

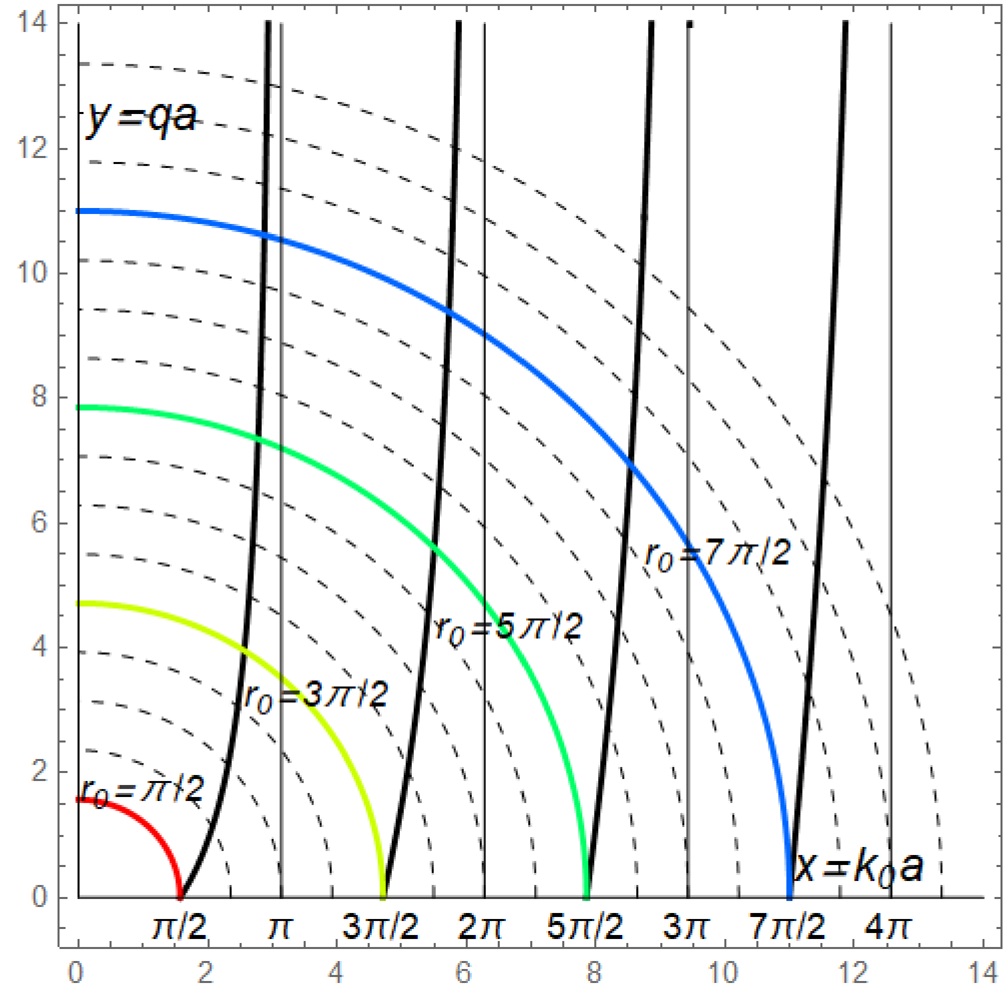


Fig.9 l = 0. A plot of $y = -x\cot(x)$ and $x^2 + y^2 = \frac{2\mu V_0 a^2}{\hbar^2} = r_0^{\,2}$. $x = k_0 a$. $y = qa$. The intersection of two curves leads to the solution (graphically solved). $E = -\frac{\hbar^2 q^2}{2\mu}$. The curve [$y = -x\cot(x)$] crosses the ***y***=0 line at $x = z(-1,n) = \frac{\pi}{2}(2n-1)$; x = $\pi/2$, $3\pi/2$, $5\pi/2$.

There is no bound state for $$r_0^{\,2} = \frac{2\mu V_0 a^2}{\hbar^2} < \left(\frac{\pi}{2}\right)^2 = 2.4674.$$

There is a single bound state for $$\left(\frac{\pi}{2}\right)^2 < r_0^{\,2} < \left(\frac{3\pi}{2}\right)^2.$$

There are two bound states for $\left(\frac{3\pi}{2}\right)^2 < r_0^{\ 2} < \left(\frac{5\pi}{2}\right)^2$.

4. The binding energy and potential depth for deuteron

A. Goswami Quantum Mechanics second edition (Wavelend, 2003).

The binding energy of deuteron can be evaluated as follows.

$M(\mathrm{H}^1)$: mass of proton	938.27208816 MeV
$M(n)$: mass of neutron	939.56542052 MeV
$M(\mathrm{d})$: mass of deuteron	1875.612928 MeV
$M(\mathrm{e})$; mass of electron	0.510998950 MeV

Binding energy = $M(\mathrm{H}^1) + M(\mathrm{n}) - M(\mathrm{d})$ 2.22458 MeV.

$M(\mathrm{H}^1) + M(e)$ =	938.783087 MeV
$M(n)$- $[M(\mathrm{H}^1) + M(e)]$ =	0.7823335 MeV

The stability of the deuteron is an important part of the story of the universe. In the Big Bang model it is presumed that in early stages there were equal numbers of neutrons and protons since the available energies were much higher than the 0.782335 MeV required to convert a proton and electron to a neutron. When the temperature dropped to the point where neutrons could no longer be produced from protons, the decay of free neutrons began to diminish their population. Those which combined with protons to form deuterons were protected from further decay. This is fortunate for us because if all the neutrons had decayed, there would be no universe as we know it, and we would not be here!
http://hyperphysics.phy-astr.gsu.edu/hbase/Particles/deuteron.html

It is known that the deuteron bound state has an energy

$|E| = 2.226\ \text{MeV}.$ (we use this value hereafter).

The range of nuclear potential is approximately equal to the Compton wavelength of the pion, which, to a first approximation, meditates the interaction between nucleons, that is

$$a \simeq \frac{\hbar}{m_\pi c} = 1.46\ \text{fm}$$

where $m_\pi = 135.0$ MeV/c^2. Using this value of a, and $\mu = \dfrac{M_p M_n}{M_p + M_n}$, we have

$$\boxed{V_0 = \frac{\hbar^2}{2\mu a^2}\left(\frac{\pi}{2}\right)^2 = 47.89 \text{ MeV}}$$

which is much larger than $|E_0|$. We make a plot of $r_0^{\,2} = \dfrac{2\mu}{\hbar^2}V_0 a^2$ as a function of $y = qa = \sqrt{\dfrac{2\mu|E|}{\hbar^2}}a$. This can be obtained using the ContourPlot of the Mathematica;

$$y = -\sqrt{r_0^{\,2} - y^2}\cot(\sqrt{r_0^{\,2} - y^2}).$$

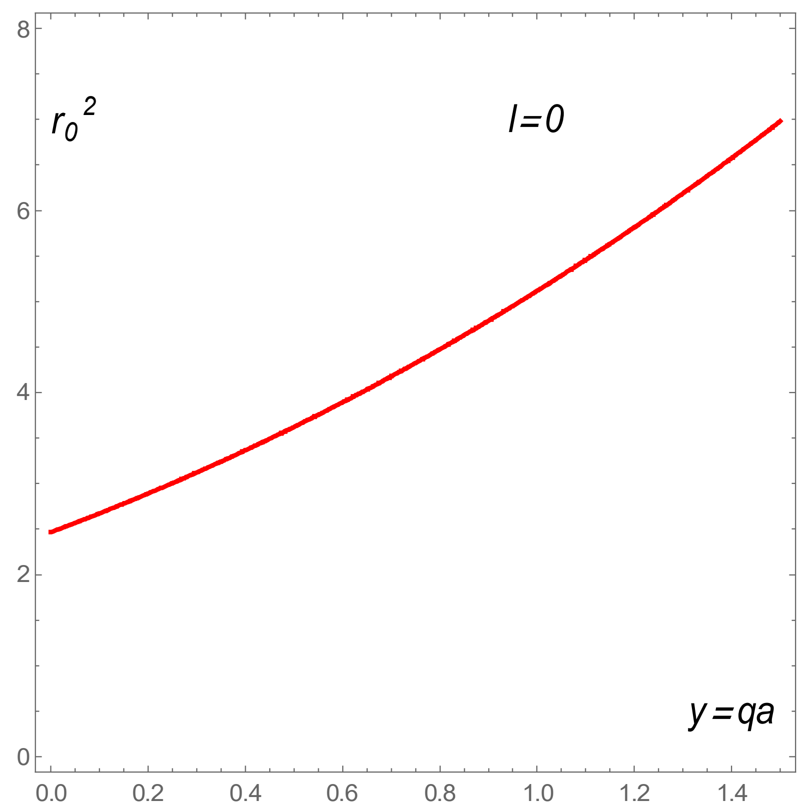


Fig.10 The range-depth relation for the square-well potential. The plot of $r_0^{\,2} = \dfrac{2\mu}{\hbar^2}V_0 a^2$ as a function of $y = qa = \sqrt{\dfrac{2\mu|E|}{\hbar^2}}a$. This figure is the same as that shown by **Goswami** in his book (Quantum Mechanics, 2[nd] edition, Wavelend, 2003), except for the scale along the y axis ($r_0^{\,2}$).

((**Mathematica**))

The radius *a*:

$$
\begin{aligned}
a &= 1.46168\times 10^{-13}\,\text{cm} \\
&= 1.46168\times 10^{-15}\,\text{m} \\
&= 1.46168\ \text{fm}
\end{aligned}
$$

The reduced mass μ:

$$
\begin{aligned}
\mu &= 8.36887\times 10^{-25}\,\text{g} \\
&= 8.36887\times 10^{-28}\,\text{kg}
\end{aligned}
$$

The potential penetration depth V_0:

$$V_0 = 47.894\ \text{MeV}.$$

We use the cgs unit for Mathematica.

```
Clear["Global`*"];
rule1 = {ħ → 1.054571628 × 10^-27, c → 2.99792 × 10^10,
   mp → 1.672621637 × 10^-24, mn → 1.674927211 × 10^-24,
   qe → 4.8032068 × 10^-10, MeV → 1.602176487 × 10^-6};
```

Radius a

$$a1 = \frac{\hbar\, c}{135\, MeV} \text{ //. rule1}$$

1.46168×10^{-13}

Reduced mass μ

$$\mu 1 = \frac{mp\, mn}{mp + mn} \text{ //. rule1}$$

8.36887×10^{-25}

Potential depth V0

$$V0 = \frac{\frac{\pi^2}{4}\, \hbar^2}{2\, \mu 1\, a1^2\, MeV} \text{ //. rule1}$$

47.894

5. Solution of $u(r)$ for $r<a$ with l

We solve the differential equation for $r<a$

$$\frac{d^2}{dr^2}u(r) - \frac{l(l+1)}{r^2}u(r) + \frac{2\mu}{\hbar^2}(E+V_0)u(r) = 0 \qquad \text{(for } r<a\text{)}.$$

or

$$\frac{d^2}{dr^2}u(r) + [k_0{}^2 - \frac{l(l+1)}{r^2}]u(r) = 0,$$

where the wave number k_0 is defined as

$$E+V_0=\frac{\hbar^2}{2\mu}k_0{}^2 .$$

Now we introduce a dimensionless variable ρ,

$$\rho=k_0 r .$$

Then we have

$$[\frac{d^2}{d\rho^2}-\frac{l(l+1)}{\rho^2}+1]u_{k,l}(\rho)=0 .$$

The solution of this differential equation is obtained as

$$u_{k,l}(\rho)=A_1\rho j_l(\rho)+A_2\rho n_l(\rho) .$$

where the spherical Bessel function and spherical Neumann function are defined by

$$j_l(\rho)=\sqrt{\frac{\pi}{2\rho}}J_{l+\frac{1}{2}}(\rho) ,$$

$$n_l(\rho)=\sqrt{\frac{\pi}{2\rho}}N_{l+\frac{1}{2}}(\rho) .$$

Note that $n_l(\rho)$ becomes infinity in the limit of $\rho\to 0$. So we choose the first term

$$R_{k,l}(r)=A_l j_l(k_0 r) .$$

where $rR_{k,l}(r)=u_{k,l}(r)$ and A_l is constant.

((**Mathematica**))

```
Clear["Global`*"];
eq1 = y''[x] + (1 - L (L + 1)/x^2) y[x] == 0;
DSolve[eq1, y[x], x]
{{y[x] → √x BesselJ[1/2 (1 + 2 L), x] C[1] +
    √x BesselY[1/2 (1 + 2 L), x] C[2]}}
```

((**Note**))

We note that

$$u(\rho) \propto \sqrt{\rho} J_{l+\frac{1}{2}}(\rho) = \rho \frac{J_{l+\frac{1}{2}}(\rho)}{\sqrt{\rho}} = \sqrt{\frac{2}{\pi}} \rho j_l(\rho) .$$

Thus we have

$$R(r) = \frac{u(r)}{r} \propto \frac{u(\rho)}{\rho} = j_l(\rho) .$$

6. Solution of *u*(*r*) for *r*>*a* with finite *l*

We solve the differential equation for $r>a$

$$\frac{d^2}{dr^2} u(r) - \frac{l(l+1)}{r^2} u(r) + \frac{2\mu}{\hbar^2} E u(r) = 0 , \qquad \text{(for } r>a\text{)}.$$

where

$$E = -\frac{\hbar^2}{2\mu} q^2 ,$$

where q is the wavenumber.

$$\frac{d^2}{dr^2} u(r) + [(iq)^2 - \frac{l(l+1)}{r^2}] u(r) = 0 .$$

Now we introduce a dimensionless variable ρ,

$$\rho = iqr .$$

Then we get

$$\frac{d^2}{d\rho^2} u_{k,l}(\rho) + [1 - \frac{l(l+1)}{\rho^2}] u_{k,l}(\rho) = 0 .$$

The solution of this differential equation is obtained as

$$\begin{aligned} R_{k,l}(r) &= B_1' j_l(iqr) + B_2' n_l(iqr) \\ &= B_1 h_l^{(1)}(iqr) + B_2 h_l^{(2)}(iqr) \end{aligned}$$

where $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ are the spherical Hankel function of the first and second kind.

$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(1)}(x) = j_l(x) + i n_l(x),$$

$$h_l^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(2)}(x) = j_l(x) - i n_l(x).$$

The asymptotic forms of $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ are given by

$$h_\ell^{(1)}(x) \approx -i \frac{e^{i(x - l\pi/2)}}{x},$$

$$h_\ell^{(2)}(x) \approx i \frac{e^{-i(x - l\pi/2)}}{x},$$

in the limit of large x. Then we get

$$h_\ell^{(1)}(iqr) \approx -i \frac{e^{i(iqr - l\pi/2)}}{iqr} = -\frac{e^{(-qr - il\pi/2)}}{qr},$$

$$h_\ell^{(2)}(iqr) \approx i \frac{e^{-i(iqr - l\pi/2)}}{iqr} = \frac{e^{(qr + il\pi/2)}}{qr},$$

which means that $h_\ell^{(2)}(iqr)$ becomes diverging for large r, while $h_\ell^{(1)}(iqr)$ becomes zero for large r. So, we choose $h_\ell^{(1)}(iqr)$ as the solution of $R_{k,l}(r)$ for $r>a$,

$$R_{k,l}(r) = B_l h_l^{(1)}(iqr).$$

where B_1 is constant.

10. Boundary conditions with finite *l*

Using the boundary condition at $r = a$, we determine the energy eigenvalues. We note that the wave function and its derivative should be continuous at $r = a$.

$$A_l j_l(k_0 a) = B_l h_l^{(1)}(iqa),$$

$$A_l k_0 j_l'(k_0 r)|_{r=a} = B_l (iq) h_l^{(1)}{}'(iqr)|_{r=a},$$

leading to

$$\frac{k_0}{j_l(k_0 a)} j_l\,'(k_0 a) = \frac{iq}{h_l^{(1)}(iqa)} h_l^{(1)}\,'(iqa),$$

where

$$\frac{\partial j_l(z)}{\partial z} = j_l\,'(z)\,, \qquad \frac{\partial h_l^{(1)}(z)}{\partial z} = h_l^{(1)}\,'(z)\,.$$

Spherical Bessel function;

$$j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z)$$ SphericalBesselJ[l, z] (Mathematica)

Spherical Hankel function:

$$h_l^{(1)}(z) = \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(1)}(z)$$ SphericalHankel1[l, z] (Mathematica)

Here we use the recursion formula for $f_l(\rho) = j_l(\rho),\ h_l^{(1)}(\rho),\ ,$

$$f_{l-1}(\rho) + f_{l+1}(\rho) = \frac{2l+1}{\rho} f_l(\rho),$$

$$lf_{l-1}(\rho) - (l+1) f_{l+1}(\rho) = (2l+1) f_l\,'(\rho).$$

From these equations, we have

$$f_l\,'(\rho) = f_{l-1}(\rho) - \frac{(l+1)}{\rho} f_l(\rho)$$

Leading to the relations,

$$j_l\,'(z) = j_{l-1}(z) - \frac{l+1}{z} j_l(z)$$

$$h_l^{(1)}\,'(z) = h_{l-1}^{(1)}(z) - \frac{l+1}{z} h_l^{(1)}(z)$$

Then, the boundary condition can be rewritten as

$$\frac{k_0[j_{l-1}(k_0a)-\frac{l+1}{k_0a}j_l(k_0a)}{j_l(k_0a)}=\frac{iq[h_{l-1}^{(1)}(iqa)-\frac{l+1}{iqa}h_l^{(1)}(iqa)]}{h_l^{(1)}(iqa)}$$

or

$$\boxed{\frac{k_0 j_{l-1}(k_0a)}{j_l(k_0a)}=\frac{iqh_{l-1}^{(1)}(iqa)}{h_l^{(1)}(iqa)}} \qquad (l = 1, 2, 3, \quad).$$

Finally, we get

$$\frac{xj_{l-1}(x)}{j_l(x)}=\frac{iyh_{l-1}^{(1)}(iy)}{h_l^{(1)}(iy)}$$

$$x^2+y^2=r_0^2=\frac{2\mu a^2}{\hbar^2}V_0$$

(**L.I. Schiff, Quantum mechanics**)

Note that $j_{-1}(x)=\frac{\cos x}{x}$ (**APPENDIX-A**)

In summary, we have

(a) $l = 0$

$$y=-x\cot x \qquad \text{for } l = 0$$

$$\frac{xj_{-1}(x)}{j_0(x)}=\frac{x\left(\frac{\cos x}{x}\right)}{\frac{\sin x}{x}}=x\cot x=\frac{iyh_{-1}^{(1)}(iy)}{h_0(iy)}.$$

(b) $l = 1, 2, 3, 4, \ldots$

$$\frac{xj_{l-1}(x)}{j_l(x)}=\frac{iyh_{l-1}^{(1)}(iy)}{h_l^{(1)}(iy)}, \qquad \text{for } l = 1, 2, 3,\ldots$$

with $x^2+y^2=r_0^2$. We make a plot of $y^2=(qa)^2=\frac{2\mu}{\hbar^2}|E|a^2$ as a function of $r_0^2=\frac{2\mu}{\hbar^2}V_0a^2$ with the use of ContourPlot of the Mathematica.

9. The condition for the bound state energy $E = 0$

Using these boundary conditions, we can determine the values of V_0 and l such that the energy of the bound state E becomes zero. We note that the energy $E = 0$ is equivalent to $y = 0$. Note that $l = 0$ (s), 1 (p), 2 (d), 3 (f), and 4 (g).

For $l = 0$, we get the condition $\cos x = 0$, leading to

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

or

$$z[0-1, n] = \frac{\pi}{2}(2n-1) \text{ with } n = 1, 2, 3, \ldots..$$

and

$$\frac{2\mu a^2 V_0^{(0)}}{\hbar^2} = [z(0-1,\ n)]^2 = [\frac{\pi}{2}(2n-1)]^2 \qquad (l = 0)$$

For $l = 0, 1, 2, 3, \ldots$, we have $j_{l-1}(x) = 0$. The zeros of $j_{l-1}(x)$ is defined by

$$x = z(l-1, n),$$

or

$$\frac{2\mu a^2 V_0^{(0)}}{\hbar^2} = [z(l-1, n,)]^2$$

where n = 1, 2, 3,,…, $V_0^{(0)}$ is the potential depth for $E = 0$, l (=1, 2, 3,…) is the orbital angular momentum, and $\varsigma(l-1, n,)$ is the zero of the spherical Bessel function $j_{l-1}(x)$.

Table 1: Zeros of the spherical Bessel functions.

Number of zero; n	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$
1	3.14159	4.49341	5.76346	6.98793	8.18256
2	6.28319	7.72525	9.09501	10.4171	11.7049
3	9.42478	10.9041	12.3229	13.6980	15.0397
4	12.5664	14.0662	15.5146	16.9236	18.3013
5	15.7080	17.2208	18.6890	20.1218	21.5254

Table 2: Zeros of the spherical Bessel functions.

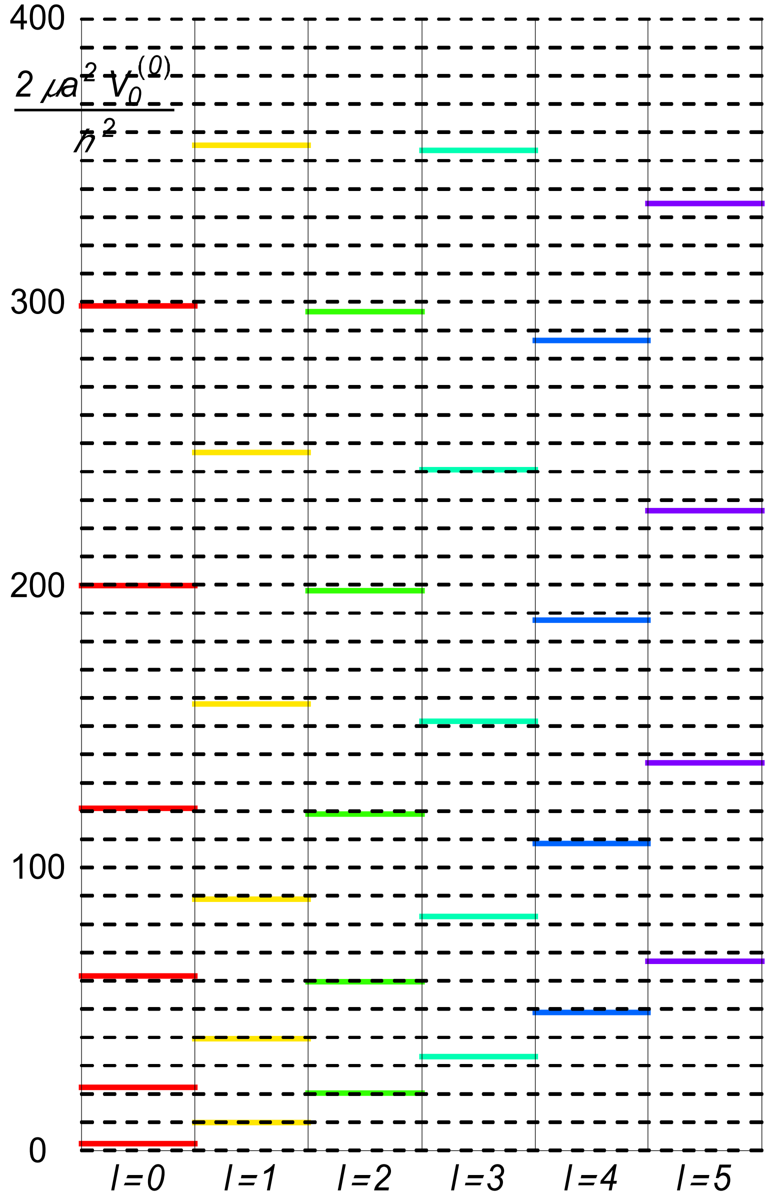
Number of zero n	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$
1	π	1.4303π	1.83457π	2.22433π	2.60459π
2	2π	2.4590π	2.89503π	3.31587π	3.72579π
3	3π	3.4709π	3.9225π	4.36022π	4.78727π
4	4π	4.4774π	4.9385π	5.38696π	5.82547π
5	5π	5.4815π	5.9489π	6.40497π	6.85175π

((**Mathematica**))
Zeroes of the Spherical Bessel functions

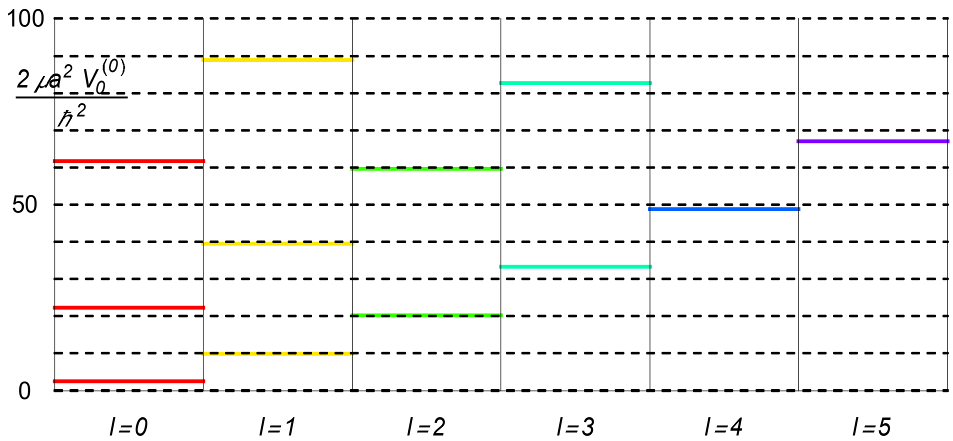
```
Clear["Global`*"];
x[L_, n_] := Module[{g1, x1}, g1 = BesselJZero[L + 0.5, n];
  x1 = g1];
fm = Prepend[Table[{0, n, (π / 2) (2 n - 1) // N}, {n, 1, 5, 1}],
   {"L=-1", " n", "z[L=-1,n]"}] // TableForm;
f0 = Prepend[Table[{0, n, x[0, n] // N}, {n, 1, 5, 1}],
   {"L=0", " n", "z[L=0,n]"}] // TableForm;
f1 = Prepend[Table[{1, n, x[1, n] // N}, {n, 1, 5, 1}],
   {"L=1", " n", "z[L=1,n]"}] // TableForm;
f2 = Prepend[Table[{2, n, x[2, n] // N}, {n, 1, 5, 1}],
   {"L=2", " n", "z[L=2,n]"}] // TableForm;
f3 = Prepend[Table[{L = 3, n, x[3, n] // N}, {n, 1, 5, 1}],
   {"L=3", " n", "z(L=3,n)"}] // TableForm;
f4 = Prepend[Table[{L = 4, n, x[4, n] // N}, {n, 1, 5, 1}],
   {"L=4", " n", "z(L=4,n)"}] // TableForm;
Grid[{{fm}, {f0}, { f1}, {f2}, {f3}, {f4}}, Frame → All]
```

L=-1	n	z[L=-1,n]
0	1	1.5708
0	2	4.71239
0	3	7.85398
0	4	10.9956
0	5	14.1372
L=0	n	z[L=0,n]
0	1	3.14159
0	2	6.28319
0	3	9.42478
0	4	12.5664
0	5	15.708
L=1	n	z[L=1,n]
1	1	4.49341
1	2	7.72525
1	3	10.9041
1	4	14.0662
1	5	17.2208
L=2	n	z[L=2,n]
2	1	5.76346
2	2	9.09501
2	3	12.3229
2	4	15.5146
2	5	18.689
L=3	n	z(L=3,n)
3	1	6.98793
3	2	10.4171
3	3	13.698
3	4	16.9236
3	5	20.1218
L=4	n	z(L=4,n)
4	1	8.18256
4	2	11.7049
4	3	15.0397
4	4	18.3013
4	5	21.5254

Here we make a plot of $\frac{2\mu a^2 V_0^{(0)}}{\hbar^2}$ where $E = 0$, as a function of l and n.



(a)



(b)

Fig.11(a) (b) Plot of $\dfrac{2\mu a^2 V_0^{(0)}}{\hbar^2} = r_0^{\,2}$ where E = 0, as a function of l and n: l = 0, 1, 2, 3,…. n = 1, 2, 3, 4, ….. Note that the bound state (negative energy eigenvalue) exists only when $r_0^{\,2} > l(l+1)$ from the discussion of the effective potential.

(a) $l = 0$: $j_{-1}(z) = \dfrac{\cos z}{z} = 0$ $\cos z = 0$

n	$z(0-1,n)$	$[z(0-1,n)]^2$
1	$\pi/2\,(=\zeta_{01})$	2.46740
2	$3\pi/2\,(=\zeta_{02})$	22.2066
3	$5\pi/2\,(=\zeta_{03})$	61.6850
4	$7\pi/2\,(=\zeta_{04})$	120.903
5	$9\pi/2\,(=\zeta_{05})$	199.859

(b) $l = 1$: $j_0(z) = \dfrac{\sin z}{z} = 0$; $\sin z = 0$

n	$z(1-1,n)$	$[z(1-1,n)]^2$
1	$\pi\,(=\zeta_{11})$	9.86960
2	$2\pi\,(=\zeta_{12})$	39.4784
3	$3\pi\,(=\zeta_{13})$	88.8264
4	$4\pi\,(=\zeta_{14})$	157.914
5	$5\pi\,(=\zeta_{15})$	246.740

(c) $l = 2$: $j_1(z) = 0 \rightarrow z = z(2-1,n)$

n	$z(2-1,n)$	$[z(2-1,n)]^2$
1	4.49341 $(=\zeta_{21})$	20.19073
2	7.72525 $(=\zeta_{22})$	59.6795
3	10.9041 $(=\zeta_{23})$	118.8994
4	14.0662 $(=\zeta_{24})$	197.8580
5	17.2208 $(=\zeta_{25})$	296.5560

(d) $l=3$: $j_2(z)=0 \to z=z(3-1,n)$;

n	$z(3-1,n)$	$[z(3-1,n)]^2$
1	5.76346 $(=\zeta_{31})$	33.2175
2	9.09501 $(=\zeta_{32})$	82.7192
3	12.3229 $(=\zeta_{33})$	151.854
4	15.5146 $(=\zeta_{34})$	240.703
5	18.6890 $(=\zeta_{35})$	349.279

(e) $l=4$: $j_3(z)=0 \to z=z(4-1,n)$;

n	$z(4-1,n)$	$[z(4-1,n)]^2$
1	6.9879 $(=\zeta_{41})$	48.83075
2	10.4171 $(=\zeta_{42})$	108.5160
3	13.6980 $(=\zeta_{43})$	187.6352
4	16.9236 $(=\zeta_{44})$	286.4082
5	20.1218 $(=\zeta_{45})$	404.8868

9. Solution with *l* = 0 (*s*-wave)

We now discuss the solution of the Schrödinger equation with $l=0$ (s wave). We have the wave function,

$$u(r) = A\sin(k_0 r) = A\sin[(k_0 a)\frac{r}{a}] = A\sin(\alpha x) \qquad \text{for } r \le a.$$

$$u(r) = C\exp(-qr) = C\exp[-(qa)\frac{r}{a}] = C\exp[-\beta x] \qquad \text{for } r > a$$

$$\alpha = k_0 a, \qquad \beta = qa$$

$$\alpha^2 + \beta^2 = (k_0 a)^2 + (qa)^2 = \frac{2\mu}{\hbar^2} V_0 a^2 = r_0^{\,2}$$

The boundary condition:

$$A\sin(k_0 a) = C\exp(-qa),$$

$$A\sin(\alpha) = C\exp(-\beta),$$

$$Ak_0\cos(k_0 a) = C(-q)\exp(-qa), \qquad A\alpha\cos(\alpha) = -C\beta\exp(-\beta),$$

$$\beta = -\alpha\cot\alpha.$$

Normalization:

$$1 = \int_0^a \frac{A^2\sin^2(k_0 r)}{r^2}4\pi r^2 dr + \int_a^\infty \frac{C^2\exp(-2qr)}{r^2}4\pi r^2 dr,$$

or

$$\int_0^a \frac{A^2\sin^2(k_0 r)}{r^2}4\pi r^2 dr = 4\pi A^2\int_0^a \sin^2(k_0 r)dr$$
$$= 4\pi A^2 a[\frac{2\alpha - \sin(2\alpha)}{4\alpha}]$$

$$\int_a^\infty \frac{C^2\exp(-2qr)}{r^2}4\pi r^2 dr = 4\pi C^2\int_a^\infty \exp(-2qr)dr$$
$$= 4\pi C^2\frac{e^{-2\beta}}{2\beta}$$

or

$$1 = 4\pi A^2 a\left[\frac{2\alpha - \sin(2\alpha)}{4\alpha}\right] + 4\pi C^2 a\left(\frac{e^{-2\beta}}{2\beta}\right),$$

$$A\sin(\alpha) = C\exp(-\beta),$$

$$A\sqrt{a} = \frac{\sqrt{\alpha\beta}}{\sqrt{\pi}\sqrt{-2\beta\sin\alpha\cos\alpha + 2\alpha(\beta + \sin^2\alpha)}},$$

$$C\sqrt{a} = \frac{e^{\beta}\sqrt{\alpha\beta}\sin\alpha}{\sqrt{\pi}\sqrt{-2\beta\sin\alpha\cos\alpha + 2\alpha(\beta + \sin^2\alpha)}}.$$

10. The energy eigenvalue $|E|$ vs potential depth with $l = 0$.

We now use the ContourPlot to determine the relation of y^2 vs $r_0^{\ 2}$, from the equation

$$y = -x\cot x = -\sqrt{r_0^{\ 2} - y^2}\cot\sqrt{r_0^{\ 2} - y^2} \qquad \text{with} \qquad x = \sqrt{r_0^{\ 2} - y^2}$$

which depends only on y^2 and $r_0^{\ 2}$. **Figure** 12 shows the scaled energy eigenvalue $y^2 = (qa)^2 = \frac{2\mu a^2}{\hbar^2}|E|$, as a function of scaled potential depth $r_0^{\ 2} = \frac{2\mu a^2 V_0}{\hbar^2}$ for $l = 0$.

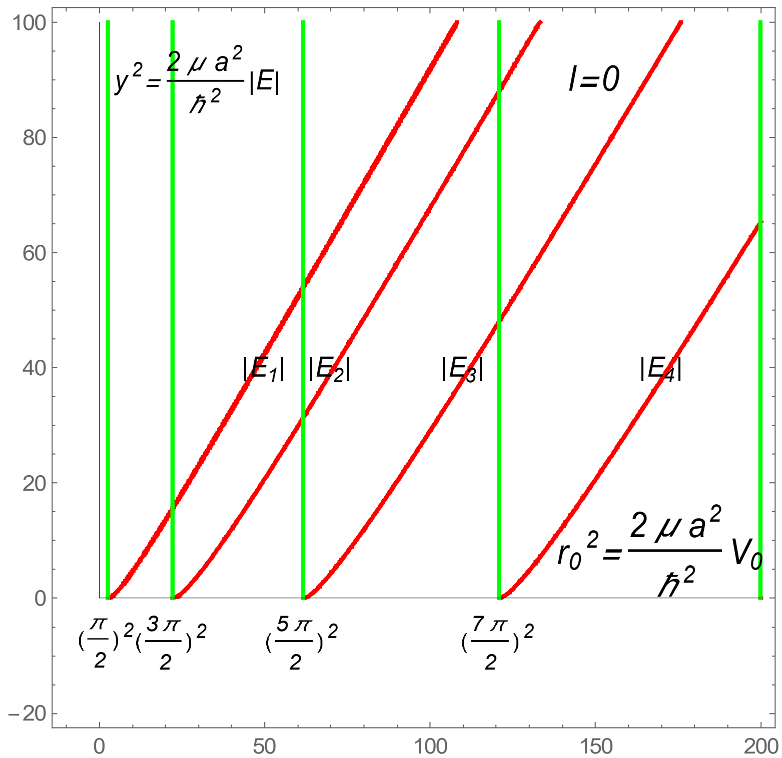


Fig.12 $\quad l = 0.\ y^2 = (qa)^2 = \frac{2\mu a^2}{\hbar^2}|E|$ vs $r_0^{\ 2} = \frac{2\mu a^2 V_0}{\hbar^2}$.

$|E| = 0$ at $r_0 = z(l-1 = 0-1, n)$; where $j_{-1}(r_0) = 0$.

$r_0 = \varsigma_{01} = \frac{\pi}{2},\ \varsigma_{02} = \frac{3\pi}{2},\ \varsigma_{03} = \frac{5\pi}{2},\ \varsigma_{04} = \frac{7\pi}{2}, \ldots$

As is shown From **Fig.10**, there is no bound state for

$$\frac{2\mu V_0 a^2}{\hbar^2} < \left(\frac{\pi}{2}\right)^2.$$

There is a single bound state (ground state) for $\left(\frac{\pi}{2}\right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left(\frac{3\pi}{2}\right)^2$. There are two bound states (ground state and the first excited state) for

$$\left(\frac{5\pi}{2}\right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left(\frac{7\pi}{2}\right)^2.$$

There are three bound states (ground state, the first excited state, and the second excited state) for

$$\left(\frac{7\pi}{2}\right)^2 < \frac{2\mu V_0 a^2}{\hbar^2} < \left(\frac{9\pi}{2}\right)^2.$$

Here we define the minimum energy eigenvalue $|E|$ among the allowed states for fixed V_0. The value of $|E|$ is the closest to $E = 0$ (but $E<0$). We make a plot of the minimum eigenvalue $|E|$ as a function of potential depth V_0.

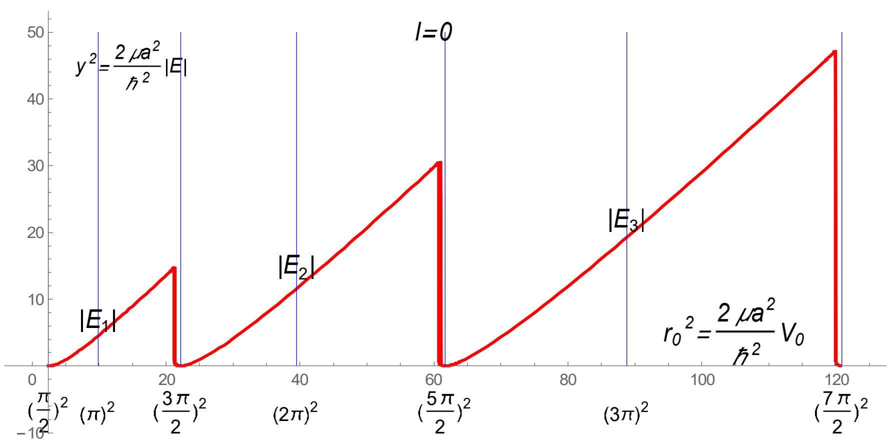


Fig.13 The energy eigenvalue ($|E|$) of the bound states which is the closest to E = 0 among bound states, but E<0. The minimum energy ($y = \frac{2\mu a^2}{\hbar^2}|E|$) as a function of the potential depth ($r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$). $r_0 = \varsigma_{01} = \frac{\pi}{2}$, $\varsigma_{02} = \frac{3\pi}{2}$, $\varsigma_{03} = \frac{5\pi}{2}$, $\varsigma_{04} = \frac{7\pi}{2}$,...

11. Wave function for *l* = 0.
(a) The wave function $u(r)$ of the ground state for *l* = 0.

Below we show the plot of the wave function $u(r)$ as a function of r/a for the ground state for $l = 0$ (s wave). The two nucleons have a substantial probability of being separated by a distance that is greater than the range of the potential well. The shape of the wave function is, correspondingly, not very sensitive to the detailed nature of the potential. The wave function for s = 1.0 ($r_0 = \frac{\pi}{2}s$)is similar to the figure obtained by Bethe and Morrison. For the ground state with $l = 0$,

$$u(r) = C\exp(-qr) = C\exp[-(qa)\frac{r}{a}] \qquad \text{for } r > a$$

The quantity $1/(qa)$ can be taken as a measure of the size of the deuteron (the effective radius of the deuteron). It is shown that the effective radius is considerably larger than the range of nuclear force.

$$\frac{1}{qa} >> 1.$$

Thus, most of the area under $u(r)$ occurs for $r/a > 1$.

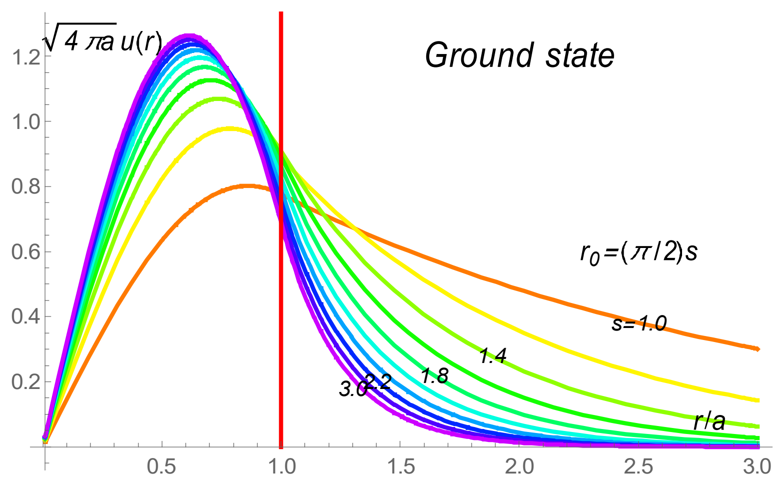


Fig.14 $l = 0$. Wave function $u(r)$ for the ground state (s = 1 -3) as a function of r/a. $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \frac{\pi}{2}s$. s is changed as a parameter. $s = 1-3$. $\Delta s = 0.2$.

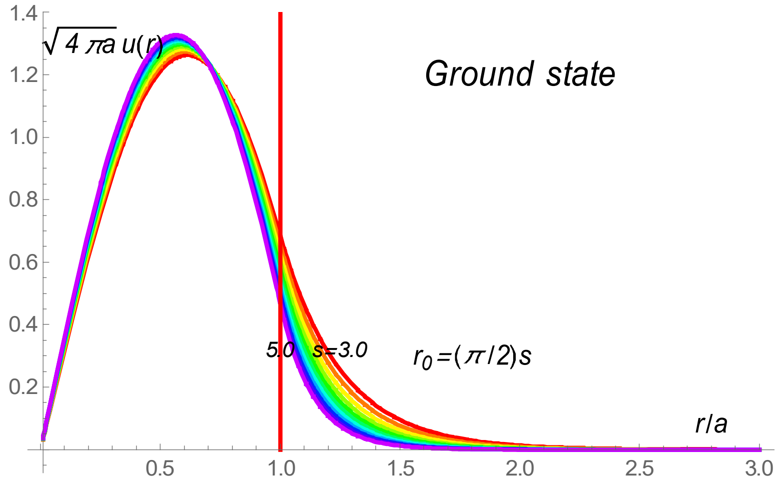


Fig.15 l = 0. Wave function $u(r)$ for the ground state (s = 3 -5) as a function of r/a. $r_0^{\ 2}=\dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0=\dfrac{\pi}{2}s$ and s is changed as a parameter; s = 3.0 – 5.0, $\Delta s = 0.2$.

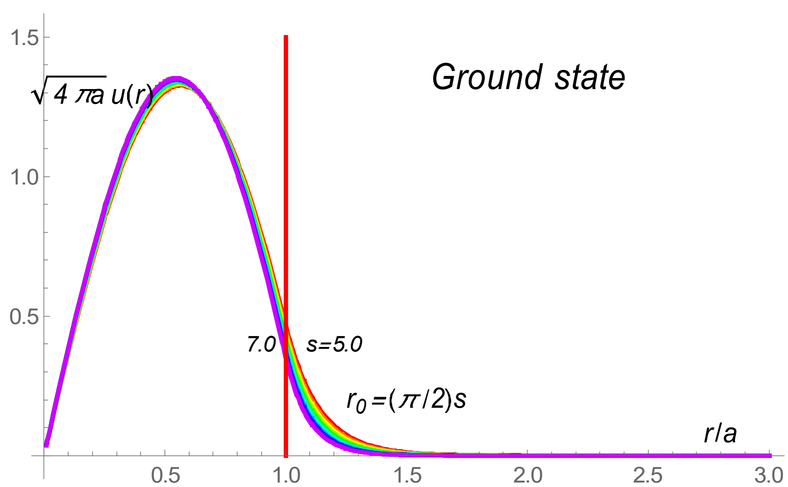


Fig.16 l = 0. Wave function $u(r)$ for the ground state (s = 5 - 7) as a function of r/a. $r_0^{\ 2}=\dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0=\dfrac{\pi}{2}s$ and s is changed as a parameter; s = 5.0 – 7.0, $\Delta s = 0.2$.

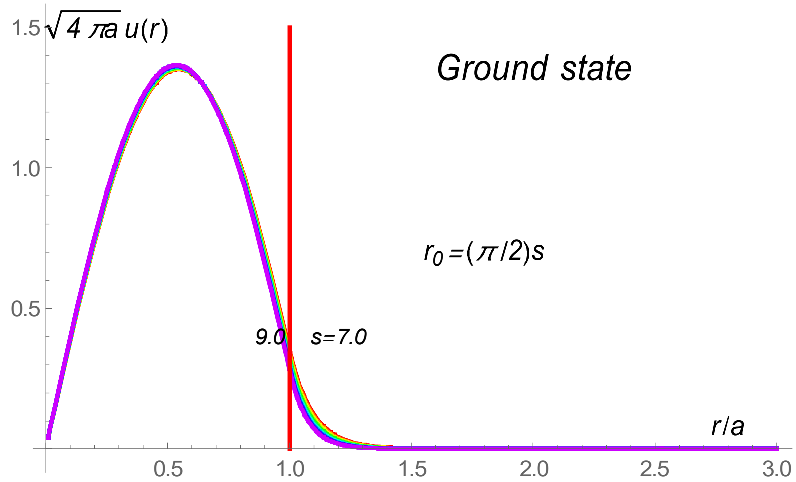


Fig.17 $l = 0$. Wave function $u(r)$ for the ground state (s = 7 - 9) as a function of r/a. $r_0^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \dfrac{\pi}{2}s$ and s is changed as a parameter; s = 7.0 – 9.0, $\Delta s = 0.2$.

(b) The wave function $u(r)$ of the first excited state for $l = 0$.

The wave function $u(r)$ of the first excited state for $l = 0$ is shown below, as a function of r/a, where r_0 is changed as a parameter. The wave unction $u(r)$ with $r_0 \simeq \dfrac{3}{2}\pi$, is similar to the figure obtained by Bethe and Morrison. Since the energy eigenvalue of the first excited state is still negative (bound state), but is very close to zero. So that, the wave function does not decay over a distance much longer than the radius a.

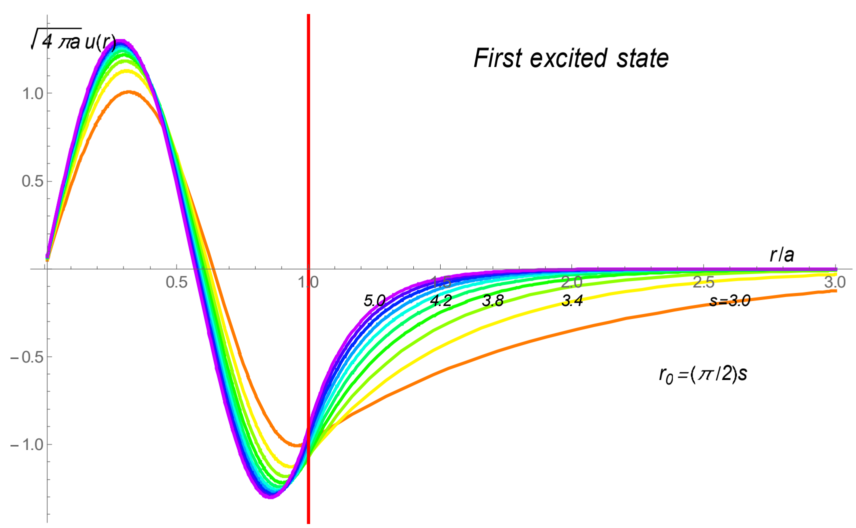


Fig.18 $l = 0$. Wave function $u(r)$ for the first excited state (s = 3 -5) as a function of r/a. $r_0^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \dfrac{\pi}{2}s$ and s is changed as a parameter; s = 3.0 – 5.0, $\Delta s = 0.2$.

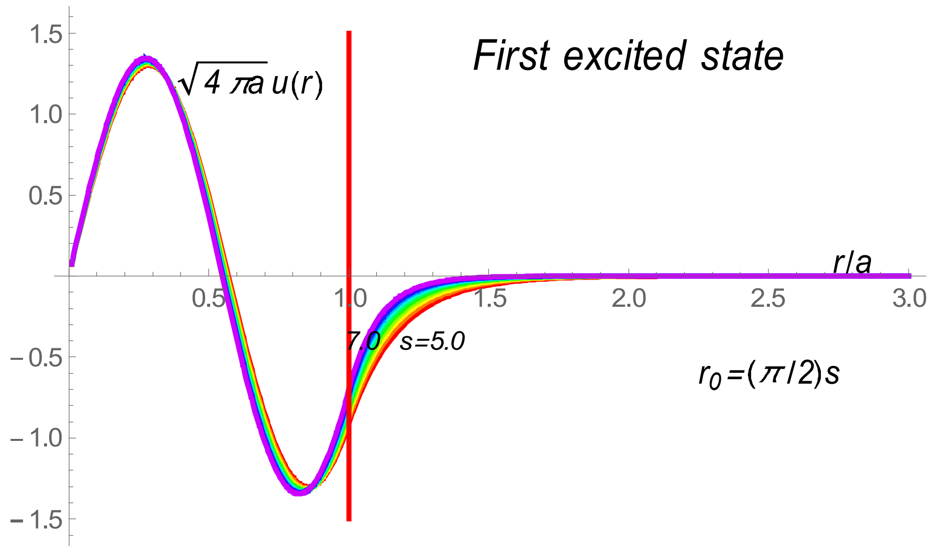


Fig.19 $l = 0$. Wave function $u(r)$ for the first excited state (s = 5 - 7) as a function of r/a. $r_0^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \dfrac{\pi}{2}s$ and s is changed as a parameter; s = 5.0 – 7.0, $\Delta s = 0.2$.

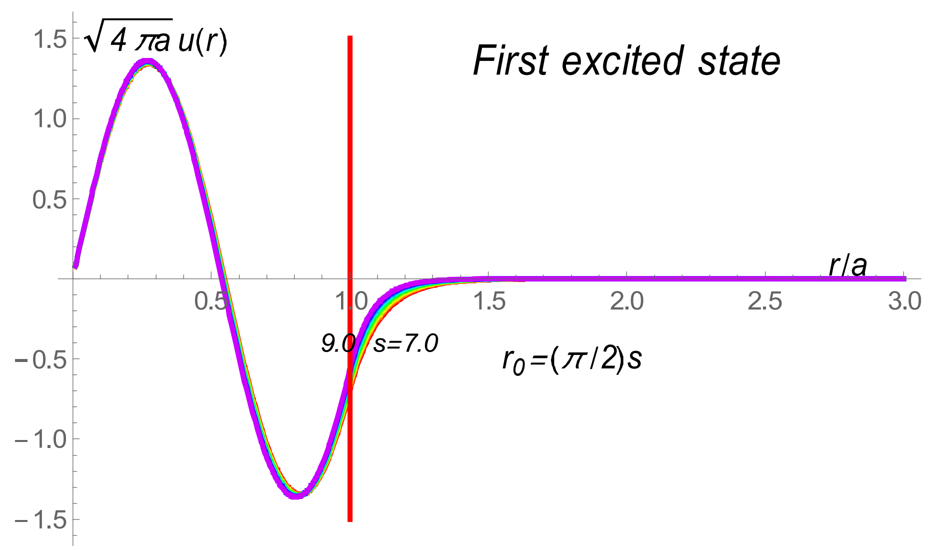


Fig.20 l = 0. Wave function $u(r)$ for the first excited state (s = 7 - 9) as a function of r/a. $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \frac{\pi}{2}s$ and s is changed as a parameter; $s = 7.0 - 9.0$, $\Delta s = 0.2$.

(c) The wave function $u(r)$ of the second excited state for l = 0.

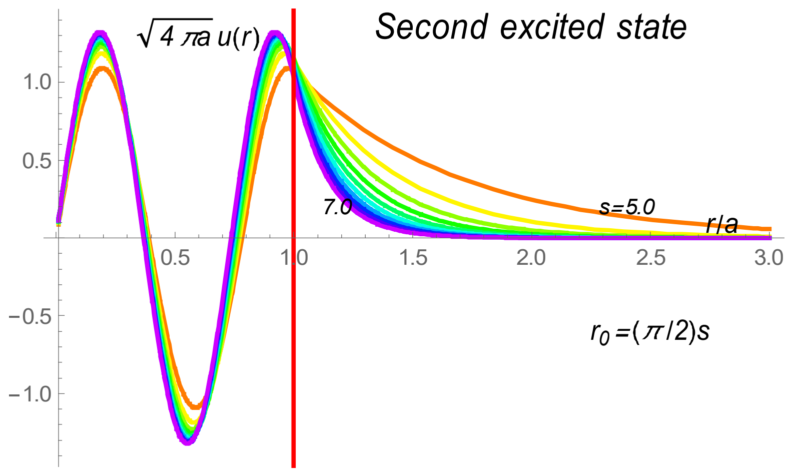


Fig.21 l = 0. Wave function $u(r)$ for the second excited state (s = 5 - 7) as a function of r/a. $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \frac{\pi}{2}s$ and s is changed as a parameter; $s = 5.0 - 7.0$, $\Delta s = 0.2$.

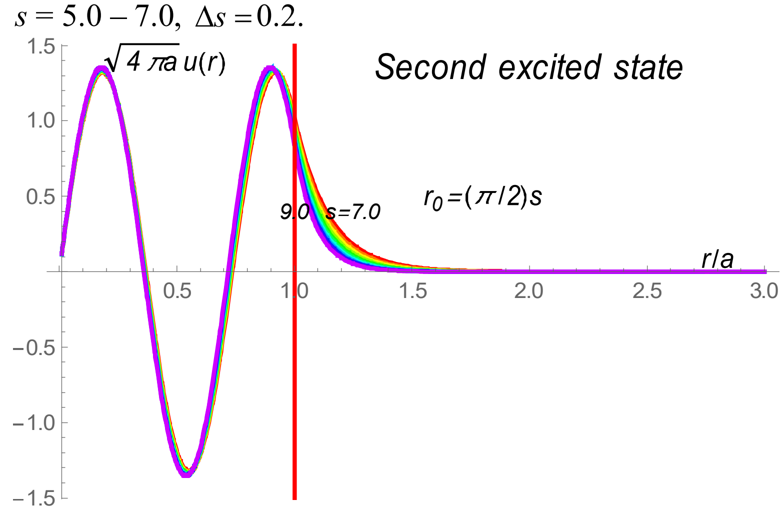


Fig.22 l = 0. Wave function $u(r)$ for the second excited state (s = 7 - 9) as a function of r/a. $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \frac{\pi}{2}s$ and s is changed as a parameter; $s = 7.0 - 9.0$, $\Delta s = 0.2$.

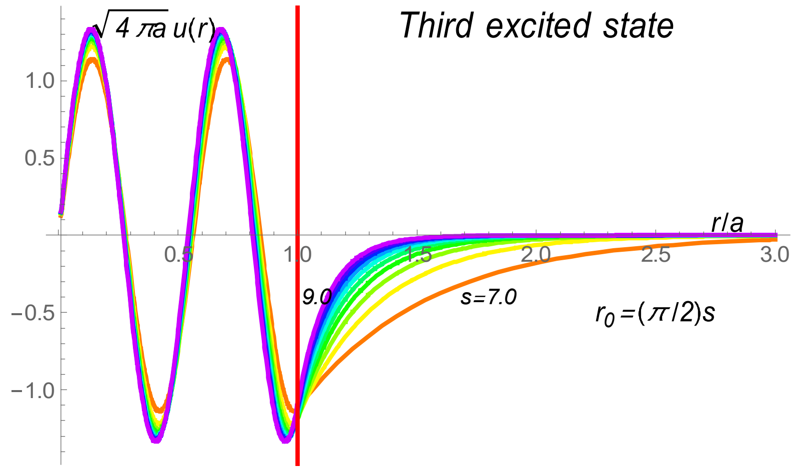


Fig.23 l = 0. Wave function $u(r)$ for the third excited state (s = 7 - 9) as a function of r/a. $r_0^{\ 2} = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0 = \dfrac{\pi}{2}s$ and s is changed as a parameter; $s = 7.0 - 9.0,\ \Delta s = 0.2$.

12. ContourPlot of *y* vs *x* for *l*

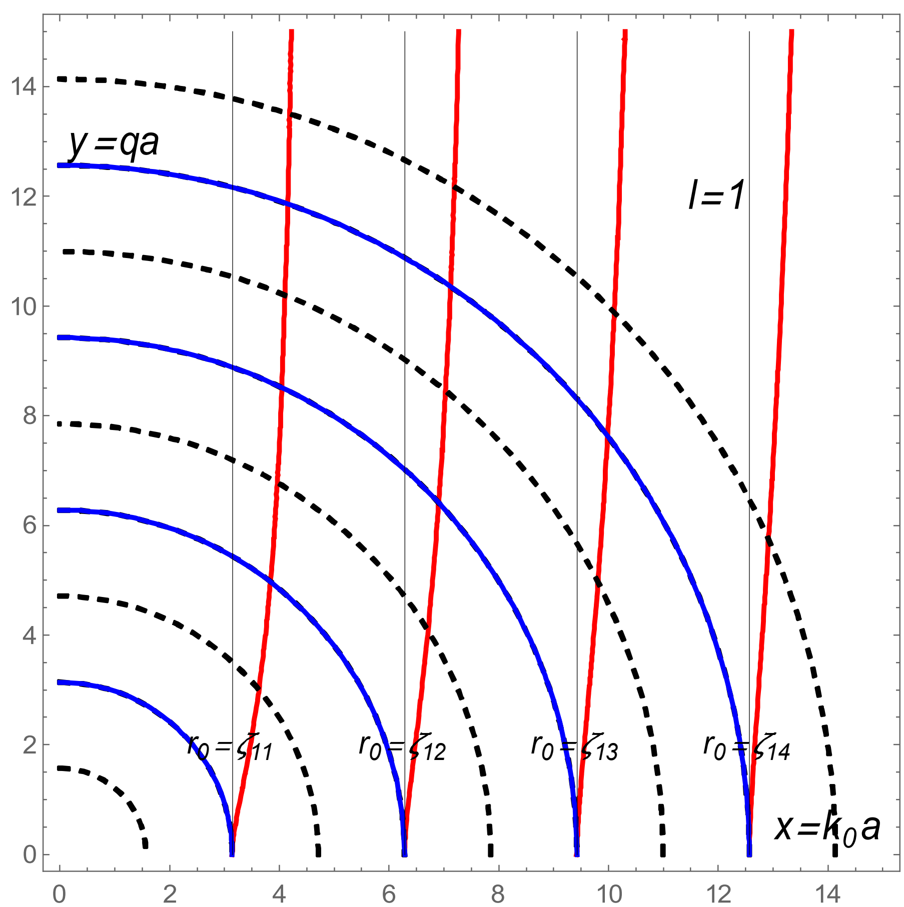


Fig.24 ContourPlot for the boundary condition. y vs x. l = 1. $E=0$ at $r_0 = z(1-1=0, n)$, at which $j_0(r_0)=0$. $r_0 = \varsigma_{11}$, ς_{12}, ς_{13}, ς_{14}

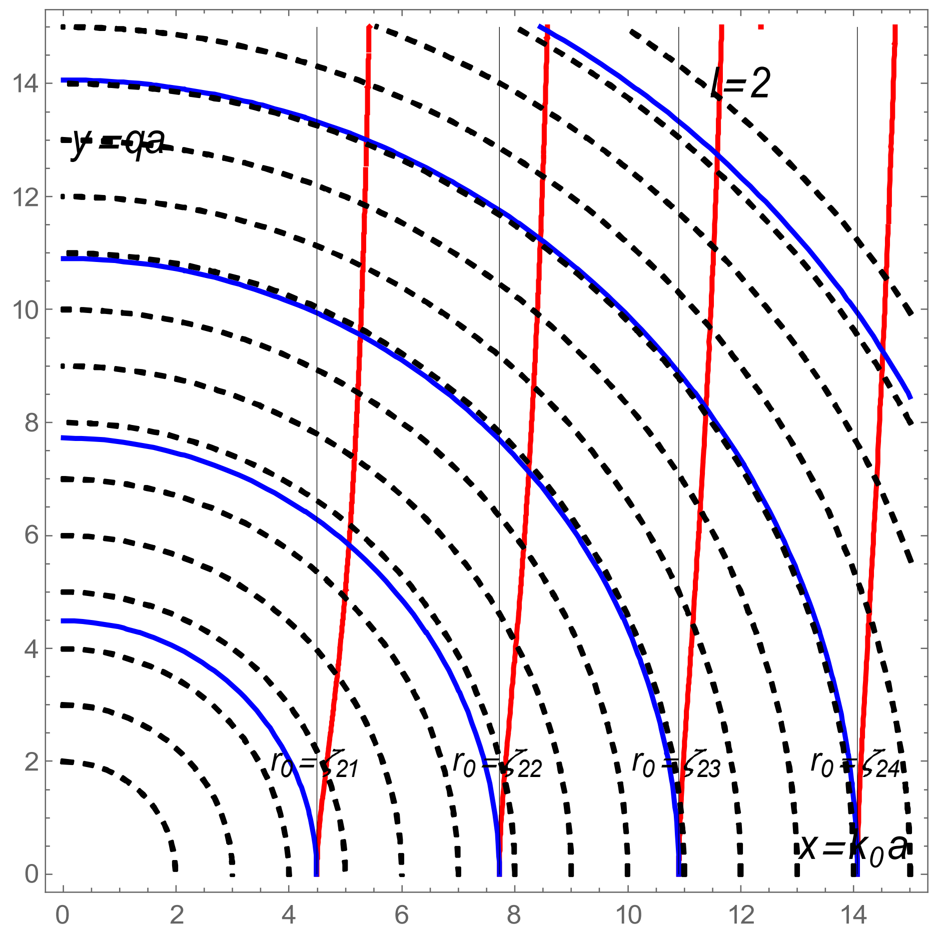


Fig.25 ContourPlot for the boundary condition. y vs x. l = 2. $E=0$ at $r_0 = z(2-1=1, n)$, at which $j_1(r_0)=0$. $r_0 = \varsigma_{21}$, ς_{22}, ς_{23}, ς_{24}.

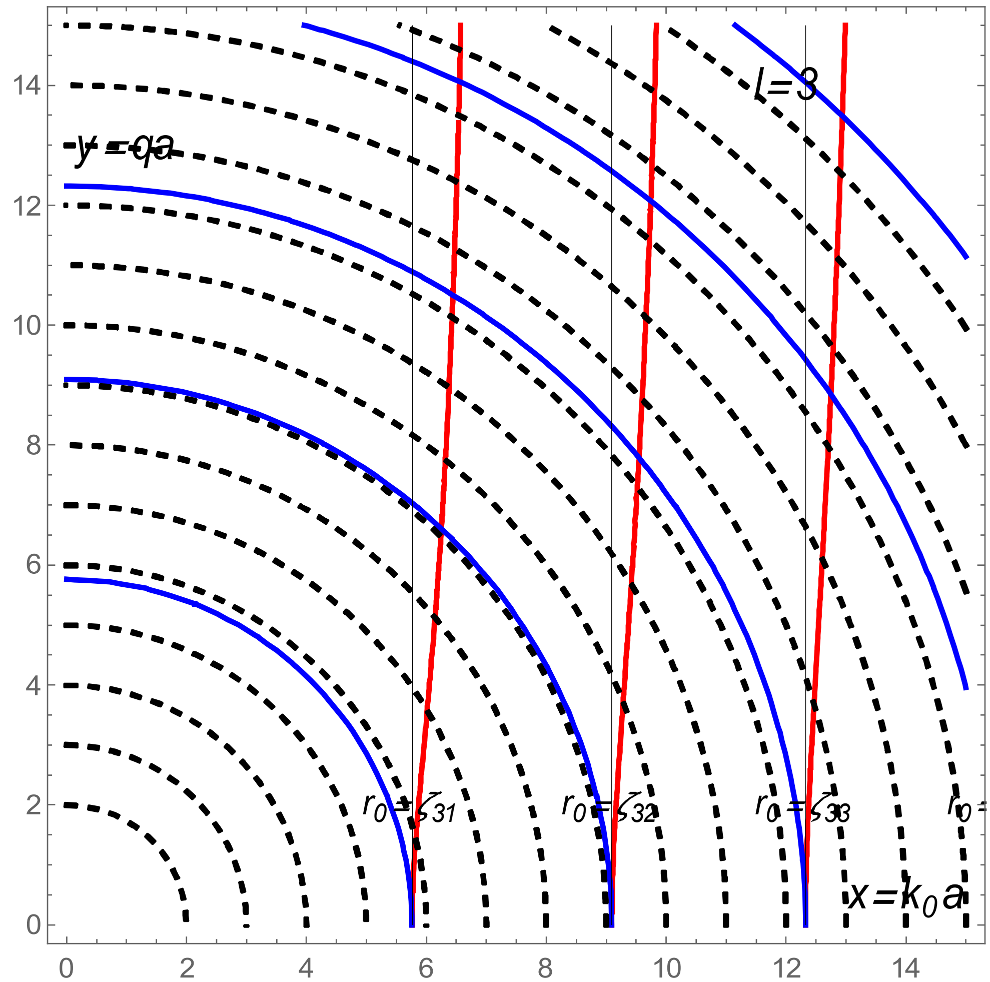


Fig.26 ContourPlot for the boundary condition. y vs x. l = 3. $E=0$ at $r_0 = z(3-1=2, n)$, at which $j_2(r_0)=0$. $r_0 = \varsigma_{31}$, ς_{32}, ς_{33}, ς_{34}.

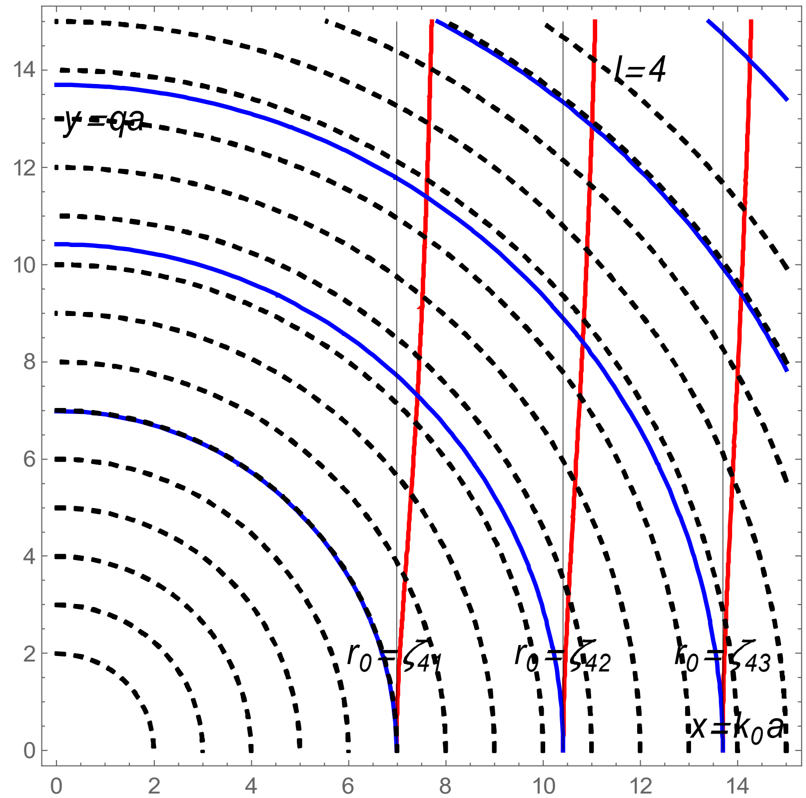


Fig.27 ContourPlot for the boundary condition. y vs x. $l=4$. $E=0$ at $r_0 = z(4-1=3,n)$, at which $j_3(r_0)=0$. $r_0 = \varsigma_{41}$, ς_{42}, ς_{43}.

13. The energy eigenvalue $|E|$ vs potential depth with l = 1, 2, 3, and 4.

We now use the ContourPlot of

$$\frac{xj_{l-1}(x)}{j_l(x)} = \frac{iyh_{l-1}^{(1)}(iy)}{h_l^{(1)}(iy)} \quad \text{with} \quad x = \sqrt{r_0^2 - y^2}$$

We get the scaled energy eigenvalue $y^2 = (qa)^2 = \frac{2\mu a^2}{\hbar^2}|E|$ as a function of scaled potential depth $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$ for l = 1, 2, 3, and 4.

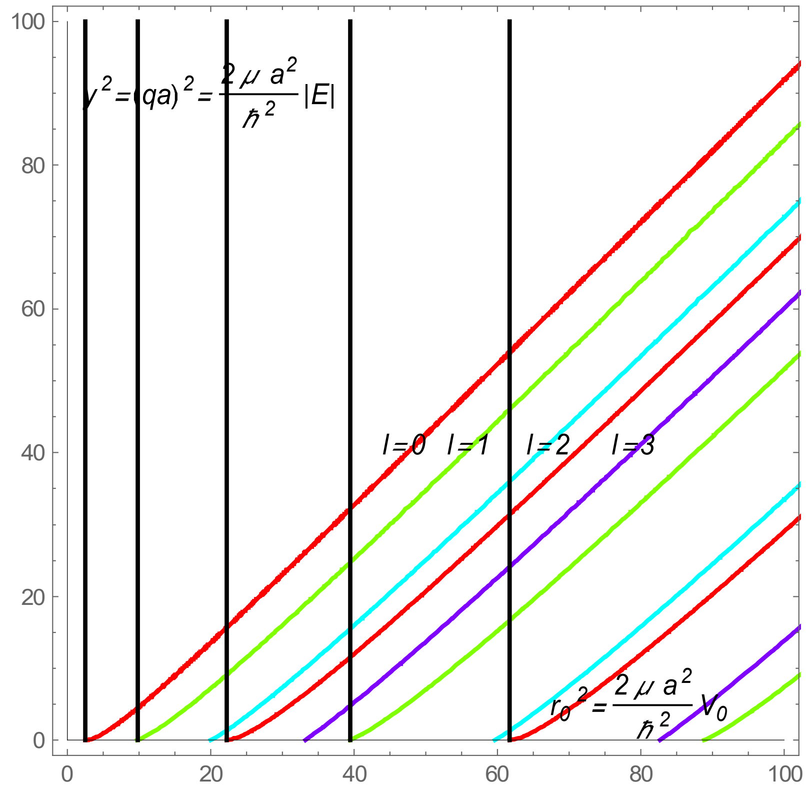


Fig.28 $y^2=(qa)^2=\dfrac{2\mu a^2}{\hbar^2}\left|E\right|$ vs $r_0{}^2=\dfrac{2\mu a^2 V_0}{\hbar^2}$ for l = 1, 2, 3, and 4. $\left|E\right|=0$ at $r_0=z(l-1,n)$, at which $j_{l-1}(r_0)=0$

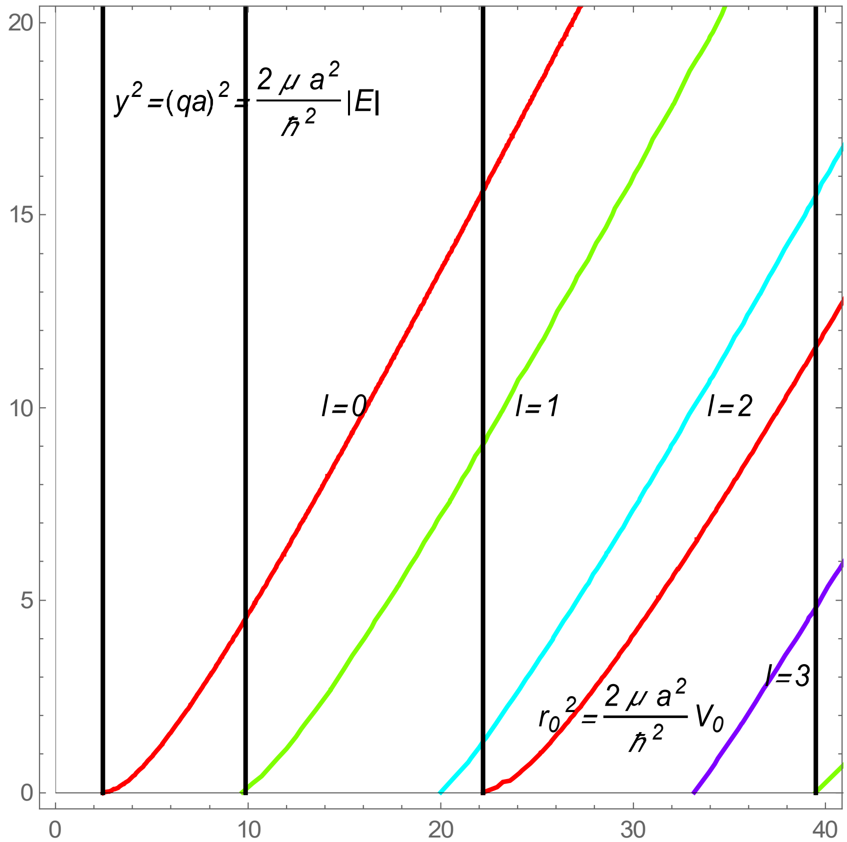


Fig.29 $y^2 = (qa)^2 = \dfrac{2\mu a^2}{\hbar^2}|E|$ vs $r_0{}^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$ for l = 0, 1, 2, 3, and 4. $|E| = 0$ at $r_0 = [z(l-1, n)]$, at which $j_{l-1}(r_0) = 0$

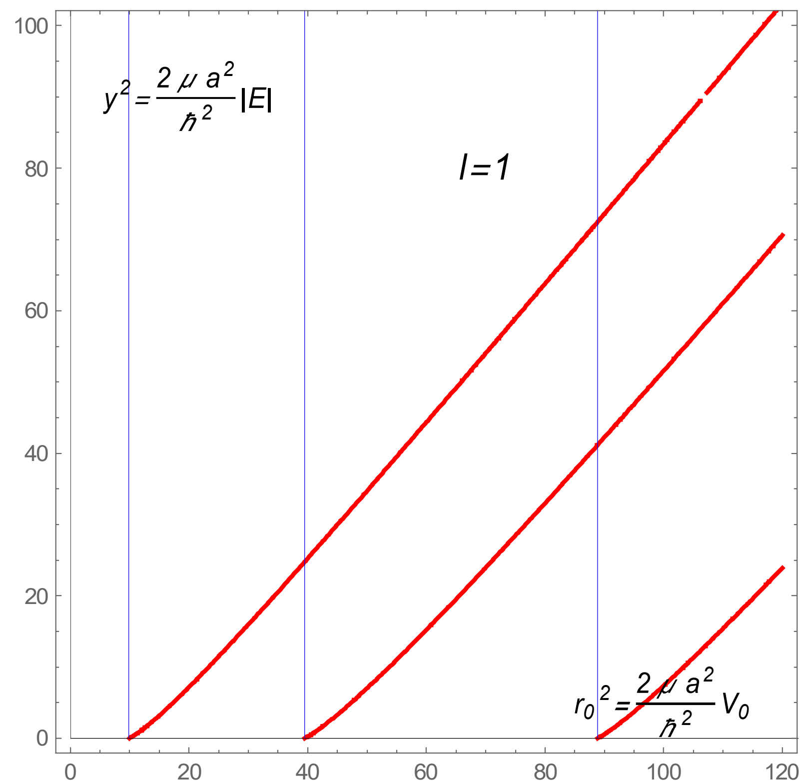


Fig.30(a) $l=1$. $y^2 = (qa)^2 = \dfrac{2\mu a^2}{\hbar^2}|E|$ vs $r_0{}^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0{}^2 = \varsigma_{11}{}^2$, $\varsigma_{12}{}^2$, $\varsigma_{13}{}^2$ (vertical blue lines).

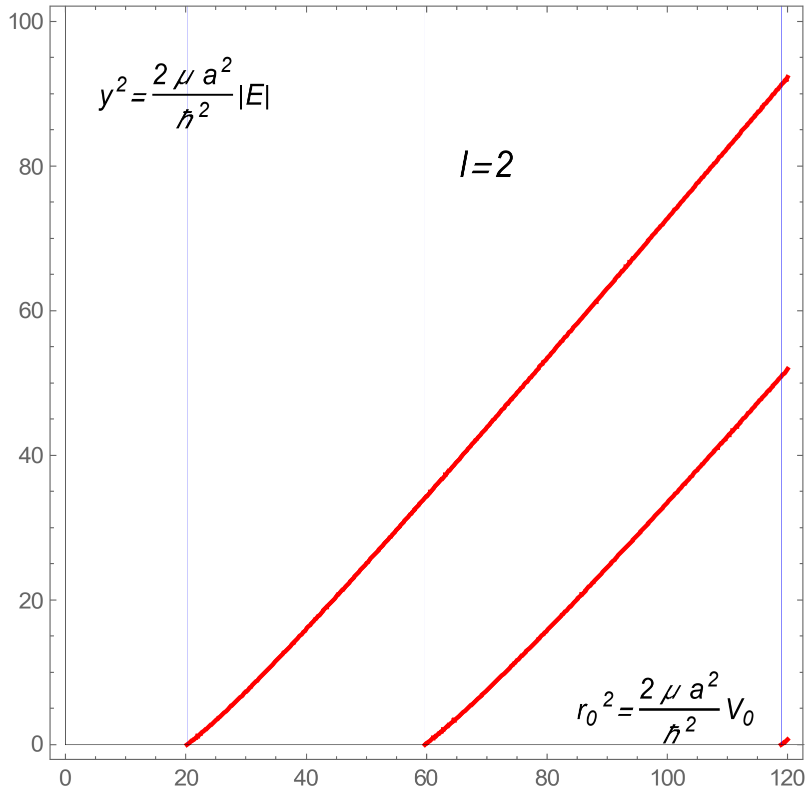


Fig.30(b) $l = 2$. $y^2 = (qa)^2 = \dfrac{2\mu a^2}{\hbar^2}|E|$ vs $r_0{}^2 = \dfrac{2\mu a^2 V_0}{\hbar^2}$. $r_0{}^2 = \varsigma_{21}{}^2,\ \varsigma_{22}{}^2,\ \varsigma_{23}{}^2$ (vertical blue lines).

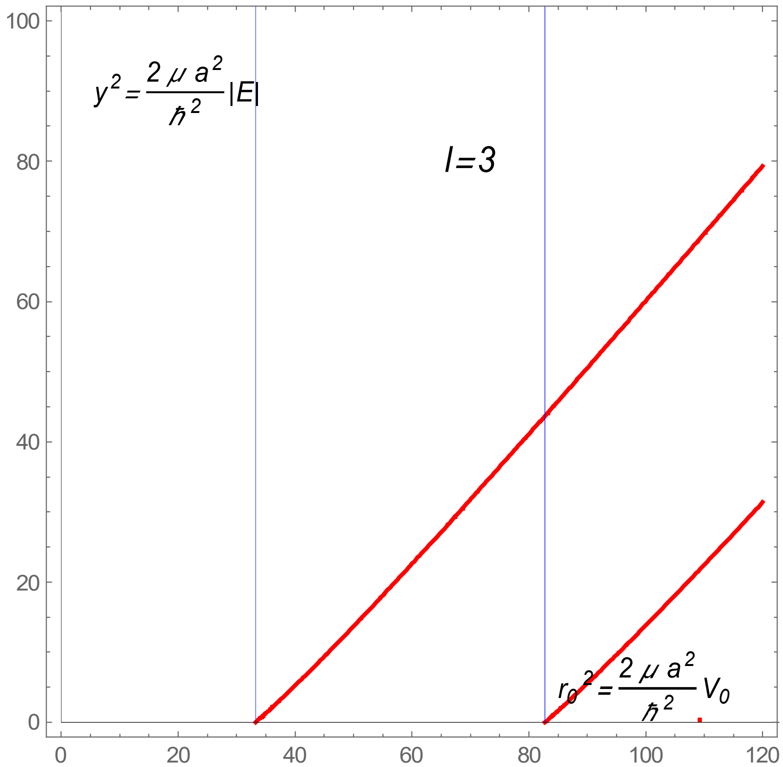


Fig.30 (c) $l = 3$. $y^2 = (qa)^2 = \frac{2\mu a^2}{\hbar^2}|E|$ vs $r_0^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0^2 = \varsigma_{31}^2$, ς_{32}^2, ς_{33}^2 (vertical blue lines).

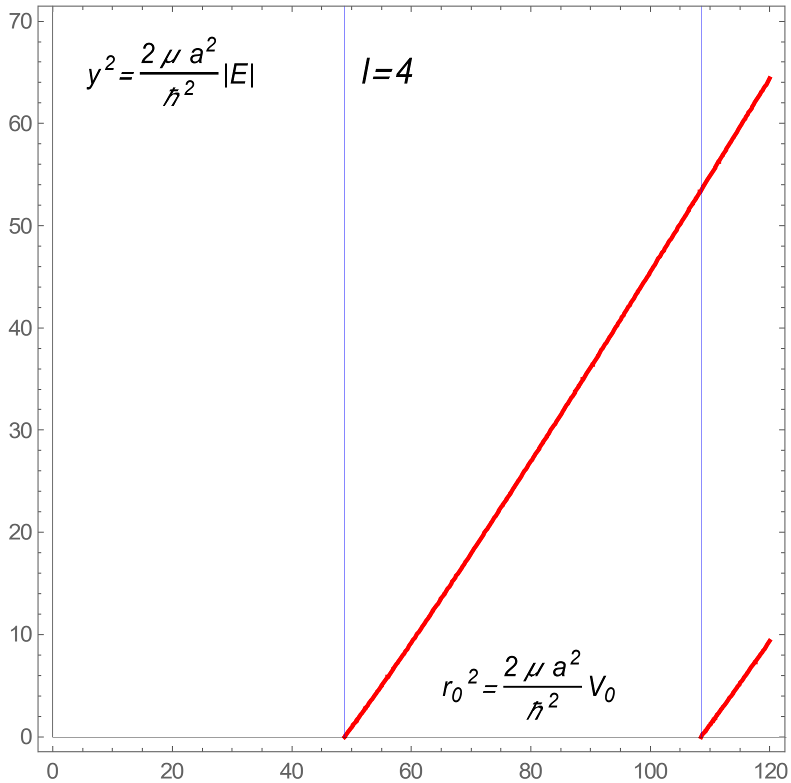


Fig.30(d) $l = 4$. $y^2 = (qa)^2 = \frac{2\mu a^2}{\hbar^2}|E|$ vs $r_0{}^2 = \frac{2\mu a^2 V_0}{\hbar^2}$. $r_0{}^2 = \varsigma_{41}{}^2$, $\varsigma_{42}{}^2$ (vertical blue lines).

As is shown From **Figs.30,** there is no bound state for

$$\frac{2\mu V_0 a^2}{\hbar^2} < [z(l-1,1)]^2 .$$

There is a single bound state (ground state) for

$$[z(l-1,1)]^2 < \frac{2\mu V_0 a^2}{\hbar^2} < [z(l-1,2)]^2 .$$

There are two bound states (ground state and the first excited state) for

$$[z(l-1,2)]^2 < \frac{2\mu V_0 a^2}{\hbar^2} < [z(l-1,3)]^2 .$$

There are three bound states (ground state, the first excited state, and the second excited state) for

$$[z(l-1,3)]^2 < \frac{2\mu V_0 a^2}{\hbar^2} < [z(l-1,4)]^2 .$$

Here we define the minimum energy eigenvalue $|E|$ among the allowed states for fixed V_0. The value of $|E|$ is the closest to $E = 0$ (but $E<0$). We make a plot of the minimum eigenvalue $|E|$ as a function of potential depth V_0.

14. The effect of the angular momentum

The minimum value of V_0a^2 for the *p*-wave binding ($l = 1$) is larger than that for the *s*-wave binding ($l = 0$), and so on.

$$\frac{2\mu V_0 a^2}{\hbar^2} \geq \varsigma_{01}{}^2 \qquad (l = 0)$$

$$\frac{2\mu V_0 a^2}{\hbar^2} \geq \varsigma_{11}{}^2 \qquad (l = 1),$$

$$\frac{2\mu V_0 a^2}{\hbar^2} \geq \varsigma_{21}{}^2 \qquad (l = 2),$$

$$\frac{2\mu V_0 a^2}{\hbar^2} \geq \varsigma_{31}{}^2 \qquad (l = 3),$$

$$\frac{2\mu V_0 a^2}{\hbar^2} \geq \varsigma_{41}{}^2 \qquad (l = 4),$$

where

$$\varsigma_{01} = \frac{\pi}{2}, \qquad \varsigma_{11} = \pi , \qquad \varsigma_{21} = 4.49341 = 1.430297\pi ,$$
$$\varsigma_{31} = 5.76346 = 1.83457\pi , \qquad \varsigma_{41} = 6.9879 = 2.22432\pi$$

Physically, the meaning of this is very clear. In the case of $l = 1$, there exists a centrifugal barrier and, therefore, a particle requires stronger attraction for binding. In fact, it can be shown that the strength of the spherical potential well, V_0a^2, required to bind a particle of arbitrary *l* increases monotonically with *l*. This system does not show any degeneracy in the *l* quantum number. (**Das**).

15. The use of Heisenberg's principle of uncertainty.

David Bohm, Quantum Theory (Dover, 1079).

The fact that no bound states are possible unless

$$\frac{2\mu a^2 V_0}{\hbar^2} \geq (\frac{\pi}{2})^2,$$

is easily understood in terms of the Heisenberg's principle of uncertainty. To have a bound state, a particle must be localized roughly within the radius of the well *a*. To have a wave function large only in a region of the size of the well, there must also be a range of momenta $p = \hbar / a$ and, therefore, the energy is

$$\frac{1}{2\mu} p^2 = \frac{1}{2\mu} (\frac{\hbar}{a})^2.$$

Before a particle can be trapped within the well, the potential energy given up when the particle enters the well must be greater than the kinetic energy that the particle obtains merely because it is localized within the radius *a*. Thus, no bound states at all are possible unless

$$E = \frac{1}{2\mu} (\frac{\hbar}{a})^2 - V_0 < 0, \qquad \text{or} \qquad \frac{2\mu a^2 V_0}{\hbar^2} > 1 \qquad \text{(bound state)}$$

If V_0 is barely great enough to provide the kinetic energy necessary to localize the particle within the well, then the binding energy $|E|$ will be very small.

If V_0 is increased, the binding energy becomes greater, and eventually V_0 becomes so great that it can supply the kinetic energy necessary to make the wave function oscillate once within the well. At this point, a new bound state becomes possible. If V_0 is made greater still, eventually a third oscillation becomes possible, then a fourth, etc. Thus, the number of bound states depends on how much deeper the well is than the minimum amount needed to contain the particle within the well. We now apply the Heisenberg's principle of uncertainty to the case of $l = 1, 2, 3, 4, \ldots$. Suppose that the linear momentum is approximated as

$$p \approx \frac{\hbar}{a} \lambda_0,$$

where λ_0 is on the order of unity. Then energy *E* is evaluated as

$$E \simeq \frac{\hbar^2}{2\mu a^2} [\lambda_0{}^2 + l(l+1)] - V_0.$$

The bound state with $E < 0$, appears, only if

$$\frac{2\mu a^2 V_0}{\hbar^2} = r_0^{\ 2} > \lambda_0^{\ 2} + l(l+1) > l(l+1).$$

which is consistent with the condition that the effective potential at $r = a$ is negative.

$$r_0^{\ 2} > l(l+1).$$

We make a plot of $r_0^{\ 2} = 2\mu a^2 V_0 / \hbar^2$ as s function of *l* (*l* = 0, 1, 2, 3, and 4).

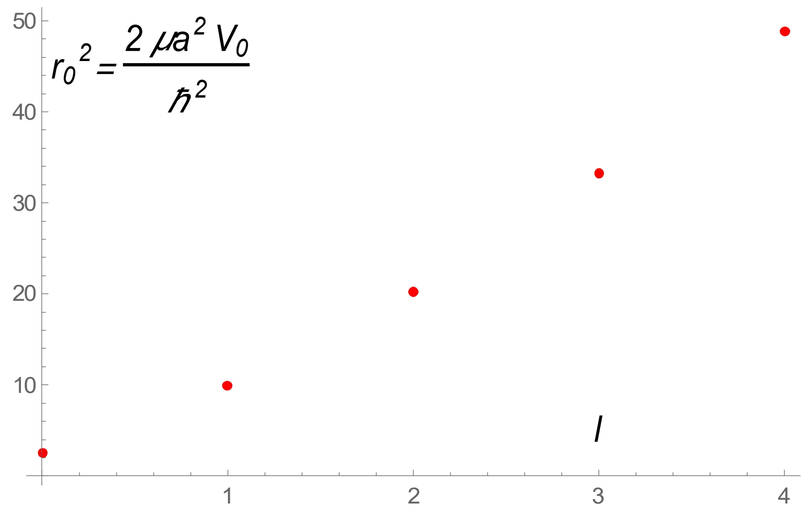


Fig.31 $r_0^{\ 2} = \dfrac{2\mu a^2 V_0}{\hbar^2}$ vs the orbital angular momentum *l*. $\varsigma_{01}^{\ 2} = 2.4674$.. $\varsigma_{11}^{\ 2} = 9.8696$. $\varsigma_{21}^{\ 2} = 20.1907$. $\varsigma_{31}^{\ 2} = 33.2175$. $\varsigma_{41}^{\ 2} = 48.8307$.

As shown in **Fig.31**, the smallest value of $r_0^{\ 2} = 2\mu a^2 V_0 / \hbar^2$ for which exists a bound state (ground state) with *l* = 1, is greater than the corresponding value of $r_0^{\ 2} = 2\mu a^2 V_0 / \hbar^2$ for $l = 0$. This is due to the additional repulsive centrifugal potential energy. A particle possessing angular momentum requires a stronger attractive potential to bind it than a particle with no angular momentum number. Indeed, it turns out that the minimum square well potential strength $r_0^{\ 2} = 2\mu a^2 V_0 / \hbar^2$ required to bind a particle of orbital angular momentum quantum number *l* increases monotonically with increasing *l*.

16. Summary

Although the spherical square well is not a realistic model for the internuclear potential it does give us some useful information on the nuclear force. We note that the proton and neutron have spin 1/2. In fact, the deuteron has an intrinsic spin (*S* = 1), but not 0;

$$D_{1/2} \times D_{1/2} = D_1 + D_0 .$$

This indicates that the nuclear force is spin dependent. The deuteron has a magnetic moment and an electric quadrupole moment (Q = 0.0027 x 10^{-24} cm^2). The existence of the quadrupole moment tells us that the system is not strictly expressed by a spherically symmetric force. However, the departure of spherical symmetry turns out not to be large. The ground state of the deuteron is a mixture of 96 % ($l = 0$) and 4% ($l = 2$) states. This mixing is due to a spin-orbit coupling in the nucleon-nucleon interaction which is neglected in our simple model. Further discussion will be given elsewhere.

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APPENDIX-1
Spherical Hankel function of the first kind

$$h_{-1}^{(1)}(x) = \frac{e^{ix}}{x} . \qquad \text{(which will be derived below)}$$

$$h_0^{(1)}(x) = -ie^{ix}\frac{1}{x} .$$

$$h_1^{(1)}(x) = -e^{ix}\left(\frac{x+i}{x^2}\right).$$

$$h_2^{(1)}(x) = ie^{ix}\left(\frac{x^2+3ix-3}{x^3}\right).$$

$$h_3^{(1)}(x) = e^{ix}\left(\frac{x^3 + 6ix^2 - 15x - 15i}{x^4}\right).$$

Spherical Bessel function

$$j_{-1}(x) = \frac{\cos x}{x}$$ (which will be derived below)

$$j_0(x) = \frac{\sin x}{x},$$

$$j_1(x) = \frac{\sin x - x\cos x}{x^2},$$

$$j_2(x) = \frac{(3 - x^2)\sin x - 3x\cos x}{x^3},$$

$$j_3(x) = \frac{3(5 - 2x^2)\sin x + x(-15 + x^2)\cos x}{x^4}.$$

Recursion formula:

$$j_{-1}(x) + j_1(x) = \frac{1}{x} j_0(x).$$

Rayleigh formula:

$$j_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}.$$

$$h_l^{(1)}(x) = -i(-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{e^{ix}}{x}.$$

Mathematica

$j_l(z)$: SphericalBesselJ[l,z]

$h_l^{(1)}(z)$: SphericalHankelH1[l,z]

((**Note**)) **Derivation of the expression of** $j_{-1}(x)$ **and** $h_{-1}^{(1)}(x)$

Using

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x), \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

we get

$$j_{-1}(x) = \sqrt{\frac{\pi}{2x}} J_{-1/2}(x) = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \cos x = \frac{\cos x}{x}.$$

We also get

$$j_{-1}(x) = \frac{1}{x} j_0(x) - j_1(x) = \frac{\cos x}{x},$$

$$h_{-1}^{(1)}(x) = \frac{1}{x} h_0^{(1)}(x) - h_1^{(1)}(x) = \frac{e^{ix}}{x}.$$

Plot of $j_{-1}(x), j_0(x), j_1(x)$ as a function of x

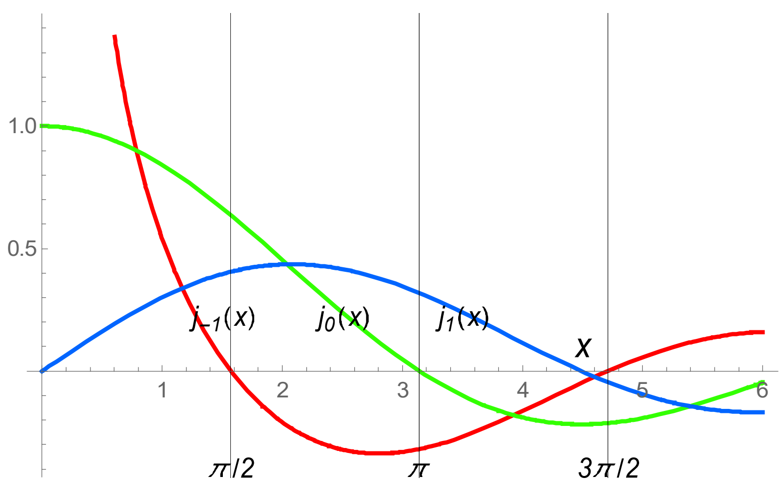


Fig.A-1 Plots of $j_{-1}(x), j_0(x), j_1(x)$, as a function of x.

Plot of $ih_{-1}^{(1)}(ix)$, $h_0^{(1)}(ix)$ as a function of x

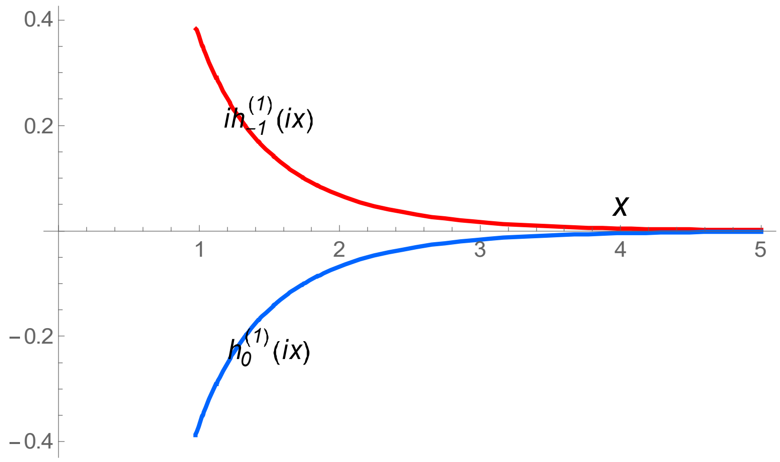


Fig.A2: Plots of $ih_{-1}^{(1)}(ix)$, $h_0^{(1)}(ix)$ as a function of x.

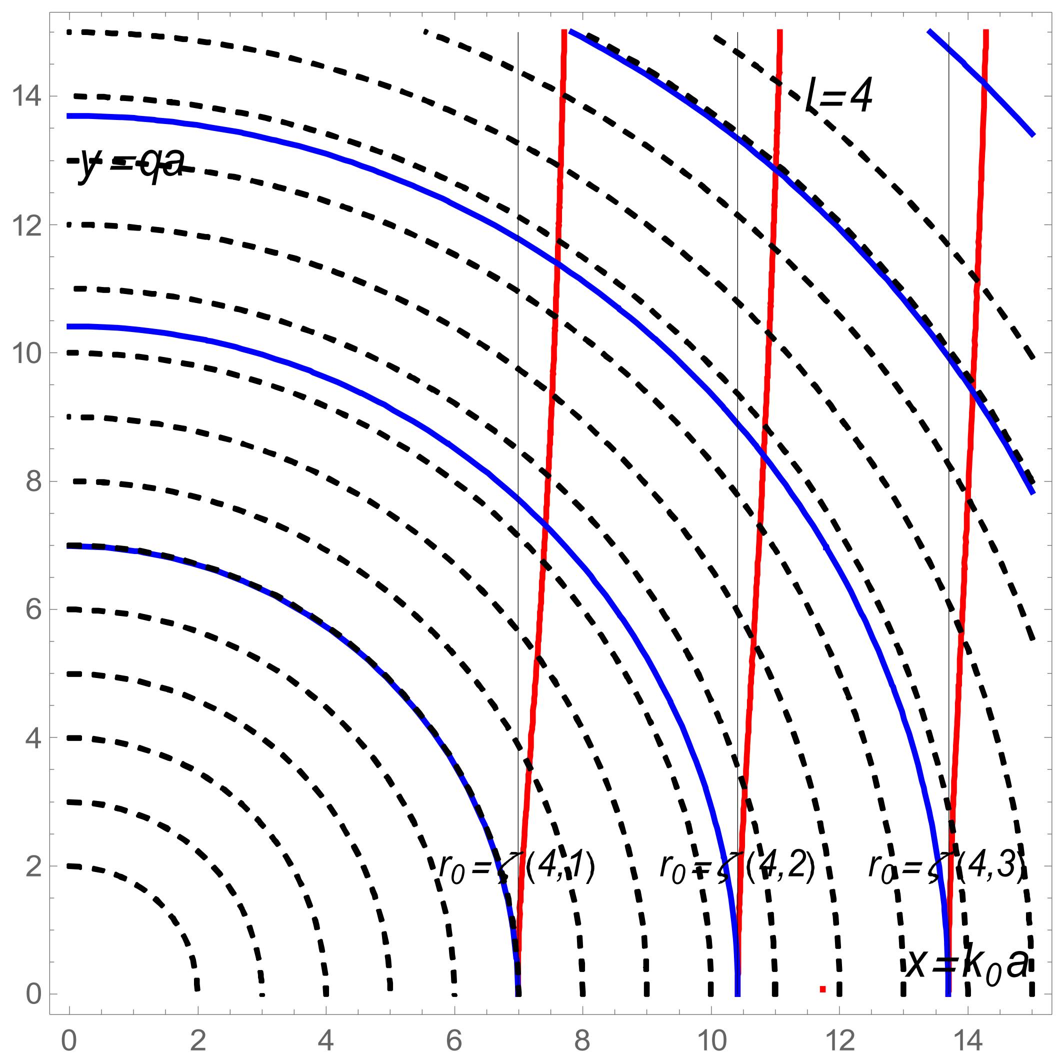
APPENDIX-2 Mathematica

((**Mathematica-1**))

$l = 1$

```
Clear["Global`*"];
eq1 = (x SphericalBesselJ[L - 1, x])/(SphericalBesselJ[L, x]) ==
   (ⅈ y SphericalHankelH1[L - 1, ⅈ y])/(SphericalHankelH1[L, ⅈ y]);
s1 = ContourPlot[Evaluate[eq1 /. L → 1], {x, 0, 15},
   {y, 0, 15}, ContourStyle → {Red, Thick}];
g1 =
  Graphics[
   {Text[Style["x=k₀a", Italic, Black, 15],
     {14, 0.5}],
    Text[Style["y=qa", Italic, Black, 15], {1, 13}],
    Text[Style["l=1", Italic, Black, 15], {12, 12}],
    Text[Style["r₀=π", Italic, Black, 12], {2, 2}],
    Text[Style["r₀=2π", Italic, Black, 12], {5, 3.3}],
    Text[Style["r₀=3π", Italic, Black, 12],
     {8.3, 4.3}],
    Text[Style["r₀=3π", Italic, Black, 12],
     {11.1, 5.1}], Black, Thin,
    Table[Line[{{n π, 0}, {n π, 15}}], {n, 1, 4, 1}]}];
```

```
s2 = ContourPlot[
  Evaluate[Table[x^2 + y^2 == r0^2,
    {r0, π / 2, 9 π / 2, π / 2}]], {x, 0, 15},
  {y, 0, 15},
  ContourStyle → Table[{Black, Dashed}, {i, 0, 2}]];
s3 = ContourPlot[
  Evaluate[Table[x^2 + y^2 == r0^2, {r0, π, 4 π, π}]],
  {x, 0, 15}, {y, 0, 15},
  ContourStyle → Table[{Green, Thick}, {i, 0, 4}]];
Show[s1, s2, s3, g1]
```



((**Mathematica-2**))

```
Clear["Global`*"];
```

$$eq0 = y == -\sqrt{s0 - y^2}\ \mathrm{Cot}\left[\sqrt{s0 - y^2}\right]\ /.\ y \to \sqrt{y0};$$

```
g0 = ContourPlot[Evaluate[eq0], {s0, 0, 200},
  {y0, 0, 100}, ContourStyle → {Red, Thick},
  PlotPoints → 100];
```

$$\mathrm{Bin}[L\_] := \mathrm{Module}\Big[\{eq1,\ g1\},$$

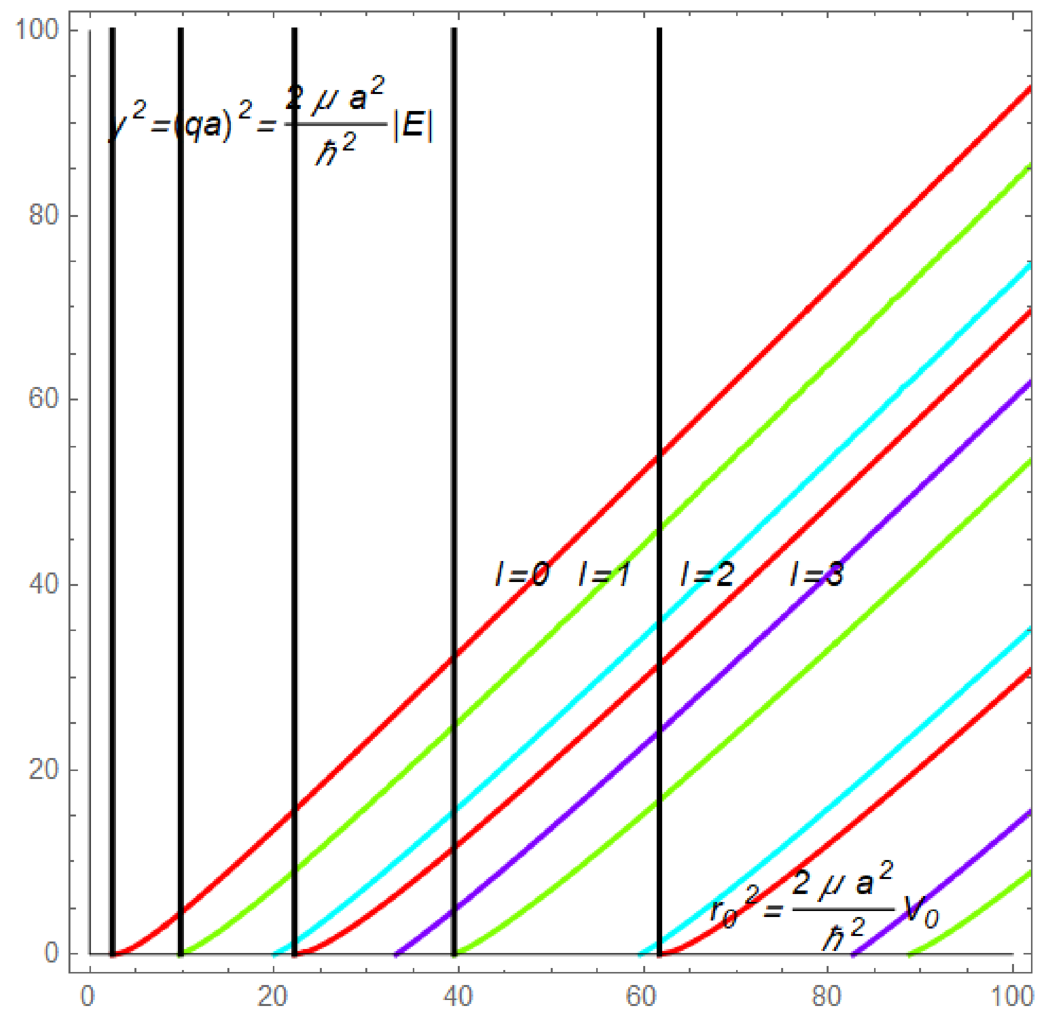
$$eq1 = \frac{\sqrt{s0 - y^2}\ \mathrm{SphericalBesselJ}\left[L - 1,\ \sqrt{s0 - y^2}\right]}{\mathrm{SphericalBesselJ}\left[L,\ \sqrt{s0 - y^2}\right]} == \frac{i\, y\ \mathrm{SphericalHankelH1}[L - 1,\ i\, y]}{\mathrm{SphericalHankelH1}[L,\ i\, y]}\ /.\ y \to \sqrt{y0};$$

```
g1 = ContourPlot[Evaluate[eq1], {s0, 0, 200},
  {y0, 0, 100},
  ContourStyle → {Hue[0.25 L], Thick}]];
```

```
r1 =
  Graphics[{Black, Thick,
    Table[Line[{{(π/2 n)^2, 0}, {(π/2 n)^2, 100}}],
     {n, 1, 5}]}];

t1 =
  Graphics[
   {Text[Style["r0^2 = (2 μ a^2)/ħ^2 V0", Black, 12, Italic],
     {80, 5}],
    Text[Style["y^2 = (qa)^2 = (2 μ a^2)/ħ^2 |E|", Black, 12,
      Italic], {20, 90}], Black, Thin,
    Line[{{0, 0}, {100, 0}}], Line[{{0, 0}, {0, 100}}],
    Text[Style["l=0", Black, 12, Italic], {47, 41}],
    Text[Style["l=1", Black, 12, Italic], {56, 41}],
    Text[Style["l=2", Black, 12, Italic], {67, 41}],
    Text[Style["l=3", Black, 12, Italic], {79, 41}]}];

Show[g0, Bin[1], Bin[2], Bin[3], r1, t1,
 PlotRange → {{0, 100}, {0, 100}}]
```



$y^2=(qa)^2=\frac{2\mu a^2}{\hbar^2}|E|$
$l=0$
$l=1$
$l=2$
$l=3$
$r_0{}^2=\frac{2\mu a^2}{\hbar^2}V_0$